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ROBUST STABILITY OF POSITIVE LINEAR TIME-DELAY SYSTEMS UNDER AFFINE PARAMETER PERTURBATIONS

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Abstract. In this paper we study robust stability of positive linear time-delay systems under arbitrary affine parameter perturbations. It is shown that real and complex stability radii of positive systems coincide for block-diagonal disturbances. Moreover, for these stability radii, estimates and computable formulae are derived which generalize to positive retarded systems known results obtained earlier for positive systems with no time-delays. Some illustrative examples are given.

1. Introduction

A dynamical system in the n-dimensional space $\mathbb{R}^n$ is called *positive* if any trajectory of the system starting from an initial position in the positive orthant $\mathbb{R}^n_+$ always remains in $\mathbb{R}^n_+$. Positive dynamical systems play an important role in the modeling of dynamical phenomena whose variables are restricted to be nonnegative. Their applications can be found in many areas such as economics, population dynamics and ecology. Mathematical theory of positive systems is based essentially on the theory of nonnegative matrices developed by Perron and Frobenius. As references we mention [2], [13].

In this paper we study the asymptotic stability of *uncertain* positive systems described by the linear differential-difference equation of the form

$$\dot{x}(t) = A_0x(t) + A_1x(t-h), \qquad t \geq 0, \tag{1}$$

where the system matrices $A_0, A_1 \in \mathbb{R}^{n\times n}$ are subjected to arbitrary affine parameter perturbations of the form

$$A_0 \rightsquigarrow A_0 + D_0\Delta_0E_0, \qquad A_1 \rightsquigarrow A_1 + D_1\Delta_1E_1. \tag{2}$$

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Here $D_i, E_i, i = 0, 1$ are given matrices specifying the structure of the perturbations and Δ_0, Δ_1 are unknown disturbance matrices whose sizes are measured by their operator norms $\|\Delta_0\|, \|\Delta_1\|$. The main problem of robustness of stability for the system (1) is to determine the maximal $r > 0$ for which the family of systems

$$\begin{aligned} &\dot{x}(t) = (A_0 + D_0\Delta_0 E_0)x(t) + (A_1 + D_1\Delta_1 E_1)x(t-h), \\ &\gamma(\|\Delta_0\|, \|\Delta_1\|) < r \end{aligned} \tag{3}$$

is asymptotically stable. Here, $\gamma(\cdot,\cdot) : \mathbb{R}^2 \mapsto \mathbb{R}_+$ is a given function representing the aggregate size of perturbations. If disturbances Δ_i, $i = 0, 1$, are allowed to be complex, the maximal number r is called the *complex stability radius* and denoted by $r_{\mathbb{C}}$. If only real perturbations are considered, the *real stability radius* $r_{\mathbb{R}}$ is obtained. These stability radii can be considered as robustness measures of stability of the system.

In the case of dynamical systems with no delays

$$\dot{x}(t) = Ax(t), \qquad t \geq 0 \tag{4}$$

the problem of computing stability radii has been studied over the last decade in a series of works initiated by Hinrichsen and Pritchard (see, e.g. [6], [7], [8], [15]). In general, the computation of $r_{\mathbb{C}}$ and $r_{\mathbb{R}}$ requires the solution of a complicated global optimization problem. For the class of positive systems of the form (4) the problem of robust stability was studied in [16], [17], where it has been shown that the real and the complex stability radii coincide and can be determined via an easily computable formula. These results have been extended in [9], [10], to arbitrary affine parameter perturbations of the form

$$A \rightsquigarrow A + \sum_{i=1}^{N} B^i \Delta^i C^i, \tag{5}$$

and

$$A \rightsquigarrow A + \sum_{i=1}^{N} \delta^i B^i. \tag{6}$$

where the matrices B^i and C^i are given nonnegative matrices defining the structure of the perturbations, Δ^i and δ^i are respectively unknown matrices and scalars representing the parameter uncertainty. Recently, the similar consideration has been done in [4] and [19] for linear positive systems $\dot{x}(t) = Ax(t)$ in Banach spaces.

The purpose of the present paper is to generalize the results of [6] and [9] to the positive linear time-delay system of the form (1). It is important to note that the problem of robust stability of delay systems has attracted attention of researchers only recently, see e.g. [3], [12], [18]. In particular, in [18] a formula of complex stability radius of a linear delay system was established. In [4] the robustness of stability of infinite-dimensional linear equation *associated* to the positive delay system (1) was studied. However, so far the problem of computing the stability radii of the *original* positive delay system under arbitrary affine perturbations as considered in [9], [10] has not been studied yet.

The organization of this paper is as follows. In the next section we recall some theorems on nonnegative matrices and derive preliminary results for later use. The main results of the paper will be presented in Section 3. We show that for positive linear time-delay systems subjected to affine perturbations of block-diagonal structure, the stability radii with respect to complex, real and nonnegative disturbances coincide. Moreover, we derive a simple formula for computing the real stability radius of the system and we illustrate the obtained results by some examples.

2. Preliminaries

Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and n, ℓ, q be positive integers. Inequalities between real matrices or vectors will be understood componentwise, i.e. for two real $\ell \times q$-matrices $A = (a_{ij})$ and $B = (b_{ij})$, the inequality $A \geq B$ means $a_{ij} \geq b_{ij}$ for $i = 1, \cdots, \ell$, $j = 1, \cdots, q$. The set of all nonnegative $\ell \times q$-matrices is denoted by $\mathbb{R}_+^{\ell \times q}$. If $x \in \mathbb{K}^n$ and $P \in \mathbb{K}^{\ell \times q}$ we define $|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$. For any matrix $A \in \mathbb{K}^{n \times n}$ the *spectral radius* and *spectral abscissa* of A are denoted, respectively, by

$$\rho(A) = \max\{|\lambda| : \ \lambda \in \sigma(A)\},$$

$$\mu(A) = \max\{\mathrm{Re}\lambda : \ \lambda \in \sigma(A)\},$$

where $\sigma(A) := \{s \in \mathbb{C} : \ \det(sI - A) = 0\}$ is the set of all eigenvalues of A. $A \in \mathbb{R}^{n \times n}$ is called a *Metzler matrix* if all the off-diagonal elements of A are nonnegative or, equivalently, $tI + A \geq 0$ for some $t \geq 0$.

