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GEOMETRIC MONODROMY OF POLYNOMIALS OF TWO COMPLEX VARIABLES

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Abstract. We establish some relations between the polar curve and the discriminant locus of a polynomial f of two complex variables. We then describe the set of bifurcation values of f via its discriminant locus. Based on the Puiseux expansions at infinity of the discriminant locus of f, we also give certain sufficient conditions for the geometric monodromy of f around a critical value at infinity to have no fixed points.

1. Introduction

Let $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a polynomial of two complex variables. For a nonzero linear form $l(x,y) := l_1x + l_2y$ of $\mathbb{C}^2$, we define a mapping $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ by $\Phi(x,y) := (l(x,y), f(x,y))$ and put

$$C(\Phi) := \Big\{(x,y) \in \mathbb{C}^2 \mid l_2 f_x - l_1 f_y = 0\Big\}.$$

The set $C(\Phi)$ (resp., $\Delta(\Phi) := \Phi(C(\Phi))$) is called the *polar curve* (resp., the *Cerf diagram* or the *discriminant locus*) of f with respect to l.

In this paper we establish some relations between the polar curve and the discriminant locus of a polynomial f of two complex variables. Besides, we shall give certain sufficient conditions for the geometric monodromy of f around a given critical value at infinity to have no fixed points.

In what follows we shall need some facts on the topology of polynomials of two variables. It is well-known that f induces a locally trivial C^∞-fibration

$$f \ : \mathbb{C}^2 \setminus f^{-1}(A(f)) \longrightarrow \mathbb{C} \setminus A(f) \tag{1.1}$$

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over the complement of the so-called *bifurcation* set $A(f)$ of f, which is the set of either critical values or atypical values coming from "the singularities at infinity of f" (see, for example, [7], [14], [16], [17]). A value $t_0 \in \mathbb{C}$ is called *regular at infinity* if there exist a small $\eta > 0$ and a compact $K \subset \mathbb{C}^2$ such that the restriction

$$f \ : \ f^{-1}(\{t \mid |t - t_0| \leq \eta\}) \setminus K \longrightarrow \{t \mid |t - t_0| \leq \eta\}$$

is a trivial C^∞-fibration ([13]). If t_0 is not regular at infinity, it is called a *critical value at infinity* of f. If we denote by A_f (resp., A_∞) the set of critical values (resp., the set of critical values at infinity) of f, then (see, for example, [7])

$$A(f) = A_f \cup A_\infty.$$

In view of (1.1) one can introduce the geometric monodromy of f over a circle small enough around a given critical value at infinity. More precisely, consider the restriction

$$f \ : \ f^{-1}(\{t \mid |t - t_0| = \eta\}) \longrightarrow \{t \mid |t - t_0| = \eta\},$$

where $t_0 \in A_\infty$ and $\eta > 0$ small enough. The map associated with the path

$$[0, 1] \longrightarrow \{t \mid |t - t_0| = \eta\}, \qquad s \mapsto \eta e^{2\pi\sqrt{-1}s} + t_0,$$

is a diffeomorphism from $f^{-1}(\eta)$ onto itself, which is called the *geometric monodromy* of f.

The paper is organized as follows. In Section 2 we describe a geometric characterization of the set $A(f)$ via the discriminant locus $\Delta(\Phi)$. We then give normal forms for polynomials with particular *minimal discriminants*. In Section 3 we establish a relation between the Puiseux exponents at infinity of the polar curve and that of the discriminant locus, which is a version at infinity of a result of [12]. Finally, in Section 4, based on the carrousel method of Lê D. T. [10], we give certain sufficient conditions for the geometric monodromy of f around a critical value at infinity to have no fixed points. As a corollary, we obtain an analogue of A'Campo's result on Lefschetz's number of the local monodromy of Milnor's fibration [2].

2. Geometric characterization of the bifurcation values

Assume that the polynomial f is reduced and $n := \deg f - 1$. The map Φ is said to be simple if the inverse image $f^{-1}(t)$ consists of n distinct points for every critical value $(x, t) \in \mathbb{C}^2$ of Φ. Let

$$\Delta(x, t) := \operatorname{disc}_y(f(x, y) - t)$$

be the discriminant of f with respect to y. From the properties of resultants (see, for example, [18]) it follows that $\Delta(x,t)$ does not vanish identically.

Lemma 2.1. *Suppose that $l(x,y)=x$ is a generic linear form with respect to f. Then*

$$\Delta(\Phi)=\{(x,t)\in\mathbb{C}^2 \mid \Delta(x,t)=0\}.$$

Moreover, if Φ is simple then it induces a homeomorphism from $C(\Phi)$ onto $\Delta(\Phi)$.

