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POINCARÉ SERIES OF QUANTUM SPACES ASSOCIATED TO HECKE OPERATORS

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Dedicated to Prof. Dr. Bodo Pareigis on his 60th birthday

Abstract. We prove that the dimension of the homogeneous components of the quadratic algebras associated to a Hecke operator is a Pólya frequency (P-) sequence. This result enables us to give some characterizations on the Poincaré series of these quadratic algebras.

1. Introduction

Subject of this work are the quadratic algebras associated to a Hecke operator. Hecke operator on a vector space V is a linear operator on $V \otimes V$ that satisfies the Yang-Baxter equation and the Hecke equation $(x+1)(x-q) = 0$ and hence induces a representation of the Hecke algebra of the general linear group on the tensor power $V^{\otimes n}$ of V for every $n \geq 2$. Given a Hecke operator R on a vector space V, one can define the quantum anti-symmetric and symmetric tensor algebras on V, which, in case R is the usual permuting operator, are the usual anti-symmetric (i.e. exterior) and symmetric tensor algebras over V. Thus one can think of these quadratic algebras as the anti-symmetric and symmetric tensor algebras on certain quantum space. Then one can define an algebra that coacts universally upon this pair of quadratic algebras, which turns out to be a quadratic algebra, too. This algebra can be considered as the function algebra over the matrix quantum semi-group of "symmetries" of the quantum space mentioned above [16, 14, 7, 17].

In this work we study the Poincaré series of the above quadratic algebras. The first attempt in this study was made by Lyubashenko, who proved that (in case of $q = 1$) if $P_\Lambda(t)$ is a polynomial then it is a reciprocal polynomial iff R is closed [14]. This result was generalized by Gurevich for arbitrary q [7]. He also gave an example of an even Hecke operator whose rank differs from the dimension of the vector space it is defined on.

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Here we show that the rank should be less than or equal to the dimension of the corresponding vector space. We also study the case of odd-even Hecke operator and analogously show that its super rank should have the weight (the sum of two components of the rank) less than or equal to the dimension of the vector space it is defined on. Our main theorem (Theorem 3.5) states that the Poincaré series of quadratic algebras associated to a Hecke operator is a rational function having negative roots and positive poles.[1)]

Using the theory of symmetric functions we give a combinatorial formula for the dimension of irreducible representation of the matrix quantum semi-group as functions on the dimension of homogeneous components of the quantum space. Whence we recover the relation between the Poincaré series of the quantum space and of its matrix quantum semi-group, which was found in [8].

In this work, a theorem of Edrei on Pólya frequency (or P-) sequences [5] plays an important role, from which almost all results follow. Our method bases on the theory of Schur symmetric functions [15] as well as the quantum version of Schur's double centralizer theorem [8].

2. Quantum spaces and quantum semi-groups

Let $\mathbb{K}$ be a field of characteristic zero, V be a vector space over $\mathbb{K}$ of finite dimension d. A Hecke operator on V is a $\mathbb{K}$-linear operator $R : V \otimes V \longrightarrow V \otimes V$, satisfying the following relations:

$$(R\otimes I)(I\otimes R)(R\otimes I) = (I\otimes R)(R\otimes I)(I\otimes R), \tag{1}$$

$$(R+I)(R-q) = 0, \quad 0 \neq q \in \mathbb{K}. \tag{2}$$

We shall alway assume that q is not a root of unity of order greater than 1.

The name Hecke operator comes from the fact, that an operator, satisfying the above equations, induces a representation of the Hecke algebra $\mathcal{H}_n = \mathcal{H}_{q,n}$ on $V^{\otimes n}$, $\forall n \geq 2$. The reader is referred to [3] for the definition and main properties of Hecke algebras.

A quadratic algebra is a pair (A, V) consisting of a finite dimensional vector space V and a factor algebra A of the tensor algebra $T(V)$ by an ideal, generated by elements of $V \otimes V$. Given a Hecke operator R on $V \otimes V$, one can define the following quadratic algebras.

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1) After the work had been accepted for publication, the author learned that Theorem 3.5 had been also obtained by A. Davydov for q=1 [2].

$$(\Lambda, V) = (\Lambda_R, V) := (T(V)/ < \mathrm{Im}(R+1) >, V) \tag{3}$$

$$(S, V) = (S_R, V) := (T(V)/ < \mathrm{Im}(R-q) >, V) \tag{4}$$

$$(E, V^* \otimes V) = (E_R, V^* \otimes V) := (T(V^* \otimes V)/ < \mathrm{Im}(\overline{R}-1) >, V^* \otimes V), \tag{5}$$

where $\overline{R} = \theta(R^{*-1} \otimes R)\theta$, acting on $V^* \otimes V \otimes V^* \otimes V$, θ denotes the operator, that interchanges the elements in the 2^{nd} and 3^{rd} positions of the tensor products. Fix V and R. we shall refer to these algebras as Λ, S and E, respectively. Λ and S are considered as the "anti-symmetric and symmetric tensor algebras" upon certain quantum space, while E is considered as the algebra of functions on the "semi-group of symmetries of this quantum space", i.e., the matrix quantum semi-group.

