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ON THE UNIFORMITY OF MEROMORPHIC FUNCTIONS

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Abstract. The paper gives, in terms of the linear topological invariants, some conditions under which every F'-valued meromorphic function on the dual space of a Frechet-Montel space is of uniform type.

1. Introduction

For locally convex spaces E, F we denote by $\mathcal{M}(E,F)$ the vector space of F-valued meromorphic functions on E. A F-valued meromorphic function f on E is said to be of *uniform type* if f can be meromorphically factorized through a Banach space. This means that there exists a continuous semi-norm ρ on E and a meromorphic function g from E_ρ, the canonical Banach space associated to ρ, into F such that $f = g\omega_\rho$, where $\omega_\rho : E \longrightarrow E_\rho$ is the canonical map.

Put $\mathcal{M}_u(E,F) = \big\{ f \in \mathcal{M}(E,F) \;\big|\, f \text{ is of uniform type}\big\}$. We are interested in the equality

$$(MUN) \quad \mathcal{M}(E,F) = \mathcal{M}_u(E,F).$$

Let's recall that in the case of the holomorphic functions, the analogous identity

$$(HUN) \quad \mathcal{H}(E,F) = \mathcal{H}_u(E,F)$$

was investigated by many authors. Here $\mathcal{H}(E,F)$ denotes the space of F-valued entire functions on E equipped with the compact-open topology and

$$\mathcal{H}_u(E,F) = \big\{ f \in \mathcal{H}(E,F) \;\big|\; f \text{ is of uniform type} \big\}.$$

Colombeau and Mujica [1] have shown that (HUN) holds in the case where E is a dual Frechet-Montel space and F a Frechet space. The case where E and F are either Frechet spaces or dual Frechet spaces was investigated by Meise and Vogt [7] and recently by Le Mau Hai [5]. In [7]

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Meise and Vogt have proved that (HUN) holds for the scalar entire functions on a nuclear Frechet space E having $(\widetilde{\Omega})$. Next, Le Mau Hai [5] has extended this result by proving that (HUN) holds for every nuclear Frechet space $E \in (\widetilde{\Omega})$ and for every Frechet space $F \in (DN)$. Observe that this is also true for the dual Frechet case with a suitable hypothesis, for example $E' \in (DN)$ and $F' \in (\widetilde{\Omega})$. However, the equality (MUN) was only considered recently by Le Mau Hai for the case where E is a dual Frechet-Schwartz space with an absolute basis [4]. He has proved that if $E' \in (DN)$ and F is a dual Frechet space with $F' \in (\widetilde{\Omega})$, then (MUN) holds.

The main aim of this paper is to investigate some sufficient and necessary conditions for E and F such that (MUN) holds. Unfortunately, a result of Meise-Vogt type for the meromorphic case remains to be found.

We shall use the standard notations from the theory of locally convex spaces as presented in the books of Pietsch [9] and Schaefer [10].

Let E be a Frechet space with a fundamental system of semi-norms $\{\, \|\bullet\|_k \}$. For a subset B of E, put $\|u\|_B^* = \sup\{\, |u(x)| \ :\ x \in B\}$ for $u \in E'$. Write $\|\bullet\|_k^*$ for $B = U_k = \{\, x \in E \ :\ \|x\|_k < 1\,\}$.

By using these notations we say that E has the property

(DN) if $\exists p\ \forall q, d > 0\ \exists k, C > 0,\quad \|\bullet\|_q^{1+d} \leq C\ \|\bullet\|_k\ \|\bullet\|_p^d.$

$(\underline{DN})$ if $\exists p\ \forall q\ \ \exists k, d, C > 0,\quad \|\bullet\|_q^{1+d} \leq C\ \|\bullet\|_k\ \|\bullet\|_p^d.$

$(\overline{\overline{\Omega}})$ if $\forall p\ \exists q\ \forall k, d > 0\ \ \exists C > 0,\quad \|\bullet\|_q^{*1+d} \leq C\ \|\bullet\|^*_k\, \|\bullet\|^{*d}_p.$

(LB^∞) if $\forall \rho_n \uparrow \infty\quad \forall p\ \exists q\ \forall k\ \exists n_k, C > 0\ \forall u \in E'\ \exists n_u \in [k; n_k]$,

$$\|u\|_q^{*1+\rho_{n_u}} \leq C\ \|u\|_{n_u}^*\ \ \|u\|^{*\rho_{n_u}}_p .$$

The above properties were introduced and investigated by Vogt (see [12], [13]).