Proof. Since $n+1=\deg f$, we may write

$$f(x,y)=a_0(x)y^{n+1}+\cdots+a_{n+1}(x),$$

where $a_i\in\mathbb{C}[x]$, $\deg a_i\le i$, $i=0,\ldots,n+1$. Since $l(x,y)=x$ is a generic linear form with respect to f,

$$a_0(x)=\text{const}\neq 0.$$

On the other hand, by definition, $(x,t)\in\Delta(\Phi)$ if and only if there exists $y\in\mathbb{C}$ satisfying the system of equations:

$$\begin{cases} f(x,y)-t=0, \\ f_y(x,y)=0. \end{cases} \tag{2.1}$$

Or equivalently, by definition, $\Delta(x,t)=0$. Moreover, if Φ is simple, then for any $(x,t)\in\mathbb{C}^2$, the system of equations (2.1) has a unique solution y in $\mathbb{C}$. Hence Φ induces a homeomorphism from $C(\Phi)$ onto $\Delta(\Phi)$. □

From now on there is no loss of generality in assuming that $l(x,y)=x$ is a generic linear form with respect to f.

Definition 2.2. The line $t-t_0=0$ is said to be *contained* in the tangent cone of the discriminant locus $\Delta(x,t)=0$ if and only if the following conditions hold:

(i) there exists x_0 in $\mathbb{C}$ such that $\Delta(x_0,t_0)=0$, and
(ii) if the Taylor expansion of $\Delta(x,t)$ at (x_0,t_0) is

$$\Delta=\Delta_j+\Delta_{j+1}+\cdots,$$

Δ_i being a homogeneous polynomial of degree i, then $\Delta_j(x,t_0)\equiv 0$.

The next theorem describes the set of bifurcation values $A(f)$ of the polynomial f via the discriminant locus.

Theorem 2.3. *With the notations as above we have:*

(i) t_0 *is a critical value of* f *if and only if the line* $t - t_0 = 0$ *is contained in the tangent cone of the discriminant locus* $\{\Delta(x,t) = 0\}$.

(ii) t_0 *is a critical value at infinity of* f *if and only if the line* $t - t_0 = 0$ *is an asymptote of* $\{\Delta(x,t) = 0\}$.

Proof. The second part of Theorem 2.3 is essentially a result of [6]. Suppose that $t_0 \in A_f$, i.e., there exists $(x_0, y_0) \in \mathbb{C}^2$ such that

$$\begin{cases} f(x_0, y_0) = t_0, \\ f_x(x_0, y_0) = f_y(x_0, y_0) = 0. \end{cases}$$

Let

$$p \;:\; (\mathbb{C}, 0) \longrightarrow (C(\Phi), (x_0, y_0)), \quad \tau \mapsto (x(\tau), y(\tau)),$$

be a parametrization of the polar curve $C(\Phi)$ in a small neighborhood of (x_0, y_0). Then the map

$$(\mathbb{C}, 0) \longrightarrow (\Delta(\Phi), (x_0, t_0)), \quad \tau \mapsto (x(\tau), t(\tau) := f(x(\tau), y(\tau))),$$

is a parametrization of $\Delta(\Phi)$ in a small neighborhood of (x_0, t_0). We have

$$\begin{aligned} \frac{dt(\tau)}{d\tau} &= \frac{df(p(\tau))}{d\tau} \\ &= f_x(p(\tau))\dot{x}(\tau) + f_y(p(\tau))\dot{y}(\tau) \\ &= f_x(p(\tau))\dot{x}(\tau). \end{aligned}$$

It follows that

$$\lim_{\tau \to 0} \frac{\dot{t}(\tau)}{\dot{x}(\tau)} = \lim_{\tau \to 0} f_x(p(\tau)) = f_x(x_0, y_0) = 0.$$

In other words, the line $t - t_0 = 0$ is contained in the tangent cone of $\Delta(x,t) = 0$.

Conversely, suppose that the line $t - t_0 = 0$ is contained in the tangent cone of $\Delta(x,t) = 0$. By definition, Lemma 2.1 implies that there exist $(x_0, y_0) \in C(\Phi) \cap f^{-1}(t_0)$ and a parametrization of $C(\Phi)$ in a small neighborhood of (x_0, y_0)

$$p \ : \ (\mathbb{C}, 0) \longrightarrow (C(\Phi), (x_0, y_0)), \quad \tau \mapsto (x(\tau), y(\tau)),$$

such that

$$\lim_{\tau \to 0} \frac{\dot{t}(\tau)}{\dot{x}(\tau)} = 0,$$

where $t(\tau) := f(x(\tau), y(\tau))$. It follows that

$$f_x(x_0, y_0) = \lim_{\tau \to 0} f_x(p(\tau)) = \lim_{\tau \to 0} \frac{\dot{t}(\tau)}{\dot{x}(\tau)} = 0,$$

i.e., $t_0 \in A_f$. □

From Theorem 2.3 we obtain the following corollary.