A norm $\|\cdot\|$ on $\mathbb{K}^n$ is said to be *monotonic* if $\|x\| = \|\,|x|\,\|$ for all $x \in \mathbb{K}^n$. Every p-norm on $\mathbb{K}^n, 1 \leq p \leq \infty$, is monotonic. Throughout the paper, unless otherwise stated, the norm $\|M\|$ of a matrix $M \in \mathbb{K}^{\ell\times q}$ is always understood as the operator norm defined by $\|M\| = \max_{\|y\|=1} \|My\|$, where $\mathbb{K}^q$ and $\mathbb{K}^\ell$ are provided with some monotonic vector norms. Then, the following monotonicity property holds, see e.g. [17],

$$P \in \mathbb{K}^{l\times q},\ Q \in \mathbb{R}_+^{l\times q},\ |P| \le Q \ \Rightarrow\ \|P\| \le \|\,|P|\,\| \le \|Q\|. \tag{7}$$

In order to facilitate the presentation, we summarize some existing results on properties of nonnegative matrices and Metzler matrices which will be used in the sequel (see, e.g. [2], [13], [17]).

Theorem 2.1. *Suppose that $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix. Then*

(i) *(Perron-Frobenius) $\mu(A)$ is an eigenvalue of A and there exists a nonnegative eigenvector $x \ge 0$, $x \ne 0$ such that $Ax = \mu(A)x$.*

(ii) *Given $\alpha \in \mathbb{R}$, there exists a nonzero vector $x \ge 0$ such that $Ax \ge \alpha x$ if and only if $\mu(A) \ge \alpha$.*

(iii) *$(tI_n - A)^{-1}$ exists and is nonnegative if and only if $t > \mu(A)$.*

(iv) *Given $B \in \mathbb{R}_+^{n\times n}$, $C \in \mathbb{C}^{n\times n}$. Then*

$$|C| \le B \quad \Longrightarrow \quad \mu(A + C) \le \mu(A + B). \tag{8}$$

It follows from Theorem 2.1 (i) that for any $A \in \mathbb{R}_+^{n\times n}$,

$$\rho(A) = \mu(A). \tag{9}$$

Consider the linear delay system (1) and denote by $\sigma(A_0, A_1)$ the set of all roots of the characteristic polynomial:

$$\sigma(A_0, A_1) = \{\lambda \in \mathbb{C} : \det(\lambda I - A_0 - A_1 e^{-h\lambda}) = 0\}.$$

Then

$$\mu_0 := \mu(A_0, A_1) := \max\{\mathrm{Re}\ \ \lambda :\ \lambda \in \sigma(A_0, A_1)\}$$

is called the *spectral abscissa of the linear delay system* (1). It is well-known that the system (1) is asymptotically stable if and only if $\sigma(A_0, A_1) \subset \mathbb{C}^-$ or, equivalently, $\mu(A_0, A_1) < 0$, see e.g. [1], [5]. For a given $\phi \in C([-h, 0], \mathbb{R}^n)$ the unique solution of the equation (1) satisfying the initial condition

$$x(t) = \phi(t), \qquad t \in [-h, 0], \tag{10}$$

is defined by

$$x(t) = e^{A_0 t}\phi(0) + \int_0^t e^{A_0(t-\tau)} A_1 x(\tau - h)\, d\tau, \tag{11}$$

for $t \ge 0$.

Definition 2.2. The system (1) is said to be positive if, for any nonnegative initial condition $\phi \in C([-h,0],\mathbb{R}^n_+)$, the corresponding solution $x(\cdot)$ satisfies $x(t) \in \mathbb{R}^n_+$ for every $t \geq 0$.

It is well-known that the system (1) is positive if and only if A_0 is a Metzler matrix and $A_1 \geq 0$, see e.g. [14].

We shall need the following properties of the spectral abscissa of positive delay systems.

Lemma 2.3. *Let $A_0 \in \mathbb{R}^{n\times n}$ be a Metzler matrix and $A_1 \in \mathbb{R}^{n\times n}_+$. Then the spectral abscissa μ_0 of the delay system* (1) *is equal to the spectral abscissa μ_1 of the Metzler matrix $A_0 + A_1 e^{-h\mu_0}$, that is*

$$\mu_0 := \mu(A_0, A_1) = \mu(A_0 + A_1 e^{-h\mu_0}) =: \mu_1. \tag{12}$$

Proof. By definition, there exist $\gamma_0 \in \mathbb{R}, y_0 \in \mathbb{C}^n, y_0 \neq 0$ such that

$$(A_0 + A_1 e^{-h(\mu_0 + \imath\gamma_0)})y_0 = (\mu_0 + \imath\gamma_0)y_0.$$

Since A_0 is a Metzler matrix, there exists $t_0 \geq 0$ such that $t_0 I + A_0 \geq 0$. We deduce

$$\begin{aligned}(\mu_0 + t_0)|y_0| &\leq |(\lambda + t_0)y_0| = |(t_0 I + A_0)y_0 + A_1 e^{-h(\mu_0+\imath\gamma_0)}y_0| \\ &\leq (t_0 I + A_0)|y_0| + A_1 e^{-h\mu_0}|y_0|.\end{aligned}$$

It implies $(A_0 + A_1 e^{-h\mu_0})|y_0| \geq \mu_0 |y_0|$. By Theorem 2.1 (ii), we have $\mu_1 := \mu(A_0 + A_1 e^{-h\mu_0}) \geq \mu_0$. By Theorem 2.1 (iv),

$$\mu(A_0 + A_1 e^{-h\theta}) \leq \mu(A_0 + A_1 e^{-h\mu_0}) = \mu_1 \text{ for all } \theta \geq \mu_0. \tag{13}$$

Consider the continuous real function $f(\theta) = \theta - \mu(A_0 + A_1 e^{-h\theta})$ for $\theta \in [\mu_0, +\infty)$. We have, by (13),

$$f(\mu_0) = \mu_0 - \mu(A_0 + A_1 e^{-h\mu_0}) = \mu_0 - \mu_1 \leq 0.$$

Assume $f(\mu_0) < 0$. Since $\lim\limits_{\theta\to+\infty} f(\theta) = +\infty, f(\theta_0) = 0$ for some $\theta_0 > \mu_0$, so that $\theta_0 = \mu(A_0 + A_1 e^{-h\theta_0})$. It follows, by Theorem 2.1 (i), that θ_0 is an eigenvalue of the Metzler matrix $A_0 + A_1 e^{-h\theta_0}$ or, equivalently, $\det(\theta_0 I - A_0 - A_1 e^{-h\theta_0}) = 0$, with $\theta_0 > \mu_0$. This, however, conflicts with the definition of μ_0. Thus, $f(\mu_0) = 0$ and hence $\mu_0 = \mu(A_0 + A_1 e^{-h\mu_0}) = \mu_1$.