Let E, F be two locally convex spaces and let $D \subset E$ be an open subset. A function $f\ :\ D \longrightarrow F$ is called holomorphic if f is continuous and if for every $y \in F'$, the dual space of F, the function $y \circ f \in F'$ is Gâteaux holomorphic. By $\mathcal{H}(D, F)$ we denote the space of F-valued holomorphic function on D equipped with the compact-open topology. A holomorphic function $f\ :\ D_\circ \longrightarrow F$, where $D_\circ$ is a dense open subset of D, is said to be *meromorphic* on D if for every $z \in D$ there exist a neighbourhood U of z and holomorphic functions $h\ :\ U \longrightarrow F,\ \ \sigma\ :\ U \longrightarrow C$ $(\sigma \neq 0)$ such that

$$f\,\big|_{D_\circ \cap U} = \frac{h}{\sigma}\bigg|_{D_\circ \cap U}.$$

By $\mathcal{M}(D, F)$ we denote the vector space of F-valued meromorphic functions on D. For details concerning holomorphic functions on locally convex spaces we refer to the book of Dineen [3].

We shall prove the following assertions.

Theorem 1.1. (i) *Let E be a nuclear Frechet space. Then $\mathcal{M}(E', F') = \mathcal{M}_u(E', F')$ for every Frechet space $F \in (LB^\infty)$ if and only if $E \in (DN)$.*

(ii) *Let F be a Frechet space. Then $\mathcal{M}(E', F') = \mathcal{M}_u(E', F')$ for every nuclear Frechet space $E \in (DN)$ if and only if $F \in (LB^\infty)$.*

Theorem 1.2. *Let E be a Frechet-Montel space with the property (DN) and F a Frechet space with the property $(\overline{\overline{\Omega}})$. Then $\mathcal{M}(E', F') = \mathcal{M}_u(E', F')$.*

2. Proof of Theorem 1.1

Lemma 2.1. *Let E be a nuclear Frechet space with the property (DN) and F a Frechet space with the property (LB^∞). Assume that $f : E' \longrightarrow F'$ is a holomorphic function. Then f is of uniform type.*

Proof. Consider the continuous linear map $\hat{f} : \mathcal{H}_b(F') \longrightarrow \mathcal{H}(E')$ associated to f:

$$\hat{f}(\varphi)(u) = \varphi(f(u)) \text{ for } \varphi \in \mathcal{H}_b(F') \text{ and } u \in E'.$$

Since $F \in (LB^\infty)$ and $\mathcal{H}(E') \in (DN)$ [8], we can find by [10] a neighbourhood V of $0 \in F$ such that $\hat{f}(V)$ is bounded. Then, for every bounded subset B in E', we have

$$\sup\{|f(u)(y)| \ : \ u \in B, \ y \in V\} = \sup\left\{|\hat{f}(y)(u)| \ : \ u \in B, \ y \in V\right\} < \infty.$$

Thus, $f : E' \longrightarrow F'_V$, where F_V is the Banach space associated to V, is bounded and Gâteaux holomorphic. Hence $f : E' \longrightarrow F'_V$ is holomorphic. By Colombeau and Mujica [1], f is of uniform type. $\square$