Corollary 2.4. *If t_0 is a bifurcation value of f, i.e., $t_0 \in A(f)$, then*

$$\#\Big(\{t = t_0\} \cap \Delta(\Phi)\Big) < m = \deg_x(\Delta(x, t)).$$

Theorem 2.3 allows us to make the following definition.

Definition 2.5. The discriminant $\Delta(x, t)$ of the polynomial f is *minimal* if the factorization of $\Delta(x, t)$ into irreducible factors in $\mathbb{C}[x, t]$ is of the form $\Delta = \Delta_1^{\alpha_1} \cdots \Delta_r^{\alpha_r}$ such that for any $i = 1, \ldots, r$ there exists $t_0 \in \mathbb{C}$ with the property that either the line $t - t_0 = 0$ is contained in the tangent cone of the curve $\Delta_i = 0$ or it is an asymptote of $\Delta_i = 0$.

For a polynomial function $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$, the degree of f depends on the coordinate system of $\mathbb{C}^2$: if φ is an algebraic isomorphism of $\mathbb{C}^2$, then it may happen that $\deg(f) \neq \deg(f \circ \varphi)$. Following [13], we define the *intrinsic degree* of f to be

$$\deg_{\text{int}}(f) := \min\{\deg(f \circ \varphi) \mid \varphi \in \operatorname{Aut}(\mathbb{C}^2)\}.$$

For each $\varphi \in \operatorname{Aut}(\mathbb{C}^2)$, we will denote by $\Delta_\varphi(x, t)$ the discriminant of $(f \circ \varphi - t)$ with respect to y. Obviously, $\Delta_{\text{id}}(x, t) = \Delta(x, t)$, where id is the identity map.

By [8], [9], for any $\varphi \in \operatorname{Aut}(\mathbb{C}^2)$ such that the map

$$\mathbb{C}^2 \longrightarrow \mathbb{C}^2, \qquad (x, y) \mapsto (x, f \circ \varphi(x, y)),$$

is proper, we have

$$\deg_x \Delta_\varphi = \deg_y (f \circ \varphi) - \chi((f \circ \varphi)^{-1}(t)),$$

where $\chi((f \circ \varphi)^{-1}(t))$ is the Euler-Poincaré charateristic of the fibre $(f \circ \varphi)^{-1}(t)$ for t generic.

Therefore, one might hope that if $\varphi \in \mathrm{Aut}(\mathbb{C}^2)$, with $\deg(f \circ \varphi) = \deg_{\mathrm{int}}(f)$, then Δ_φ is a minimal discriminant of f. But the following example shows that this is not true.

Example 2.6. Let $f(x, y) = y^3 - 3x^2 y + 2x^3 - 12x$. We have $A_\infty = \emptyset$ and $A(f) = A_f = \{-8, -16\}$. Thus, it is easy to check that $\deg(f) = \deg_{\mathrm{int}} f = 3$. But, by definition, the discriminant of f

$$\Delta(x, t) = 27(t + 12x)(t + 12x - 4x^3)$$

is not a minimal discriminant.

The following theorem provides the normal forms for some classes of minimal discriminants.

Theorem 2.7. *Let*

$$\Delta(x, t) = c \prod_{i=1}^{r} (t - P(x) - c_i)^{\alpha_i}$$

be the discriminant of f, where $c \neq 0$, $P \in \mathbb{C}[x]$, $P(0) = 0$, $c_i \neq c_j$ $(i \neq j)$. Moreover, let the map Φ be simple. Then there exists an algebraic isomorphism $\varphi \in Aut\, \mathbb{C}^2$ such that

$$(f \circ \varphi)(x, y) = g(x) + h(y),$$

where g, h are some polynomials of one complex variable.

Proof. By the properties of the discriminant $\Delta(x, t)$ (see, for instance, §9, Chap. I, [18]), it may be concluded that there exist polynomials $H, G \in \mathbb{C}[x, t]$ such that

$$H(x, t)y = G(x, t),$$

where $x, y, t \in \mathbb{C}$ satisfy the following system of equations

$$\begin{cases} f_x(x, y) = t, \\ f_y(x, y) = 0. \end{cases}$$

So, if all solutions $t = t(x)$ of the equation $\Delta(x,t) = 0$ are polynomial functions, then all solutions $y = y(x)$ of $f_y(x,y) = 0$ are rational functions. Hence, by the assumption, all solutions $y = y(x)$ of the equation $f_y(x,y) = 0$ are rational functions of x.

On the other hand, since $l = x$ is a generic linear form with respect to f, the map Φ is proper. Hence these solutions are polynomial functions.

Moreover, from the definition of resultants (see [18]), it is easy to check that

$$\deg_t(\Delta(x,t)) = \deg_y(f) - 1 = n.$$

Therefore, we may write

$$f_y(x,y) = c' \prod_{i=1}^{k} (y - y_i(x))^{n_i},$$

where $c' \neq 0$, $y_i \in \mathbb{C}[x]$, $\sum\limits_{i=1}^{k} n_i = n$.