Lemma 2.4. *Let* $A_0 \in \mathbb{R}^{n\times n}$ *be a Metzler matrix and* $A_1 \in \mathbb{R}_+^{n\times n}$. *Assume* $\Delta_0, \Delta_1 \in \mathbb{C}^{n\times n}, B_0, B_1 \in \mathbb{R}_+^{n\times n}$ *satisfy* $|\Delta_0| \le B_0, \quad |\Delta_1| \le B_1$. *Then*

$$\mu(A_0 + \Delta_0, A_1 + \Delta_1) \le \mu(A_0 + B_0, A_1 + B_1). \tag{14}$$

Proof. Set $\mu_\delta = \mu(A_0 + \Delta_0, A_1 + \Delta_1)$ and $\mu_b = \mu(A_0 + B_0, A_1 + B_1)$. Then, by defintion, there exist $\omega_\delta \in \mathbb{R}$ and $x_\delta \in \mathbb{C}^n$, $x_\delta \neq 0$ such that

$$\left(A_0 + \Delta_0 + (A_1 + \Delta_1)e^{-h(\mu_\delta + \imath\omega_\delta)}\right) x_\delta = (\mu_\delta + \imath\omega_\delta)x_\delta.$$

and hence,

$$\left(A_0 + |\Delta_0| + (A_1 + |\Delta_1|)e^{-h\mu_\delta}\right) |x_\delta| \ge \mu_\delta |x_\delta|.$$

It follows from Theorem 2.1 (iv) that

$$\mu(A_0 + |\Delta_0| + (A_1 + |\Delta_1|)e^{-h\mu_\delta}) \ge \mu_\delta. \tag{15}$$

On the other hand, by Lemma 2.3, we have

$$\mu_b = \mu(A_0 + B_0 + (A_1 + B_1)e^{-h\mu_b}). \tag{16}$$

Therefore, by Theorem 2.1 (iii) and from (15), (16) it follows

$$\mu_b \ge \mu(A_0 + |\Delta_0| + (A_1 + |\Delta_1|)e^{-h\mu_\delta}) \ge \mu_\delta,$$

completing the proof.

As noted in the introduction, the main problem in the study of robust stability of dynamical systems is to examine to which extent stability of a norminal system is preserved under parameter perturbations of certain classes and structure. Among perturbation classes, the one of the *block-diagonal structure* defined below is most well-known in control theory and will includes all perturbation classes studied in this paper as particular cases.

Definition 2.5. We say that the perturbation class $\mathcal{D} \subset \mathbb{C}^{\ell\times q}$ is of block-diagonal structure if there exist integers $\ell_i \ge 1$, $q_i \ge 1$ for $i \in \underline{N} := \{1, 2, \cdots, N\}$ and a subset $J \subset \underline{N}$ such that $\ell = \sum_{i=1}^{N} \ell_i$, $q = \sum_{i=1}^{N} q_i$,

$$\mathcal{D} = \left\{\operatorname{diag}(\Delta^1, \ldots, \Delta^N); \Delta^i \in \mathcal{D}^i,\ i \in \underline{N}\right\}, \tag{17}$$
$$\mathcal{D}^i = \begin{cases} \mathbb{C}^{\ell_i \times q_i} & \text{if } i \in J, \\ \mathbb{C} I_{q_i} & \text{if } i \in \underline{N} \setminus J, \end{cases}$$

and $\mathcal{D}$ is endowed with the norm

$$\|\Delta\| := \|\Delta\|_{\mathcal{D}} := \|(\|\Delta^1\|, \|\Delta^2\|, \ldots, \|\Delta^N\|)\|_{\mathbb{R}^N} \tag{18}$$

where $\|\cdot\|_{\mathbb{R}^N}$ is a given monotonic norm on $\mathbb{R}^N$.

The following property of the perturbation class of block-diagonal structure will be used in the next section for deriving one of the main results. The proof based on Hahn-Banach Theorem is given in [9].

Lemma 2.6. *If $\mathcal{D} \subset \mathbb{C}^{\ell \times q}$ is a perturbation class of block-diagonal structure, then for each $\Delta \in \mathcal{D}$ and $y \in \mathbb{C}^q$ there exists $\tilde{\Delta} \in \mathcal{D}$ satisfying*

$$\tilde{\Delta} y = \Delta y, \quad |\tilde{\Delta}| \in \mathcal{D} \quad \text{and} \quad \|\,|\tilde{\Delta}|\,\| \le \|\Delta\|. \tag{19}$$

Consider a dynamical system described by the linear differential-difference equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \qquad t \ge 0, \tag{20}$$

where $A_0, A_1 \in \mathbb{R}^{n \times n}$ and $h > 0$ are given. We assume that the system (20) is asymptotically stable and the system matrices A_0, A_1 are subjected to affine parameter perturbations of the following type

$$\begin{aligned} A_0 &\rightsquigarrow A_0 + D_0 \Delta_0 E_0, \qquad \Delta_0 \in \mathcal{D}_0 \subset \mathbb{C}^{\ell^0 \times q^0}, \\ A_1 &\rightsquigarrow A_1 + D_1 \Delta_1 E_1, \qquad \Delta_1 \in \mathcal{D}_1 \subset \mathbb{C}^{\ell^1 \times q^1}, \end{aligned} \tag{21}$$

where $D_i \in \mathbb{R}^{n \times l^i}, E_i \in \mathbb{R}^{q^i \times n}, i = 0, 1$ are given matrices defining the *affine structure* of perturbations and $\mathcal{D}_i \subset \mathbb{C}^{\ell^i \times q^i}$, $i = 0, 1$ are given *perturbation classes* of block-diagonal structure.

The size of each perturbation $\Delta := (\Delta_0, \Delta_1) \in \mathcal{D}_0 \times \mathcal{D}_1$ is measured either by

$$\gamma_1(\|\Delta_0\|, \|\Delta_1\|) = \|\Delta_0\| + \|\Delta_1\|$$

or by

$$\gamma_\infty(\|\Delta_0\|, \|\Delta_1\|) = \max(\|\Delta_0\|, \|\Delta_1\|),$$

where $\|\Delta_0\| = \|\Delta_0\|_{\mathcal{D}_0}$ and $\|\Delta_1\| = \|\Delta_1\|_{\mathcal{D}_1}$ are defined by (18). For simplicity, here and in what follows, the index $\mathcal{D}_i$ for the norm of $\Delta_i, i = 0, 1$ is omitted.