Lemma 2.2. *Let β and σ be holomorphic functions on an open set D in a locally convex space and let g be a holomorphic function with values in a locally convex space. Assume that $\dfrac{\beta g}{\sigma}$ is holomorphic on D and* $\operatorname{codim} Z(g, \sigma) \geq 2$. *Then $\dfrac{\beta}{\sigma}$ is holomorphic on D.*

Proof. Given $z_o \in D$. Since the local ring $\mathcal{O}_{z_o}$ of germs of holomorphic functions at z_o is factorial [6], we can write $\sigma = \sigma_1^{m_1}\sigma_2^{m_2}\ldots\sigma_p^{m_p}$ in a neighbourhood U of z_o such that $\sigma_{1z_o}, \sigma_{2z_o}, \ldots, \sigma_{pz_o}$ are irreducible. By the hypothesis and by the equality

$$\frac{\beta g}{\sigma_1} = \frac{\beta g}{\sigma}\sigma_1^{m_1-1}\ldots\sigma_p^{m_p},$$

it follows that $\dfrac{\beta g}{\sigma_1}$ is holomorphic at z_o. On the other hand, from the hypothesis codim $Z(g,\sigma) \geq 2$ and $Z(\sigma) = \bigcup\limits_{i=1}^{p} Z(\sigma_i)$ it follows that codim $Z(g,\sigma_i) \geq 2$ for $i = 1, \ldots, p$. Hence, by the irreducibility of σ_{1z_o} we infer that

$$Z\big(\sigma_1\big)_{z_o} \subseteq Z\big(\beta\big)_{z_o}.$$

This again implies $\beta = \beta_1\sigma_1$ at z_o. By continuing this process we infer that $\dfrac{\beta}{\sigma}$ is holomorphic at z_o. $\square$

Proof of Theorem 1.1.

(i) Assume that $E \in (DN)$ and $F \in (LB^\infty)$. Given $f : E' \longrightarrow F'$ a meromorphic function. By $\mathcal{O}_{E'}$ (resp. $\mathcal{M}_{E'}$) we denote the sheaf of germs of holomorphic (resp. meromorphic) functions on E'. Let

$$\begin{aligned}
\mathcal{O}^*_{E'} &= \{\sigma \in \mathcal{O}_{E'} : \sigma \text{ is invertible}\},\\
M^*_{E'} &= M_{E'} \setminus \{0\},\\
D_{E'} &= M^*_{E'}/\mathcal{O}^*_{E'}.
\end{aligned}$$

Then we have the two exact sequences on E':

$$0 \longrightarrow Z \longrightarrow \mathcal{O}_{E'} \xrightarrow{\exp} \mathcal{O}^*_{E'} \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}^*_{E'} \longrightarrow M^*_{E'} \xrightarrow{\eta} D_{E'} \longrightarrow 0,$$

where $\exp(\sigma) = e^{2\pi i\sigma}$ and η is the canonical map. By [2] , $H^1(E', \mathcal{O}_{E'}) = 0$. On the other hand, since $H^2(E', Z) = 0$, the exact cohomology sequences associated to the above exact sheaf sequences give that for every divisor $d \in H^o(E', D_{E'})$, there exists a meromorphic function $\tau \in H^o(E', M^*_{E'})$ such that $\eta(\tau) = d$.

By the meromorphicity of f, for every $z \in E'$ we can choose a neighbourhood V_1 of z and the holomorphic functions $h : V_1 \longrightarrow F'$, $\sigma : V_1 \longrightarrow \mathbf{C}$, $\sigma \neq 0$, such that

$$f|_{V_1} = \frac{h}{\sigma}\ .$$

Write $\sigma = \sigma_1^{m_1}\sigma_2^{m_2}\dots\sigma_p^{m_p}$ in a neighbourhood V_2 of z in V_1 such that the germs $\sigma_{1z}, \sigma_{2z}, \dots, \sigma_{pz}$ at z are irreducible [6]. Without loss of generality we may assume that h_z can be not divisible by $\sigma_{1z}, \sigma_{2z}, \dots, \sigma_{pz}$. Then there exists a neighbourhood U of z in V_2 such that