Let

$$\Gamma_i := \{(x,y) \in \mathbb{C}^2 \mid y = y_i(x)\}, \quad i = 1, \dots, k,$$

and

$$D_i := \{(x,t) \in \mathbb{C}^2 \mid t = P(x) + c_i\}, \quad i = 1, \dots, r.$$

By Lemma 2.1, Φ induces a homeomorphism from $C(\Phi) = \bigcup\limits_{i=1}^{k} \Gamma_i$ onto $\Delta(\Phi) = \bigcup\limits_{i=1}^{r} D_i$. But $D_i \cap D_j = \emptyset$ $(i \neq j)$, so

$$r = k, \quad n_i = \alpha_i, \quad \Gamma_i \cap \Gamma_j = \emptyset \quad (i \neq j).$$

Moreover, by reindexing if necessary, we can assume that the restrictions

$$\Phi|_{\Gamma_i} \; : \; \Gamma_i \longrightarrow D_i, \qquad i = 1, \dots, k,$$

are homeomorphisms.

From $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$, we have $y_i(x) - y_j(x) = \text{const} \neq 0$. On the other hand, since $y_i(x)$, $i = 1, \dots, k$, are polynomials, it follows that there exist a polynomial function $Q \in \mathbb{C}[x]$ and constants $b_i, i = 1, \dots, k$, $b_i \neq b_j$ $(i \neq j)$, such that $y_i(x) = Q(x) + b_i$. Therefore, one may rewrite

$$f_y(x,y) = c' \prod_{i=1}^{k} (y - Q(x) - b_i)^{n_i}.$$

It follows that there exists a polynomial $\bar{g} \in \mathbb{C}[x]$ such that

$$f(x,y) = \bar{g}(x) + \int_0^y c' \prod_{i=1}^k (u - Q(x) - b_i)^{n_i} du$$
$$= \bar{g}(x) + \int_{-Q(x)}^{y-Q(x)} c' \prod_{i=1}^k (z - b_i)^{n_i} dz.$$

From this we conclude that

$$f(x,y) = \bar{g}(x) + h(y - Q(x)) - h(-Q(x)),$$

where h is some polynomial of one complex variable with $\deg h = n+1$. Let

$$\varphi \; : \; \mathbb{C}^2 \longrightarrow \mathbb{C}^2, \qquad (x,y) \mapsto (x, y + Q(x)).$$

Then φ is an algebraic isomorphism of $\mathbb{C}^2$. It is easy to check that

$$(f \circ \varphi)(x,y) = g(x) + h(y),$$

where $g(x) := \bar{g}(x) - h(-Q(x))$ □

The following corollary shows that Theorem 2.7, in a certain case, agrees with a result of Ahbyankar-Moh (see [1], [3]).

Corollary 2.8. *Under the assumptions of Theorem 2.7, if moreover* $P = ax$ $(a = const \neq 0)$, *then*

$$f \sim x \qquad (\operatorname{Aut} \mathbb{C}^2).$$

Proof. Actually, by Theorem 2.7, there exists an algebraic isomorphism $\varphi \in \operatorname{Aut} \mathbb{C}^2$ such that

$$(f \circ \varphi)(x,y) = g(x) + h(y),$$

where g, h $(\deg h = n+1)$ are some polynomials of one complex variable.

On the other hand, by the assumption and Theorem 2.3, $A_f = \emptyset$. Therefore, the system of equations

$$\begin{cases} g_x(x) = 0, \\ h_y(y) = 0 \end{cases}$$

has no solution. So $\deg g = 1$. In other words, we may write

$$g(x) = \alpha x + \beta \qquad (\alpha \neq 0).$$

Hence, the map

$$(x, y) \mapsto (g(x) + h(y), y)$$

is an algebraic isomorphism of $\mathbb{C}^2$, and so $f \sim x$ $(\operatorname{Aut} \mathbb{C}^2)$. $\square$

3. The Puiseux exponents at infinity of disciminants

In this section, we will establish a relation between the Puiseux exponents at infinity of the polar curve and that of the discriminant locus, which is a version at infinity of [12]. First, we recall the definition of the Puiseux exponents at infinity of a plane curve (see [5]).