Definition 2.7. The complex stability radius of the system (20) with respect to perturbations of the form (21) is defined by

$$r_{\mathbb{C}} = \inf\{\gamma(\|\Delta_0\|, \|\Delta_1\|) : \Delta_i \in \mathcal{D}_i, i = 0,1, \\ \mu(A_0 + D_0\Delta_0 E_0, A_1 + D_1\Delta_1 E_1) \geq 0\}, \tag{22}$$

where, $\gamma = \gamma_1$ or $\gamma = \gamma_\infty$ and by definition, $\inf \emptyset = \infty$.

If in this definition, instead of the whole class of complex perturbations $\mathcal{D}_i$, we take its subsets of *real* and of *nonnegative* perturbations defined respectively by

$$\mathcal{D}_i^{\mathbb{R}} := \mathcal{D}_i \cap \mathbb{R}^{\ell^i \times q^i}, \quad \mathcal{D}_i^{+} = \mathcal{D}_i \cap \mathbb{R}_+^{\ell^i \times q^i}, \quad i = 0,1, \tag{23}$$

then we have the notions of *real stability radius* and *positive stability radius* which will be denoted by $r_{\mathbb{R}}$ and r_+, respectively.

We illustrate the above general definitions by two examples.

Example 2.8 (Single perturbations). Suppose $\mathcal{D}_0 = \mathbb{C}^{\ell^0 \times q^0}$, $\mathcal{D}_1 = \mathbb{C}^{\ell^1 \times q^1}$, i.e. we consider perturbations of the form

$$\begin{aligned} A_0 &\rightsquigarrow A_0 + D_0\Delta_0 E_0, & \Delta_0 &\in \mathbb{C}^{\ell^0 \times q^0}, \\ A_1 &\rightsquigarrow A_1 + D_1\Delta_1 E_1, & \Delta_1 &\in \mathbb{C}^{\ell^1 \times q^1}. \end{aligned} \tag{24}$$

Let $\gamma = \gamma_1$. Then the complex stability radius of the system (20) is given by :

$$r_{\mathbb{C}} = \inf\{\|\Delta_0\| + \|\Delta_1\| : \Delta_i \in \mathbb{C}^{\ell^i \times q^i}, i = 0,1, \\ \mu(A_0 + D_0\Delta_0 E_0, A_1 + D_1\Delta_1 E_1) \geq 0\} \tag{25}$$

where, by definition, $\inf \emptyset = \infty$. The real stability radius $r_{\mathbb{R}}$ and the positive stability radius r_+ are obtained by replacing $\mathbb{C}^{\ell^i \times q^i}$, $i = 0,1$ in (25) by $\mathbb{R}^{\ell^i \times q^i}$, $i = 0,1$ and $\mathbb{R}_+^{\ell^i \times q^i}$, $i = 0,1$, respectively.

In this particular case, a computable formula for $r_{\mathbb{C}}$ is given by the following theorem which is an extension of the main result in [6] to delay systems.

Through the paper we set $\infty^{-1} = 0$ and $0^{-1} = \infty$.

Theorem 2.9. *Let*

$$G_{ij}^{(s)} = E_i(sI - A_0 - A_1 e^{-hs})^{-1} D_j, \quad i,j = 0,1$$

be the associated transfer matrix functions of the linear delay system (20) *which is subjected to parameter perturbations* (24). *Assume that the complex stability radius* $r_{\mathbb{C}}$ *of the system is defined by* (25). *Then*

$$\frac{1}{\max\limits_{\omega\in\mathbb{R}}\left\{\|G_{ij}(\imath\omega)\| : i,j=0,1\right\}} \le r_{\mathbb{C}} \le \frac{1}{\max\limits_{\omega\in\mathbb{R}}\left\{\|G_{ii}(\imath\omega)\| : i=0,1\right\}}. \tag{26}$$

In particular, if $D := D_0 = D_1$ *or* $E := E_0 = E_1$ *then*

$$r_{\mathbb{C}} = \frac{1}{\max\limits_{\omega\in\mathbb{R}}\left\{\|G_{00}(\imath\omega)\|, \|G_{11}(\imath\omega)\|\right\}}. \tag{27}$$

The above theorem was proved in [18] for the general linear retarded system with multi delays. In the case when $D := D_0 = D_1$ and $E := E_0 = E_1$ and the size function is $\gamma_\infty(\cdot,\cdot)$, the same proof applies, with a minor modification, and yields the following formula for the complex stability radius of the system (20) with rerspect to the size function $\gamma_\infty = \max\{\|\Delta_0\|, \|\Delta_1\|\}$:

$$r^{\infty}_{\mathbb{C}} = \frac{1}{2\max\limits_{\omega\in\mathbb{R}}\{\|G_{00}(\imath\omega)\|, \|G_{11}(\imath\omega)\|\}}. \tag{28}$$

We note that, by appropriate choice of the structure matrices D_0, E_0, D_1, E_1, the case of single perturbations considered in Example 2.8 covers a large class of parameter perturbations, e.g. unstructured disturbances (i.e. when $D_i = E_i = I_n$, $i = 0,1$) or disturbances of individual elements, rows and/or columns of A_0, A_1. However, not all affine perturbations can be represented in the form (24) (for instance, perturbations which affect only diagonal entries of A_0 and A_1). A more general class of perturbations which allows us to deal with such a situation is the class of affine multi-perturbations as shown in the following

Example 2.10 (Multi-perturbations). Suppose the system matrices of (21) are perturbed as follows:

$$\begin{aligned} A_0 &\rightsquigarrow A_0 + \sum_{i=1}^{N_0} B_0^i\Delta_0^i C_0^i, \quad \Delta_0^i \in \mathbb{C}^{\ell_i^0\times q_i^0},\ i\in \underline{N_0} := \{1,\dots N_0\},\\ A_1 &\rightsquigarrow A_1 + \sum_{i=1}^{N_1} B_1^i\Delta_1^i C_1^i, \quad \Delta_1^i \in \mathbb{C}^{\ell_i^1\times q_i^1},\ i\in \underline{N_1} := \{1,\dots N_1\} \end{aligned} \tag{29}$$

where $B_j^i \in \mathbb{C}^{n\times \ell_i^j}, C_j^i \in \mathbb{C}^{q_i^j\times n}, \quad i \in \underline{N_j}, \quad j = 0,1$ are given matrices defining the scaling and structure of the parameter uncertainty and Δ_0^i, Δ_1^i are unknown disturbance matrices.