$$f|_U = \frac{h}{\sigma}$$

and codim $Z(h,\sigma) \geq 2$ in U (where $Z(h,\sigma) = h^{-1}(0) \cap \sigma^{-1}(0)$). Thus, we can find an open cover $\{U_j\}$ of E' and holomorphic functions $h_j : U_j \longrightarrow F'$, $\sigma_j : U_j \longrightarrow \mathbf{C}$ such that

$$f|_{U_j} = \frac{h_j}{\sigma_j}$$

and codim $Z(h_j,\sigma_j) \geq 2$ for $j \geq 1$.

Since $\dfrac{h_i}{\sigma_i} = \dfrac{h_j}{\sigma_j}$ on $U_i \cap U_j$ for all $i, j \geq 1$, Lemma 2.2 implies that the form $z \mapsto \big(\sigma_j\big)_z \mathcal{O}^*_{E',z}$ for $z \in U_j$ defines a divisor d on E'. Thus, there exists a meromorphic function β on E' such that $\beta \neq 0$ and $\dfrac{\beta_z}{d_z} \in O^*_{E',z}$ for $z \in E'$. It is easy to see that β is holomorphic on E' and hence $h = \beta f$ is holomorphic on E'. From Lemma 2.1, we infer that h, β are of uniform type, and hence so is f.

Conversely, assume that E is a nuclear Frechet space such that $\mathcal{M}(E',F') = \mathcal{M}_u(E',F')$ for every Frechet space $F \in (LB^\infty)$. By Vogt [12], in order to prove $E \in (DN)$ it suffices to prove that each continuous linear mapping T from $\mathcal{H}(\Delta)$ into E is bounded on some neighbourhood of 0, where $\mathcal{H}(\Delta)$ denotes the space of holomorphic functions on the open unit disc Δ in $\mathbf{C}$.

Since $\mathcal{H}(\Delta) \in (LB^\infty)$ [12], by the hypothesis we obtain $\mathcal{M}(E',[\mathcal{H}(\Delta)]') = \mathcal{M}_u(E',[\mathcal{H}(\Delta)]')$. Let $T' : E' \longrightarrow [\mathcal{H}(\Delta)]'$ be the dual mapping of $T : \mathcal{H}(\Delta) \longrightarrow E$. Obviously, $T' \in \mathcal{M}(E',[\mathcal{H}(\Delta)]')$ and hence $T' \in \mathcal{M}_u(E',[\mathcal{H}(\Delta)]')$. Therefore we have $T' = g \circ \omega_q$, where ω_q is the canonical mapping from E' into E'_q, the Banach space associated with E', and $g : E'_q \longrightarrow [\mathcal{H}(\Delta)]'$ is a meromorphic function. Because T', ω_q are linear and ω_q is surjective, we have the linearity of g.

Put $V = \omega_q^{-1}(U)$ where U is the open unit ball of E'_q. Then V is a neighbourhood of $0 \in E'$. We have $T'(V) = g \circ \omega_q(V) \subset g(U)$, which is bounded in $[\mathcal{H}(\Delta)]'$. This means T' is bounded on a neighbourhood of in $\mathcal{H}(\Delta)$ and hence T is also bounded on a neighbourhood of in E.

(ii) The sufficiency follows from (i). By the (DN)-characterization of Vogt [12] and by applying the equality $\mathcal{M}(E', F') = \mathcal{M}_u(E', F')$ to $E = \mathcal{H}(\mathbf{C})$ which has (DN) [12], the necessity can be proved as in (i). The proof of Therem 1.1 is now complete.