Let P be a polynomial of two complex variables. Denote by $\overline{V} \subset \mathbb{C}\mathbb{P}^2$ the compactification of the curve $V := \{P(x, y) = 0\}$. Let

$$\{Z_1, \dots, Z_r\} := \overline{V} \cap \{z = 0\}.$$

Assume that the curve $\overline{V}$ is irreducible at all the points $Z_i, i = 1, \dots, r$, with the same geometrical multiplicity m. Then, according to [5], for x sufficiently large we can write

$$P(x, y) = c \prod_{i=1}^{r} \prod_{l=1}^{n_i} \big(y - \varphi_i(e^{\frac{2\pi\sqrt{-1}}{n_i}\ell} x)\big)^m, \tag{3.1}$$

where $c = \text{const} \neq 0$, $m\Big(\sum_{i=1}^{r} n_i\Big) = \deg P$, and $\varphi_i(x)$, $i = 1, \dots, r$, are of the form

$$\varphi_i(x) = c_i x + x\varphi_{i0}(x^{-1}) + \sum_{j=1}^{g_i} x^{1-\frac{\beta_{ij}}{n_i}} \varphi_{ij}\big(x^{-\frac{e_{ij}}{n_i}}\big),$$

where $c_i \neq c_j$ $(i \neq j)$, $\varphi_{ij}(0) \neq 0$ $(j > 0)$,

$$n_i = e_{i0}, \quad e_{i0} = n_{i1}e_{i1}, \quad e_{i1} = n_{i2}e_{i2}, \quad e_{ig_i-1} = n_{ig_i}e_{ig_i}, \quad e_{ig_i} = 1,$$

$$\beta_{i1} = m_{i1}e_{i1} < \beta_{i2} = m_{i2}e_{i2} < \cdots < \beta_{ig_i} = m_{ig_i}e_{ig_i},$$

and m_{ij} and n_{ij} are relatively prime.

Let $\gamma_{ij} = 1 - \dfrac{\beta_{ij}}{n_i}$.

Definition 3.1. The tuples $(n_i, \gamma_{i1}, \gamma_{i2}, \dots, \gamma_{ig_i})$, $i = 1, \dots, r$, are called *Puiseux exponents at infinity* of the curve V.

We now formulate the main result of this section.

Theorem 3.2. *Suppose that* $(n_i, \gamma_{i1}, \dots, \gamma_{ig_i})$ *(resp.,* $(n_i, \gamma'_{i1}, \dots, \gamma'_{ig_i})$*) are Puiseux exponents at infinity of the polar curve* $C(\Phi)$ *(resp., the discriminant locus* $\Delta(\Phi)$*). Then*

$$\gamma'_{i1} = (mn_i + 1)\gamma_{i1} + (n - mn_i),$$
$$\gamma'_{ij} = (me_{ij-1} + 1)\gamma_{ij} + m(h_{i1}\gamma_{i1} + \dots + h_{ij-1}\gamma_{ij-1}) + (n - mn_i),\ j > 1,$$

where $h_{ij} := e_{ij-1} - e_{ij}$.

Following Maisonobe [12], we divide the proof into a sequence of lemmas. We begin with a definition. Let

$$\mathrm{val}(\psi(x)) := \frac{r_0}{n_0},$$

where $\psi(x)$ is of the form $\psi(x) = \sum_{j=r_0}^{-\infty} a_j x^{\frac{j}{n_0}}$ $(n_0 > 0,\ a_{r_0} \neq 0)$.

Lemma 3.3. (i) *For each* $i = 1, \dots, r, j = 1, \dots, g_i$, *we have*

$$\varphi_i(\varepsilon^l x) - \varphi_i(x) \sim (\varepsilon^{l\gamma_{ij}} - 1)\varphi_{ij}(0)x^{\gamma_{ij}} \qquad (|x| \gg 1)$$

if and only if l *is not a multiplicity of* $n_{i1}n_{i2}\dots n_{ij}$ *but of* $n_{i1}n_{i2}\dots n_{ij-1}$, *where* $\varepsilon = e^{2\pi\sqrt{-1}}$. *Moreover,*

$$h_{ij} = \#\{l \mid 1 \le l \le n_i - 1, \mathrm{val}(\varphi_i(\varepsilon^l x) - \varphi_i(x)) = \gamma_{ij}\}.$$

(ii) *For each* $i, j = 1, \dots, r$ $(i \neq j)$, $l = 0, \dots, n_j - 1$, *we have*

$$\mathrm{val}(\varphi_j(\varepsilon^l x) - \varphi_i(x)) = 1.$$

Proof. The proof follows from the definition. □

To calculate the Puiseux exponents at infinity of the polar curve $C(\Phi)$, it is sufficient by Lemma 3.3 to compute the valuation of $\varphi_i(\varepsilon^l x) - \varphi_i(x)$, $l = 0, \dots, n_i - 1$.