These perturbations can also be represented in the form (21) with $\mathcal{D}_0$, $\mathcal{D}_1$ being of block-diagonal structure. To see this it is enough to define

$$\ell^j := \sum_{i=1}^{N_j} \ell_i^j, \quad q^j := \sum_{i=1}^{N_j} q_i^j,$$

$$D_j := [B_j^1, \cdots, B_j^{N_j}] \in \mathbb{C}^{n\times \ell^j}, \quad E_j := [C_j^1, \cdots, C_j^{N_j}]^\top \in \mathbb{C}^{q^j\times n},$$

and

$$\Delta_j \in \mathcal{D}_j := \{\mathrm{diag}(\Delta_j^1, \dots, \Delta_j^{N_j}) : \Delta_j^i \in \mathbb{C}^{\ell_i^j\times q_i^j}, i \in \underline{N_j}\},$$

for $j = 0, 1$.

It is clear that the defined perturbation classes $\mathcal{D}_0$, $\mathcal{D}_1$ are of block-diagonal structure and (29) is equivalently represented in the form (21).

The robust stability of the system (with no time-delay) $\dot{x}(t) = Ax$ subjected to affine multi-perturbation was considered first in [8], but so far, in the literature, there have not been available results on computation of the complex stability radius for this case.

3. Main Results

It is clear from Definition 2.7 that the stability radii of the system (20) with respect to perturbations of the type (21) satisfy

$$r_{\mathbb{C}} \le r_{\mathbb{R}} \le r_+. \tag{30}$$

In general, these stability radii can be arbitrarily distinct (see e.g. [8]). As far as their computation concerns, while Theorem 2.9 reduces the computation of the complex stability radius to a global optimization problem over the real line, the problem for the real stability radius is much more difficult and a very complicated solution is known only for the case where $A_1 = 0$, see [15]. This is therefore natural to investigate for which kind of systems these three radii coincide. Motivated by results of [9], in this section we show that, for positive linear time-delay systems, the equalities in (30) hold and moreover, for certain classes of block-diagonal perturbations, the computation of these stability radii is straightfoward.

Consider a positive dynamical system described by the linear differential-difference equation in $\mathbb{R}^n$:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \qquad t \geq 0,\ h > 0. \tag{31}$$

Then, as noted in the previous section, $A_0 \in \mathbb{R}^{n\times n}$ is a Metzler matrix and $A_1 \in \mathbb{R}_+^{n\times n}$. We assume that the system matrices are subjected to affine parameter perturbations of the form

$$\begin{aligned} A_0 &\rightsquigarrow A_0 + D_0\Delta_0 E_0, \qquad \Delta_0 \in \mathcal{D}_0 \\ A_1 &\rightsquigarrow A_1 + D_1\Delta_1 E_1, \qquad \Delta_1 \in \mathcal{D}_1. \end{aligned} \tag{32}$$

Theorem 3.1. *Let the positive linear retarded system* (31) *be asymptotically stable. Assume that the system matrices A_0, A_1 are subjected to parameter perturbations of the form* (32)*, where $D_i \in \mathbb{R}_+^{n\times l_i}$, $E_i \in \mathbb{R}_+^{q_i\times n}$, $i = 0,1$ and $\mathcal{D}_0$, $\mathcal{D}_1$ are given perturbation classes of block-diagonal structure. Then*

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_+. \tag{33}$$

Proof. By Definition 2.7 we always have (30). Therefore, assuming $r_{\mathbb{C}} < +\infty$ it suffices to prove that $r_+ \leq r_{\mathbb{C}}$. Let $(\Delta_0, \Delta_1) \in \mathcal{D}_0 \times \mathcal{D}_1$ be a destabilizing disturbance, that is

$$\mu := \mu(A_0 + D_0\Delta_0E_0, A_1 + D_1\Delta_1E_1) \geq 0.$$

Then, by the definition of the spectral abscissa μ,

$$\Big(A_0 + D_0\Delta_0E_0 + (A_1 + D_1\Delta_1E_1)e^{-h(\mu+\imath\omega)}\Big)\, x = (\mu + \imath\omega)x \tag{34}$$

for some $x \in \mathbb{C}^n$, $x \neq 0$ and $\omega \in \mathbb{R}$. By Lemma 2.6, there exist $\tilde{\Delta}_0 \in \mathcal{D}_0$, $\tilde{\Delta}_1 \in \mathcal{D}_1$ such that

$$|\tilde{\Delta}_i| \in \mathcal{D}_i, \quad \|\,|\tilde{\Delta}_i|\,\| \leq \|\Delta_i\|, \quad \tilde{\Delta}_iE_ix = \Delta_iE_ix, \quad i = 0,1. \tag{35}$$

Since $|D_i\tilde{\Delta}_iE_i| \leq D_i|\tilde{\Delta}_i|E_i$, $i = 0,1$, it follows from Lemma 2.4 that

$$\mu(A_0 + D_0|\tilde{\Delta}_0|E_0, A_1 + D_1|\tilde{\Delta}_1|E_1) \geq \mu(A_0 + D_0\tilde{\Delta}_0E_0, A_1 + D_1\tilde{\Delta}_1E_1).$$

On the other hand, from (34), (35), we have

$$(A_0 + D_0\tilde{\Delta}_0E_0 + (A_1 + D_1\tilde{\Delta}_1E_1)e^{-h(\mu+\imath\omega)}\Big)\, x =$$

$$= \Big(A_0+D_0\Delta_0E_0+(A_1+D_1\Delta_1E_1)e^{-h(\mu+\imath\omega)}\Big)\, x = (\mu+\imath\omega)x. \tag{36}$$

Therefore, $\mu(A_0 + D_0|\tilde{\Delta}_0|E_0,\, A_1 + D_1|\tilde{\Delta}_1|E_1) \geq \mu \geq 0$. Thus, $|\tilde{\Delta}_0| \in \mathcal{D}_0^+$, $|\tilde{\Delta}_1| \in \mathcal{D}_1^+$ are destabilizing positive perturbations and, by monotonicity of γ,

$$\gamma(\|\,|\tilde{\Delta}_0|\,\|, \|\,|\tilde{\Delta}_1|\,\|) \leq \gamma(\|\,\tilde{\Delta}_0\,\|, \|\,\tilde{\Delta}_1\,\|),$$

which implies, by the definition of the stability radii $r_{\mathbb{C}}$, r_+, that $r_+ \leq r_{\mathbb{C}}$, as to be shown. This concludes the proof.