3. Proof of Theorem 1.2

Let $\Lambda(A)$ be a nuclear Frechet-Köther space. Let $\mathbf{D}_a$, $a \in \Lambda(A)$, denote an open polydisc in $\Lambda'(A)$. Assume that E is a Banach space with the unit ball B. Put

$$\mathbf{D}_a^B = \Big\{ \sum_{j \geq 1} x_j \otimes \xi_j e_j^* \;\Big|\; \overline{x} = (x_j) \subset B\,,\ \xi = (\xi_j) \in \mathbf{D}_a \Big\}.$$

Since $\mathbf{D}_a$ is open, it is easy to see that $\mathbf{D}_a^B$ is also open in

$$E \mathop{\hat{\otimes}}_{\pi} \Lambda'(A) = \Big\{ \sum_{j \geq 1} x_j \otimes e_j^* \;\Big|\; (\|x_j\|) \in \Lambda'(A) \Big\}.$$

By $\mathcal{H}_b(\mathbf{D}_a^B)$ we denote the Frechet space of holomorphic functions f on $\mathbf{D}_a^B$ for which

$$\|f\|_K = \sup \Big\{ \Big| f(\sum_{j \geq 1} x_j \otimes \xi_j e_j^*) \Big| \;\Big|\; \overline{x} \subset B,\ \xi = (\xi_j) \in K \Big\} < \infty$$

for every compact subset $K \subset \mathbf{D}_a$.

Lemma 3.1. *There exists a matrix $Q = [q_{jk}]$, $q_{jk} \geq 0$, such that*

(i) $\forall n\ \exists k, \varepsilon > 0 \quad q_{jn}^{1+\varepsilon} \leq q_{jk} q_{j1}^{\varepsilon} \quad \forall j \geq 1$, *and*

$$\sum_{j \geq 1} \frac{q_{jn}}{q_{jk}} \;<\; \infty, \quad \frac{q_{jn}}{q_{jk}} \;<\; 1 \quad \textit{for } j \geq 1,$$

(ii) $\mathcal{H}_b(D_a^B)$ *is a subspace of the space*

$$\Lambda_B(M, Q^M) = \Big\{ (\xi_m(\bar{x}))_{m\in M, \bar{x}\in B} \,\Big|\, \|\xi_m(\bar{x})\|_k < \infty \ \ \forall k \geq 1 \Big\},$$

where $\mathbf{M} = \{ m = (m_j) \subset Z_+ / m_j \neq 0$ *only for finitely many* $j\}$, $\|\xi_m(\bar{x})\|_k = \sup\{ |\xi_m(\bar{x})| \, q_k^m \ : \ \bar{x} \in B, \ m \in \mathbf{M}\}$ *and* $q_k^m = q_{1k}^{m_1} ... q_{nk}^{m_n}$ *for* $m = (m_1, ..., m_n, 0...) \in \mathbf{M}$.

Proof. By [8] there exists a matrix $Q = [q_{jk}]$ satisfying (i). Moreover, the form

$$f \longmapsto (a_m(f) = \Big(\frac{1}{2\pi i}\Big)^n \int\limits_{|\lambda_1|=r_1} \cdots \int\limits_{|\lambda_n|=r_n} \frac{f\big(\sum\limits_{j=1}^{n} \lambda_j e_j^*\big)}{\lambda^{m+1}} d\lambda,$$

$0 < r_j < \dfrac{1}{a_j}$, $\forall j \geq 1$ defines an isomorphism of $\mathcal{H}(\mathbf{D}_a)$ and $\Lambda(\mathbf{M}, Q^{\mathbf{M}})$.

Given $f \in \mathcal{H}_b(\mathbf{D}_a^B)$. For each $\overline{x} \subset B$, we define $f_{\overline{x}} \in \mathcal{H}(\mathbf{D}_a)$ by

$$f_{\overline{x}}(\xi) = f\Big(\sum_{j\geq 1} x_j \otimes \xi_j e_j^*\Big) \quad \text{for } \xi \in \mathbf{D}_a.$$

It follows that

$$\begin{aligned} |||f|||_k &:= \sup\{|a_m(f_{\overline{x}})| q_k^m \mid \overline{x} \subset B, \ m \in M\} \\ &\leq \sup\Big\{\Big|f\big(\sum_{j\geq 1} x_j \otimes \xi_j e_j^*\big)\Big| \,\Big|\, \overline{x} \subset B, \ \xi \in N_k\Big\} \\ &= \|f\|_{N_k} := \|f\|_k \,, \end{aligned}$$

where $N_k = \{(\xi_j) \ / \ |\xi_j| \leq q_{jk} \ \forall j \geq 1\}$. Hence $|||\bullet|||_k$ is a continuous seminorm on $\mathcal{H}_b(\mathbf{D}_a^B)$ for $k \geq 1$.