According to (3.1),

$$
\begin{aligned}
(3.2)\qquad & f(x,\varphi_i(\varepsilon^l x)) - f(x,\varphi_i(x)) = \\
&= \sum_{s=m}^{mn_i} \frac{(\varphi_i(\varepsilon^l x)-\varphi_i(x))^{s+1}}{(s+1)!}\frac{\partial^s f_y}{\partial y^s}(x,\varphi_i(x)) \\
&\quad + h(x)(\varphi_i(\varepsilon^l x)-\varphi_i(x))^{mn_i+2},
\end{aligned}
$$

where

$$
\begin{aligned}
h(x) := {} & \frac{1}{(mn_i+2)!}\frac{\partial^{mn_i+1} f_y}{\partial y^{mn_i+1}}(x,\varphi_i(x)) \\
& + \frac{1}{(mn_i+3)!}\frac{\partial^{mn_i+2} f_y}{\partial y^{mn_i+2}}(x,\varphi_i(x))(\varphi_i(\varepsilon^l x)-\varphi_i(x)) + \\
& + \cdots + \frac{1}{(n+1)!}\frac{\partial^{n} f_y}{\partial y^{n}}(x,\varphi_i(x))(\varphi_i(\varepsilon^l x)-\varphi_i(x))^{n-mn_i-1}.
\end{aligned}
$$

We first compute $\mathrm{val}(f(x,\varphi_i(\varepsilon^l x)) - f(x,\varphi_i(x)))$. For this purpose we set

$$
Q_i(x,y) := \prod_{l=0}^{n_i-1}(y-\varphi_i(\varepsilon^l x))^m.
$$

Then

$$
(3.3)\qquad \frac{\partial^s Q_i}{\partial y^s}(x,\varphi_i(x)) = s! \sum_{\substack{0<l_1<\ldots<l_{s-m}<mn_i \\ l_\alpha \text{ is not a mult. of } n_i}} \frac{\prod_{l=1}^{n_i-1}(\varphi_i(x)-\varphi_i(\varepsilon^l x))^m}{\prod_{\alpha=1}^{s-m}(\varphi_i(x)-\varphi_i(\varepsilon^{l_\alpha} x))}.
$$

Suppose that

$$
m(h_{ig_i}+\cdots+h_{ij+1}) \le s-m \le m(h_{ig_i}+\cdots+h_{ij}).
$$

This means that

$$
me_{ij} \le s \le me_{ij-1},
$$

because $h_{ig_i}+\cdots+h_{ij} = e_{ij-1}-1$. By Lemma 3.3,

$$\#\{l_\alpha \mid 0 < l_\alpha < mn_i, \mathrm{val}(\varphi_i(x) - \varphi_i(\varepsilon^{l_\alpha} x)) = \gamma_{ig_i}\} = mh_{ig_i},$$

$$\dots$$

$$\#\{l_\alpha \mid 0 < l_\alpha < mn_i, \mathrm{val}(\varphi_i(x) - \varphi_i(\varepsilon^{l_\alpha} x)) = \gamma_{ij+1}\} = mh_{ij+1}.$$

Hence we can write

$$\frac{\partial^s Q_i}{\partial y^s}(x, \varphi_i(x)) \;=\; A_s^i B_s^i x^{k_s^i} + \sum_{\alpha < k_s^i} a_\alpha x^\alpha,$$

where

$$\begin{aligned} k_s^i &= m \sum_{l=1}^{n_i - 1} h_{il}\gamma_{il} - m \sum_{l=j+1}^{g_i} h_{il}\gamma_{il} - [s - m - m(h_{ig_i} + \cdots + h_{ij+1})]\gamma_{ij} \\ &= (me_{ij-1} - s)\gamma_{ij} + m(h_{i1}\gamma_{i1} + \cdots + h_{ij-1}\gamma_{ij-1}), \end{aligned}$$

$$\begin{aligned} A_s^i :=& \prod_{\substack{l \text{ is not a mult. of } n_{i1}}}^{0<l<n_i} (1 - \varepsilon^{l\gamma_{i1}})^m \cdots \\ & \prod_{\substack{l \text{ is not a mult. of } n_{i1}\cdots n_{ij-1} \\ l \text{ is a mult. of } n_{i1}\dots n_{ij-2}}}^{0<l<n_i} (1 - \varepsilon^{l\gamma_{ij-1}})^m \varphi_{i1}^{mh_{i1}}(0) \cdots \varphi_{ij-1}^{mh_{ij-1}}(0), \end{aligned}$$

and

$$\begin{aligned} B_s^i :=& s! \sum_{\substack{l_\alpha \text{ is not a mult. of } n_{i1}\cdots n_{ij} \\ l_\alpha \text{ is a mult. of } n_{i1}\dots n_{ij-1}}}^{0<l_1<\dots<l_{s-me_{ij}}<mn_i} \frac{\prod_{\substack{l \text{ is not a mult. of } n_{i1}\cdots n_{ij} \\ l \text{ is a mult. of } n_{i1}\dots n_{ij-1}}}^{0<l<n_i} (1 - \varepsilon^{l\gamma_{ij}})^m}{\prod_{\alpha=1}^{s-me_{ij}} (1 - \varepsilon^{l_\alpha\gamma_{ij}})} \times \\ & \times \varphi_{ij}(0)^{me_{ij-1}-s}. \end{aligned}$$

Lemma 3.4.

$$\prod_{\substack{l \text{ is not a mult. of } n_{i1}\cdots n_{ip} \\ l \text{ is a mult. of } n_{i1}\cdots n_{ip-1}}}^{0<l<n_i} (1 - \varepsilon^{l\gamma_{ip}}) = (n_{ip})^{e_{ip}}.$$

Proof. It is clear that if $l \in \{1, \dots, n_i - 1\}$ is not a multiplicity of $n_{i1} \cdots n_{ip}$ but of $n_{i1} \cdots n_{ip-1}$, then l should be of the form $l = n_{i1} \cdots n_{ip-1}(\alpha n_{ip} + \beta)$, where $\alpha \in \{0, \dots, e_{ip} - 1\}$, $\beta \in \{1, \dots, n_{ip} - 1\}$.