Now we are going to derive a computable formula for the stability radii of positive time-delay system (31) under affine perturbations (3.2). Define the transfer functions

$$G_{ij}(s) = E_i(sI - A_0 - A_1e^{-hs})^{-1}D_j, \quad i, j = 0, 1.$$

Obviously, G_{ij} are well defined for all $s \in \mathbb{C}$ with Re $s \geq \mu_0 := \mu(A_0, A_1)$. Since for each $t \in \mathbb{R}, A_0 + A_1e^{-ht}$ is a Metzler matrix it follows from Lemma 2.3 and Theorem 2.1 (iv) that $t > \mu_0$ implies $t > \mu(A_0+A_1e^{-h\mu_0}) \geq \mu(A_0 + A_1e^{-ht})$. Therefore, by Theorem 2.1 (iii),

$$R(t) := (tI - A_0 - A_1e^{-ht})^{-1} \geq 0$$

for all $t \geq \mu_0$. Thus, $G_{ij}(t) \geq 0$ for all $t \geq \mu_0$ and all $i, j \in \{0, 1\}$. On the other hand, it is easy to check that the following resolvent equation holds:

$$R(t_1) - R(t_2) = (t_2 - t_1)R(t_1)R(t_2) + (e^{-ht_1} - e^{-ht_2})R(t_1)A_1R(t_2).$$

Therefore, by multiplying the above equation with nonnegative matrices E_i from the left and D_j form the right, we get

Lemma 3.2. *If $t_2 > t_1 > \mu_0$, then*

$$G_{ij}(t_1) \geq G_{ij}(t_2) \geq 0 \quad \text{for} \quad i, j = 0, 1. \tag{37}$$

The following theorem gives a simple formula for the stability radii of positive linear time-delay systems with respect to the class of single perturbations considered in Example 2.8.

Theorem 3.3. *Suppose that the positive linear delay system* (31) *is asymptotically stable and subjected to parameter perturbations of the form* (32), *where* $D_i \in \mathbb{R}_+^{n\times\ell_i}$, $E_i \in \mathbb{R}_+^{q_i\times n}$, $\mathcal{D}_i = \mathbb{C}^{\ell_i\times q_i}$, $i = 0, 1$. *Assume that the stability radii of the system are defined as* (25) *with* $\gamma = \gamma_1$. *Then*

(38)
$$\frac{1}{\max\{\|G_{ij}(0)\| : i, j = 0, 1\}} \le r_{\mathbb{C}} = r_{\mathbb{R}} = r_+ \le \frac{1}{\max\{\|G_{ii}(0)\| : i = 0, 1\}}.$$

In particular, if $D := D_0 = D_1$ *or* $E := E_0 = E_1$ *then*

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_+ = \frac{1}{\max\{\|G_{00}(0)\|, \|G_{11}(0)\|\}}. \tag{39}$$

Proof. The two equalities and the right inequality in (38) are immediate from Theorem 2.9 and Theorem 3.1. To prove the left inequality it suffices to show that
$$r_+ \ge \frac{1}{\max\{\|G_{ij}(0)\| : i, j = 0, 1\}}.$$
Let $(\Delta_0, \Delta_1) \in \mathbb{R}_+^{l^0\times q^0} \times \mathbb{R}_+^{l^1\times q^1}$ be destabilizing nonnegative perturbation so that
$$\tilde{\mu}_0 := \mu(A_0 + D_0\Delta_0E_0, A_1 + D_1\Delta_1E_1) \ge 0.$$
By Lemma 2.3, $\tilde{\mu}_0 = \mu(A_0 + D_0\Delta_0E_0 + (A_1 + D_1\Delta_1E_1)e^{-h\tilde{\mu}_0})$. Hence, by Theorem 2.1 (i), there exist $x_0 \in \mathbb{R}_+^n, x_0 \ne 0$ such that
$$(A_0 + D_0\Delta_0E_0 + (A_1 + D_1\Delta_1E_1)e^{-h\tilde{\mu}_0})x_0 = \tilde{\mu}_0x_0.$$
or, equivalently,
$$(\tilde{\mu}_0I - A_0 - A_1e^{-h\tilde{\mu}_0})x_0 = D_0\Delta_0E_0x_0 + D_1\Delta_1E_1e^{-h\tilde{\mu}_0}x_0.$$
Since the system (3.1) is asymptotically stable, $(\tilde{\mu}_0I - A_0 - A_1e^{-h\tilde{\mu}_0})$ is invertible and the above equation is equivalent to
$$\begin{aligned} x_0 = {} & (\tilde{\mu}_0I - A_0 - A_1e^{-h\tilde{\mu}_0})^{-1}D_0\Delta_0E_0x_0 \\ & + (\tilde{\mu}_0I - A_0 - A_1e^{-h\tilde{\mu}_0})^{-1}D_1\Delta_1E_1e^{-h\tilde{\mu}_0}x_0. \end{aligned}$$
Let $q \in \{0, 1\}$ be such an index that $\|E_qx_0\| = \max\{\|E_ix_0\|, i = 0, 1\}$. Then from the last equation it follows that $E_qx_0 \ne 0$. Multiplying this equation with E_q from the left we get,
$$E_qx_0 = G_{q0}(\tilde{\mu}_0)\Delta_0E_0x_0 + G_{q1}(\tilde{\mu}_0)\Delta_1E_1e^{-h\tilde{\mu}_0}x_0$$

and hence

$$\|G_{q0}(\tilde{\mu}_0)\|\,\|\Delta_0\|\,\|E_0x_0\| + \|G_{q1}(\tilde{\mu}_0)\|\,\|\Delta_1\|\,\|E_1x_0\| \geq \|E_qx_0\|.$$

This implies

$$\max\{\|G_{ij}(\tilde{\mu}_0)\| : i,j=0,1\}(\|\Delta_0\|+\|\Delta_1\|)\|E_qx_0\| \geq \|E_qx_0\|$$

and, subsequently,

$$\max\{\|G_{ij}(\tilde{\mu}_0)\| : i,j=0,1\}(\|\Delta_0\|+\|\Delta_1\|) \geq 1. \tag{40}$$

On the other hand, by Lemma 3.2 we have $G_{ij}(0) \geq G_{ij}(\tilde{\mu}_0) \geq 0$ for all $i,j \in \{0,1\}$ and hence, by (7),

$$\max\{\|G_{ij}(\tilde{\mu}_0)\| : i,j=0,1\} \leq \max\{\|G_{ij}(0)\| : i,j=0,1\}.$$

Therefore, by definition and (40), we obtain

$$r_+ \geq \frac{1}{\max\{\|G_{ij}(0)\| : i,j=0,1\}}$$

as to be shown. Further, if $D_0 = D_1$ (respectively, $E_0 = E_1$) then, by definition, $G_{01}(s) = G_{00}(s)$, $G_{10}(s) = G_{11}(s)$ (respectively, $G_{01}(s) = G_{11}(s)$, $G_{10}(s) = G_{00}(s)$), so that, in this case, (38) implies (39). The proof is complete.