On the other hand, since for $n \geq 1$ there exists $k > n$ such that

$$\sum_{j\geq 1} \frac{q_{jn}}{q_{jk}} < \infty \quad \text{and} \quad \frac{q_{jn}}{q_{jk}} < 1 \quad \forall j \geq 1,$$

we have

$$\begin{aligned}
\|f\|_n &\le \sup\Big\{\sum_M |a_m(f_{\overline{x}})||\xi^m| \ \Big|\ \overline{x}\subset B,\ \xi\in N_n\Big\}\\
&\le |||f|||_k \times \sum_{m\in M}\Big(\frac{q_n}{q_k}\Big)^m\\
&= |||f|||_k \times \prod_{j\ge 1}\sum_{p=1}^{\infty}\Big(\frac{q_{jn}}{q_{jk}}\Big)^p\\
&= \frac{|||f|||_k}{\prod\limits_{j\ge 1}\Big(1-\dfrac{q_{jn}}{q_{jk}}\Big)}
\end{aligned}$$

Since $\{N_k\}$ is an exhaustion sequence of compact sets in $\mathbf{D}_a$, it follows that the form

$$f \longmapsto \big(a_m(f_{\overline{x}})\big)_{m\in\mathbf{M},\,\overline{x}\subset B}$$

defines an embedding from $\mathcal{H}_b(\mathbf{D}_a^B)$ into $\Lambda_B\big(\mathbf{M},Q^{\mathbf{M}}\big)$. $\square$

Lemma 3.2. *Let E be a Frechet space with the property $(\overline{\overline{\Omega}})$ and $Q=[q_{jk}\ge 0]$ a matrix satisfying the condition*

$$\forall n\ \exists k,\ \varepsilon>0 \quad q_{jn}^{1+\varepsilon}\le q_{jk}q_{j1}^{\varepsilon}\quad \forall j\ge 1.$$

Then every continuous linear map from E into $\Lambda_B\big(M,Q^M\big)$ is bounded on a neighbourhood of $0\in E$.

Proof. Given a sequence $K(N)$ of positive integers numbers. Since $E\in(\overline{\overline{\Omega}})$, for $K(1)$ there exists K such that

$$\forall K(N)\ \forall\varepsilon>0\ \exists C>0 \quad \|\bullet\|_K^{*\,1+\varepsilon}\le C\,\|\bullet\|_{K(N)}^{*}\,\|\bullet\|_{K(1)}^{*\,\varepsilon}.$$

Given $n\ge 1$. Choose $k\ge n$, $\varepsilon>0$ such that $q_{jn}^{1+\varepsilon}\le q_{jk}q_{j1}^{\varepsilon}$ $\forall j\ge 1$. Let $q_{jn}\,\|u\|_K^*\ge q_{j1}\,\|u\|_{K(1)}^*$. Then the inequality

$$\begin{aligned}
\|u\|_K^{*\,1+\varepsilon} &\le C\,\|u\|_{K(k)}^*\,\|u\|_{K(1)}^{*\varepsilon}\\
&\le C\,\|u\|_{K(k)}^*\,\|u\|_{K}^{*\varepsilon}\Big(\frac{q_{jn}}{q_{j1}}\Big)^{\varepsilon}
\end{aligned}$$

implies that

$$\|u\|_K^* \leq C\,\|u\|_{K(k)}^* \frac{q_{jk}}{q_{jn}}\ .$$

Hence

$$q_{jn}\,\|u\|_K^* \leq C \max_{1\leq N\leq k} q_{jN}\,\|u\|_{K(N)}^*\,, \quad \forall j\geq 1 \text{ and } \forall u\in E'.$$