On the other hand, we have

$$\frac{x^n - 1}{x - 1} = \prod_{j=1}^{n-1} (x - e^{-j\frac{2\pi\sqrt{-1}}{n}}) = 1 + x + \cdots + x^{n-1}.$$

Therefore,

$$\begin{aligned}\prod_{\substack{l \text{ is not a mult. of } n_{i1}\cdots n_{ip} \\ l \text{ is a mult. of } n_{i1}\cdots n_{ip-1}}}^{0<l<n_i} (1 - \varepsilon^{l\gamma_{ip}}) &= \prod_{\substack{l \text{ is not a mult. of } n_{i1}\cdots n_{ip} \\ l \text{ is a mult. of } n_{i1}\cdots n_{ip-1}}}^{0<l<n_i} (1 - \varepsilon^{-l\frac{\beta_{ip}}{n_i}}) \\ &= (n_{ip})^{e_{ip}}.\end{aligned}$$

□

By Lemma 3.4,

$$A_s^i = n_{i1}^{me_{i1}} \cdots n_{ij-1}^{me_{ij-1}} . \varphi_{i1}^{mh_{i1}}(0) \cdots \varphi_{ij-1}^{mh_{ij-1}}(0).$$

On the other hand, from the identify

$$\left(\frac{x^{n_{ij}} - 1}{x - 1}\right)^{me_{ij}} = \prod_{\substack{l \text{ is not a mult. of } n_{i1}\cdots n_{ij} \\ l \text{ is a mult. of } n_{i1}\ldots n_{ij-1}}}^{0<l<n_i} \left(x - \varepsilon^{l\gamma_{ij}}\right)^m,$$

we deduce that

$$B_s^i = \frac{s!}{(s - me_{ij})!} \frac{\partial^{s-me_{ij}}}{\partial x^{s-me_{ij}}} \left(\frac{x^{n_{ij}} - 1}{x - 1}\right)^{me_{ij}} (1) \varphi_{ij}^{me_{ij-1}-s}(0).$$

It follows that $A_s^i B_s^i \neq 0$. Consequently, for $i = 1, \dots, r$, $j = 1, \dots, g_i$, with

$$me_{ij} \leq s \leq me_{ij-1},$$

we have

$$\frac{\partial^s Q_i}{\partial y^s}(x, \varphi_i(x)) \sim A_s^i B_s^i x^{k_s^i} \qquad (|x| \gg 1).$$

The following lemma can be easily derived from Leibnitz's formula.

Lemma 3.5. *We have*

$$\frac{\partial^s f_y}{\partial y^s}(x,\varphi_i(x)) \ \sim \ c \prod_{\alpha\neq i}^{1\leq\alpha\leq r}(c_i-c_\alpha)A_s^iB_s^ix^{k_s^i+(n-mn_i)}$$

for each s with $me_{ij}\leq s\leq me_{ij-1}$ and x sufficiently large.

Lemma 3.6. *Suppose* $\mathrm{val}(\varphi_i(\varepsilon^l x)-\varphi_i(x))=\gamma_{ij}$ *for some $l\in\{1,\dots,n_i-1\}$, $j\in\{1,\dots,g_i\}$. Then*

$$\begin{aligned}&\mathrm{val}\left((\varphi_i(\varepsilon^l x)-\varphi_i(x))^{s+1}.\frac{\partial^s f_y}{\partial y^s}(x,\varphi_i(x))\right)\\ &\quad=(me_{ij-1}+1)\gamma_{ij}+m(h_{i1}\gamma_{i1}+\cdots+h_{ij-1}\gamma_{ij-1})+(n-mn_i)\end{aligned}$$

for each s with $me_{ij}\leq s\leq me_{ij-1}$.