We assume now that the system matrices of the positive system (31) are subjected to perturbations of the following kind:

$$\begin{aligned} A_0 &\rightsquigarrow A_0 + \sum_{i=1}^{N_0} \delta_0^i B_0^i \\ A_1 &\rightsquigarrow A_1 + \sum_{i=1}^{N_1} \delta_1^i B_1^i, \end{aligned} \tag{41}$$

where $B_j^i \in \mathbb{R}_+^{n\times n}$, $i \in \underline{N_j} := \{1,\cdots,N_j\}$, $j=0,1$ are given matrices and δ_0^i, $\delta_1^i \in \mathbb{C}$ are unknown scalar parameters. Set, for $j=0,1$,

$$D_j := [B_j^1 \cdots B_j^{N_j}] \in \mathbb{C}^{n\times(nN_j)}, \quad E_j := \begin{bmatrix} I_n \\ \vdots \\ I_n \end{bmatrix} \in \mathbb{C}^{(nN_j)\times n} \tag{42}$$

and

$$\mathcal{D}_j := \{\mathrm{diag}(\delta_j^1 I_n, \dots, \delta_j^{N_j} I_n) : \delta_j^i \in \mathbb{C}, i \in \underline{N}_j\} \subset \mathbb{C}^{nN_j \times nN_j}.$$

Then it is easy to see that perturbations (41) are equivalently represented in the form (32). Note that in this case $\mathcal{D}_j (j = 0, 1)$ are of a special block-diagonal structure (with diagonal blocks of repeated scalars on the main diagonal). If we provide $\mathcal{D}_j$, $j = 0, 1$, with the norm (18) where $\|\cdot\|_{\mathbb{R}^N}$ is ∞-norm (i.e. $\|(\zeta^1, \cdots, \zeta^N)\|_{\mathbb{R}^N} = \max_{1 \le i \le N} |\zeta_i|$) and $N = nN_j$ for $j = 0, 1$, respectively, then, for any $\Delta_j = \mathrm{diag}(\delta_j^1, \cdots, \delta_j^{N_j}) \in \mathcal{D}_j$,

$$\|\Delta_j\| = \max\{|\delta_j^i| : i \in \underline{N}_j\}, \quad j = 0, 1,$$

(see, e.g. [11]). Therefore

$$\gamma_\infty(\|\Delta_0\|, \|\Delta_1\|) = \max\{\|\Delta_0\|, \|\Delta_1\|\} = \max_{i \in \underline{N}_0, k \in \underline{N}_1} \left(|\delta_0^i|,\ |\delta_1^k|\right). \tag{43}$$

Thus, according to Definition 2.7, if we take $\gamma = \gamma_\infty$ then the corresponding complex and the real stability radii of the system (31) subjected to affine perturbations (41) are given by

$$\begin{aligned} r^a_{\mathbb{K}} = \inf\Big\{ & \max_{i \in \underline{N}_0, k \in \underline{N}_1} \left(|\delta_0^i|,\ |\delta_1^k|\right) : \delta_0^i, \delta_1^k \in \mathbb{K}, \\ & \mu(A_0 + \sum_{i=1}^{N_0} \delta_0^i B_0^i, A_1 + \sum_{k=1}^{N_1} \delta_1^k B_1^k) \ge 0 \Big\}, \end{aligned} \tag{44}$$

where $\mathbb{K} = \mathbb{C},\ \mathbb{R}$.

Similarly, the positive stability radius r^a_+ is obtained by restricting, in the above definition, the scalar disturbances δ_0^i, δ_1^k to be nonnegative.

Theorem 3.4. *Suppose the positive linear delay system* (31) *is asymptotically stable and subjected to affine perturbations of the form* (41)*. If the stability radii of the system are given by* (44) *then*

$$r^a_{\mathbb{C}} = r^a_{\mathbb{R}} = r^a_+ = \frac{1}{\mu(G(0))}, \tag{45}$$

where $G(s)$ is the associated transfer matrix defined by

$$\begin{aligned} G(s) &= \begin{bmatrix} G_{00}(s) & G_{01}(s) \\ G_{10}(s) & G_{11}(s) \end{bmatrix} \in \mathbb{C}^{n(N_0+N_1) \times n(N_0+N_1)}, \\ G_{ij}(s) &= E_i(sI - A_0 - A_1 e^{-hs})^{-1} D_j \in \mathbb{C}^{nN_i \times nN_j}, \quad i, j = 0, 1. \end{aligned} \tag{46}$$

Proof. The proof is similar to that of Theorem 17 in [19]. The first two equalities in (45) follow from Theorem 3.1. Since, by Lemma 3.2, $G(0) \geq 0$, it follows from Theorem 2.1 (i), that $\rho := \mu(G(0))$ is an eigenvalue of $G(0)$ and there exists $y \in \mathbb{R}_+^{n(N_0+N_1)}$, $y \neq 0$, such that $G(0)y = \rho y$ or, equivalently

$$\begin{aligned} E_0(-A_0 - A_1)^{-1}D_0y_0 + E_0(-A_0 - A_1)^{-1}D_1y_1 = \rho y_0 \\ E_1(-A_0 - A_1)^{-1}D_0y_0 + E_1(-A_0 - A_1)^{-1}D_1y_1 = \rho y_1 \end{aligned} \tag{47}$$

where $\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = y$. Choosing the perturbations $\Delta_j = \dfrac{1}{\rho}\mathrm{diag}(I_n, \cdots, I_n) \in \mathcal{D}_j^+$, $j = 0, 1$ and setting

$$(-A_0 - A_1)^{-1}(D_0\Delta_0y_0 + D_1\Delta_1y_1) =: x \in \mathbb{R}_+^n.$$

It follows that $y_0 = E_0x, y_1 = E_1x$ (therefore $x \neq 0$) and

$$(-A_0 - A_1)x = D_0\Delta_0E_0x + D_1\Delta_1E_1x.$$

Consequently, $(A_0 + D_0\Delta_0E_0 + A_1 + D_1\Delta_1E_1)x = 0$ which means that (Δ_0, Δ_1) is nonnegative destabilizing perturbations with

$$\gamma_\infty = \max\{\|\Delta_0\|, \|\Delta_1\|\} = \frac{1}{\rho}\cdot$$

Thus, by definition we have $r_+^a \leq \dfrac{1}{\rho}$.