From this we get

$$\begin{aligned}
\|T\|_{n,K} &= \sup\left\{\|Ty\|_n \mid \|y\|_K \leq 1\right\}\\
&= \sup\left\{|a_m(\overline{x})(Ty)|q_n^m \mid \overline{x}\subset B,\ m\in M,\ \|y\|_K\leq 1\right\}\\
&= \sup\left\{\|a_m(\overline{x})\circ T\|_K^*\, q_n^m \,\Big|\, \overline{x}\subset B,\ m\in M\right\}\\
&\leq C \max_{1\leq N\leq k}\left\{\sup\left\{\|a_m(\overline{x})\circ T\|_{K(N)}^*\, q_n^m \,\Bigg|\, \overline{x}\subset B,\ m\in M\right\}\right\}\\
&\leq C \max_{1\leq N\leq k} \|T\|_{N,K(N)}
\end{aligned}$$

for $T\in L\big(E,\Lambda_B(M,Q^M)\big)$. By [12], every $T\in L\big(E,\Lambda_B(M,Q^M)\big)$ is bounded on a neighbourhood of $0\in E$.

Lemma 3.3. *Let E and F be Frechet spaces having (DN) and $(\overline{\overline{\Omega}})$ respectively. Assume that E is a Montel space. Then every holomorphic function $f: D\longrightarrow F'$ on an open set D in E' is locally bounded.*

Proof. By Vogt [13] E is a subspace of the space $B\mathbin{\hat{\otimes}_\pi} s$ for some Banach space B. It follows that the restriction map $\mathcal{R}: [B\mathbin{\hat{\otimes}_\pi} s]'\cong B'\mathbin{\hat{\otimes}_\pi} s' \longrightarrow E'$ is open. Let $\tilde{D}=\mathcal{R}^{-1}(D)$ and $g=f\circ\mathcal{R}$. It suffices to show that g is locally bounded at every $\omega_\circ\in\tilde{D}$. Without loss of generality we may assume that $0\in\tilde{D}$ and $\omega_\circ=0$. Choose an open polydisc $\mathbf{D}_a\subset s'$ with $a=(a_j)\in s$, $a_j\geq 0$ for all $j\geq 1$, such that $\overline{\mathcal{R}(conv(V\otimes\mathbf{D}_a))}\subset D$, where V denotes the unit ball in E. Take $k\geq 1$ sufficiently large such that $\sum_{j\geq 1}\frac{1}{j^k}\leq 2$. Put $b=(2j^k a_j)\in s$. Then $\mathbf{D}_b^V$ is a neighbourhood of $0\in B'\mathbin{\hat{\otimes}_\pi} s'$ contained in $\tilde{D}$ because

$$\sum_{j\geq 1} x_j\otimes\xi_j e_j^* = \sum_{j\geq 1}\frac{1}{j^k}(x_j\otimes j^k\xi_j e_j^*)$$

and

$$x_j \otimes j^k \xi_j e_j^* \in V \otimes \mathbf{D}_a \text{ for } \xi \in \mathbf{D}_a \text{ and } (x_j) \subset V.$$

Consider the continuous linear map $\hat{g} : F \longrightarrow \mathcal{H}_b(\mathbf{D}_b^V)$ induced by g:

$$\hat{g}(z)(\omega) = g(\omega)(z) \text{ for } z \in F \text{ and } \omega \in D_b^V.$$