Proof. In fact, by Lemma 3.5, we have

$$\begin{aligned}&\mathrm{val}\left((\varphi_i(\varepsilon^l x)-\varphi_i(x))^{s+1}.\frac{\partial^s f_y}{\partial y^s}(x,\varphi_i(x))\right)\\ &\quad=(s+1)\gamma_{ij}+k_s^i+(n-mn_i)\\ &\quad=(me_{ij-1}+1)\gamma_{ij}+m(h_{i1}\gamma_{i1}+\cdots+h_{ij-1}\gamma_{ij-1})+(n-mn_i). \ \square\end{aligned}$$

Lemma 3.7. *Suppose* $\mathrm{val}(\varphi_i(\varepsilon^l x)-\varphi_i(x))=\gamma_{ij}$ *for some $j\in\{1,\dots,g_i\}$. Then*

$$\begin{aligned}&\mathrm{val}\left(h(x)(\varphi_i(\varepsilon^l x)-\varphi_i(x))^{mn_i+2}\right)\\ &\quad<(me_{ij-1}+1)\gamma_{ij}+m(h_{i1}\gamma_{i1}+\cdots+h_{ij-1}\gamma_{ij-1})+(n-mn_i).\end{aligned}$$

Proof. Since

$$\begin{aligned}m(h_{i1}\gamma_{i1}&+\cdots+h_{ij-1}\gamma_{ij-1})\\ &+(me_{ij-1}+1)\gamma_{ij}>m(h_{i1}+\cdots+h_{ij-1}+e_{ij-1})\gamma_{ij}+\gamma_{ij}\\ &\qquad\qquad=mn_i\gamma_{ij}+\gamma_{ij}\\ &\qquad\qquad=(mn_i+1)\gamma_{ij}\end{aligned}$$

and $\gamma_{ij}=1-\dfrac{\beta_{ij}}{n_i}<1$, it follows that

$$\begin{aligned}&\mathrm{val}\left(h(x)(\varphi_i(\varepsilon^l x)-\varphi_i(x))^{mn_i+2}\right)\\ &=(mn_i+2)\gamma_{ij}+(n+1)-(mn_i+2)\\ &<(mn_i+1)\gamma_{ij}+(n-mn_i)\\ &<(me_{ij-1}+1)\gamma_{ij}+m(h_{i1}\gamma_{i1}+\cdots+h_{ij-1}\gamma_{ij-1})+(n-mn_i). \ \square\end{aligned}$$

Lemma 3.8. *Suppose* $\mathrm{val}(\varphi_i(\varepsilon^l x) - \varphi_i(x)) = \gamma_{ij}$ *for some* $j \in \{1, \dots, g_i\}$ *Then, for x sufficiently large, we have*

$$f(x, \varphi_i(\varepsilon^l x)) - f(x, \varphi_i(x)) \sim$$
$$n_{i1}^{me_{i1}} \dots n_{ij-1}^{me_{ij-1}} \varphi_{i1}^{mh_{i1}}(0) \dots \varphi_{ij-1}^{mh_{ij-1}}(0).\varphi_{ij}^{me_{ij-1}+1}(0) \times$$
$$\left(\int_0^1 (u^{n_{ij}} - 1)^{me_{ij}} du \right) x^{(me_{ij-1}+1)\gamma_{ij} + m(h_{i1}\gamma_{i1} + \dots + h_{ij-1}\gamma_{ij-1}) + (n - mn_i)}.$$

Proof. From what has already been proved, it follows that for $|x| \gg 1$,

$$f(x, \varphi_i(\varepsilon^l x)) - f(x, \varphi_i(x)) \sim$$
$$D.S.x^{(me_{ij-1}+1)\gamma_{ij} + m(h_{i1}\gamma_{i1} + \dots + h_{ij-1}\gamma_{ij-1}) + (n - mn_i)},$$

where

$$D := n_{i1}^{me_{i1}} \dots n_{ij-1}^{me_{ij-1}} \varphi_{i1}^{mh_{i1}}(0) \dots \varphi_{ij-1}^{mh_{ij-1}}(0).\varphi_{ij}^{me_{ij-1}+1}(0)$$

and

$$S :=$$
$$\sum_{s=me_{ij}}^{me_{ij-1}} \frac{1}{(s - me_{ij})!} \frac{1}{(s+1)} \frac{\partial^{s-me_{ij}}}{\partial x^{s-me_{ij}}} \left(\frac{x^{n_{ij}} - 1}{x - 1} \right)^{me_{ij}} (1). \left(\varepsilon^{l\gamma_{ij}} - 1 \right)^{s+1}.$$

Let

$$S(x) :=$$
$$\sum_{u=0}^{s=m(e_{ij-1}-e_{ij})} \frac{1}{u!} \frac{1}{(u + me_{ij} + 1)} \frac{\partial^u}{\partial x^u} \left(\frac{x^{n_{ij}} - 1}{x - 1} \right)^{me_{ij}} (1).(x-1)^{u+me_{ij}+1}.$$

A trivial verification shows that

$$S = S\left(\varepsilon^{l\gamma_{ij}}\right) = S(\varepsilon^{-l\frac{\beta_{ij}}{n_i}}).$$

Moreover, by Taylor's formula,

$$S'(x) = \left(\frac{x^{n_{ij}} - 1}{x - 1} \right)^{me_{ij}} (x-1)^{me_{ij}} = (x^{n_{ij}} - 1)^{me_{ij}}.$$