Conversely, suppose $\Delta_j = \mathrm{diag}(\delta_j^1 I_n, \cdots, \delta_j^{N_j} I_n)$, $\delta_j^i \geq 0$, $i \in \underline{N}_j$, $j = 0, 1$ are destabilizing nonnegative matrices, that is $\mu_0 := \mu(A_0 + D_0\Delta_0E_0, A_1 + D_1\Delta_1E_1) \geq 0$. By Lemma 2.3 and Theorem 2.1 (i), there exists a nonzero vector $x_0 \geq 0$ such that

$$(A_0 + D_0\Delta_0E_0 + (A_1 + D_1\Delta_1E_1)e^{-h\mu_0})x_0 = \mu_0x_0.$$

which implies, as in the proof of Theorem 3.1,

$$x_0 = (\mu_0I - A_0 - A_1e^{-h\mu_0})^{-1}(D_0\Delta_0E_0x_0 + D_1\Delta_1E_1e^{-h\mu_0}x_0).$$

Multiplying the last equation with E_0 and with E_1 from the left and setting

$$y = \begin{bmatrix} E_0x_0 \\ E_1x_0 \end{bmatrix}$$

we get

$$G(\mu_0)\Delta y = y$$

where $\Delta := \mathrm{diag}(\Delta_0, e^{-h\mu_0}\Delta_1)$. Denote $\gamma := \gamma_\infty(\|\Delta_0\|, \|\Delta_1\|)$ then, by (43), $\gamma y \geq \Delta y$ and hence, by Lemma 3.2, $\gamma G(0)y \geq \gamma G(\mu_0)y \geq G(\mu_0)\Delta y = y$. Since $G(0) \geq 0$ and $y \geq 0$, we can apply Theorem 2.1 (ii) to conclude that $\gamma \geq \dfrac{1}{\mu(G(0))} = \dfrac{1}{\rho}$. Therefore, $r_+^a \geq \dfrac{1}{\rho}$, completing the proof.

If $N_0 = N_1 =: N$ then $E_0 = E_1$ and hence $G_{00} = G_{10}$, $G_{01} = G_{11}$. In this case, by an easy calculation we have that

$$\mu(G(0)) = \mu(G_{00}(0) + G_{11}(0)) = \mu\Big((-A_0 - A_1)^{-1}\Big(\sum_{i=1}^{N} B_0^i + \sum_{i=1}^{N} B_1^i\Big)\Big).$$

On the other hand, if, for instance, $N = N_0 > N_1$ then we can reduce to the last case by defining $B_i^1 := 0$, $i = N_1 + 1, \cdots, N$. Therefore we get the following explicit formula for the stability radii of the positive linear delay system (31) under the affine perturbations (41):

$$(48) \qquad r_{\mathbb{C}}^a = r_{\mathbb{R}}^a = r_+^a = \frac{1}{\mu\Big((-A_0 - A_1)^{-1}\Big(\sum\limits_{i=1}^{N_0} B_0^i + \sum\limits_{i=1}^{N_1} B_1^i\Big)\Big)}.$$

We note that the spectral abscissa μ in (48) can be replaced by the spectral radius ρ because the matrix under consideration is nonnegative. Thus, we have shown that the stability radii of the system is equal to the upper bound of stability established in [12] by another approach.

We illustrate Theorems 3.3 and 3.4 by the following two examples

Example 3.5. Consider a positive linear delay system in $\mathbb{R}^2$ described by $\dot{x}(t) = A_0x(t) + A_1x(t-1)$, $t \geq 0$, where

$$A_0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Then the characteristic equation of the system is

$$\det(zI - A_0 - A_1e^{-z}) = \det\begin{bmatrix} z+1 & 0 \\ -e^{-z} & z+1 \end{bmatrix} = (z+1)^2 = 0.$$

and hence the delay system is asymtotically stable. Assume that, the system matrices are subjected to parameter pertubations of the forms $A_0 \rightsquigarrow A_0 + D_0\Delta_0 E_0$, $A_1 \rightsquigarrow A_1 + D_1\Delta_1 E_1$, where

$$E_0 = E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad D_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad D_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then

$$G_{00}(0) = E_0(-A_0 - A_1)^{-1} D_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$G_{11}(0) = E_1(-A_0 - A_1)^{-1} D_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore, if $\mathbb{R}^2$ is endowed with the 2-norm, then by (39), $r_{\mathbb{C}} = r_{\mathbb{R}} = r_+ = \dfrac{1}{\sqrt{5}}$. If $\mathbb{R}^2$ is endowed, respectively, with the ∞-norm, then the stability radii of the system are given by $r_{\mathbb{C}} = r_{\mathbb{R}} = r_+ = \dfrac{1}{2}$.

Example 3.6. Consider a positive linear delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1), \quad t \geq 0$$

where

$$A_0 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \qquad , A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then the characteristic equation of system is

$$\det(zI - A_0 - A_1 e^{-z}) = (z^2 + 3z + 2) - e^{-z} = 0.$$

By Theorem 13.9 in [1] this equation has only roots with negative real parts. Therefore the above delay system is asymptotically stable. Suppose the system matrices A_0, A_1 are subjected to parameter perturbations of the forms $A_0 \rightsquigarrow A_0 + \delta_0 B_0$, $A_1 \rightsquigarrow A_1 + \delta_1 B_1$ where

$$B_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We have, by the notation of Theorem 3.4, $E_0 = E_1 = I_2$, $D_0 = B_0$, $D_1 = B_1$. Therefore,

$$P := (-A_0 - A_1)^{-1}(B_0 + B_1) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$$

Thus, by (48),

$$r^a_{\mathbb{C}} = r^a_{\mathbb{R}} = \frac{1}{\mu(P)} = \frac{1}{3+\sqrt{10}}\,.$$

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