By applying Lemmas 3.1 and 3.2, we can find a neighbourhood U of $0 \in F$ such that $\hat{g}(U)$ is bounded in $\mathcal{H}_b(\mathbf{D}_b^V)$. Then, for every compact set K in $\mathbf{D}_b$, we have

$$\sup \left\{ |g(\omega)(z)| \,\middle|\, \omega \in K^V,\, z \in U \right\} = \sup \left\{ |\hat{g}(z)(\omega)| \,\middle|\, \omega \in K^V,\ z \in U \right\} < \infty$$

with

$$K^V = \Big\{ \sum_{j \geq 1} x_j \otimes \xi_j e_j^* \ \Big|\ \overline{x} \subset V,\ \xi \in K \Big\}.$$

Thus $g : \mathbf{D}_b^V \longrightarrow F'_U$ is holomorphic. This yields that g is locally bounded at $0 \in \mathbf{D}_b^V$. $\square$

Proof of Theorem 1.2. Given $f \in \mathcal{M}(E', F')$. By Lemma 3.3 and by the Lindelofness of E' we can find a sequence $\{u_j\}_{j=1}^\infty \subset E'$ and a sequence of balanced convex neighbourhoods $\{U_j\}_{j=1}^\infty$ of $0 \in E'$ such that

$$E' = \bigcup_{j \geq 1} (u_j + U_j)$$

and for each $j \geq 1$ there exists bounded holomorphic functions $h_j : u_j + U_j \longrightarrow F'$, $\sigma_j : u_j + U_j \longrightarrow \mathbf{C}$ for which

$$f\,\big|_{u_j + U_j} = \frac{h_j}{\sigma_j} \cdot$$

Hence h_j and σ_j induce the bounded holomorphic functions $\hat{h}_j$ and $\hat{\sigma}_j$, respectively, on a neighbourhood W_j of $\omega_{\rho_j}(u_j + U_j)$ in E'_{ρ_j}, where ρ_j denotes the semi-norm generated by U_j and ω_{ρ_j} the canonical map from E' into E'_{ρ_j}, the Banach space associated to ρ_{U_j}.

By [1] there exists a sequence $\mu_j \nearrow +\infty$ such that $\bigcap_{j \geq 1} \mu_j U_j$ is a neighbourhood of $0 \in E'$. Let $\omega(U, U_j) : E'_{\rho_U} \longrightarrow E'_{\rho_j}$ be the canonical map.

Then the family $\left\{\dfrac{\hat{h}_j\omega(U,U_j)}{\hat{\sigma}_j\omega(U,U_j)}\right\}$ defines a meromorphic function $\hat{f}$ on a neighbourhood Z of $E'/\ker\rho_U$ in E'_{ρ_U}. Let $Z_{\hat{f}}$ be the domain of existence of $\hat{f}$ over E'_{ρ_U}. Let $Z_{\hat{f}}$ be the domain of existence of$\hat{f}$ over E'_{ρ_U}. Then $Z_{\hat{f}}$ is a pseudoconvex domain in E'_{ρ_U}. Hence the function $\varphi(z) = -\log d(z, \partial Z_{\hat{f}})$ is plurisubharmonic on $Z_{\hat{f}}$. Since every plurisubharmonic function on a nuclear dual Frechet space is of uniform type [11], we can find a continuous seminorm ρ on E' and a plurisubharmonic function on E'_ρ such that $\rho \geq \rho_U$ and $\varphi\omega_{\rho_U} = \Psi\omega_\rho$. It suffices to show that Im $\omega_{\rho\rho_U} \subset Z_{\hat{f}}$. Indeed, in the converse case we can find $z \in E'_\rho$ such that $\omega_{\rho\rho_U}(z) \in \partial Z_{\hat{f}}$. Take a sequence $\{z_n\} \subset E'$ such that $\omega_\rho(z_n) \to z$. Then

$$+\infty = \lim_{n\to\infty} \varphi\omega_{\rho\rho_U}(z_n) = \lim_{n\to\infty} \varphi\omega_{\rho_U}(z_n) = \lim_{n\to\infty} \Psi\omega_\rho(z_n) \leq \Psi(z) < +\infty.$$

This is impossible. The proof of Theorem 1.2 is now complete.

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