Λ, S and E are graded algebras, let λ_i, s_i, e_i be the $\mathbb{K}$-dimension of their i^{th} homogeneous components, respectively. For a quadratic algebra (A, V), its Poincaré series is defined to be $P_A(t) = \sum_{i=0}^{\infty}(\dim_{\mathbb{K}} A_i)t^i$, where A_i is the i^{th}-homogeneous component of A. Thus, for example, $P_\Lambda(t) = \sum_{i=0}^{\infty} \lambda_i t^i$. The Poincaré series of Λ and S satisfies the following relation [7]:

$$P_\Lambda(t)P_S(-t) = 1. \tag{6}$$

E is a bialgebra and Λ, S are (right) E-comodules. In fact E can be defined as an algebra that universally coacts on Λ and S [16]. It then immediately implies that E is a bialgebra. Also, V is an E-comodule and hence $V^{\otimes n}$ is an E-comodule. The coproduct on E satisfies $\Delta(E_n) \subset E_n \otimes E_n$, hence E_n is a coalgebra. We also have $\delta(V^{\otimes n}) \subset V^{\otimes n} \otimes E_n$, i.e., $V^{\otimes n}$ is an E_n-comodule. Therefore E_n^* is an algebra, that acts on $V^{\otimes n}$ (from the left).

On the other hand, as has been mentioned, R induces a representation, say ρ_n, of the Hecke algebra $\mathcal{H}_n$ on $V^{\otimes n}$. We shall write this action from the right. It is now time to state the fact, that plays a crucial role in the study of E.

Theorem 2.1 [8]. *The actions of E_n^* and $\mathcal{H}_n$ on $V^{\otimes n}$ are centralizers of each other in $\mathrm{End}_{\mathbb{K}}(V^{\otimes n})$. In other words, the following isomorphisms* *hold*

$$E_n^* \cong \mathrm{End}_{\mathcal{H}_n} V^{\otimes n}, \tag{7}$$

$$\rho_n(\mathcal{H}_n) \cong \mathrm{End}_{E_n^*} V^{\otimes n}. \tag{8}$$

Since $\mathcal{H}_n$ is semi-simple and hence isomorphic to the direct product of matrix rings over $\mathbb{K}$ (by comparing dimensions), these equations imply the following facts:

(i) E_n^* is semi-simple, hence E_n is cosemi-simple.

(ii) Simple E_n^*-modules are of the form $\mathrm{Im}\rho_n(e)$, where e is a primitive idempotent of $\mathcal{H}_n$. Two primitive idempotents define isomorphic modules iff they belong to the same minimal two-sided ideal.

(iii) There are minimal two-sided ideals A_i of $\mathcal{H}_{nx}$ such that

$$E_n^* \cong \oplus \mathrm{End}_{A_i} V^{\otimes n} A_i, \tag{9}$$

iv) $\mathrm{End}_{A_i}(V^{\otimes n} A_i)$ is a matrix ring of dimension equal to the square of the $\mathbb{K}$-dimension of $\mathrm{Im}\rho_n(e)$ for any primitive idempotent $e \in A_i$.

On the other hand, it is known that minimal two-sided ideals of $\mathcal{H}_n$ can be indexed by partitions of n [4]. Let λ be a partition of n (in notation $\lambda \vdash n$), that means $\lambda = (\lambda_1, \lambda_2, \dots), \lambda_1 \geq \lambda_2 \geq \dots, \sum \lambda_i = n$, and A_λ be the corresponding two-sided minimal ideal, M_λ be the simple module of E_n^* defined by any primitive idempotent from A_λ. Let m_λ denote the dimension of M_λ over $\mathbb{K}$. Then we have, according to (iv) above,

$$e_n = \dim E_n = \dim E_n^* = \sum_{\lambda \vdash n} m_\lambda^2. \tag{10}$$

Notice that by definition $m_\lambda \geq 0$ for all λ. This remark together with Theorem 3.2 implies almost all results of this paper.

Since R satisfies the Yang-Baxter equation, it induces a coquasitriangular structure on E and, therefore, a braiding in the category E-Comod of E-comodules. In particular, the braiding on $V \otimes V$ is precisely R. The reader is referred to [10, 11] for the general theory of braided categories and to [13] for the particular case of E-Comod.

3. Symmetric functions and Pólya frequency sequences

In this section we recall some properties of symmetric functions needed in our context. The reader is referred to the book of Macdonald [15] for an introduction into the theory of symmetric function. We then recall a theorem, due to Edrei [5], that plays the main role in our study.

We study the symmetric functions in a countable set of variables x_1, $x_2, \ldots$. We shall closely follow Macdonald's book [loc.cit.], up to some changes of notation for our convenience. The monomial symmetric functions will be denoted by $\mathbf{k}_\lambda$, $\lambda \in \mathcal{P}$, where $\mathcal{P}$ denotes the set of partitions. The elementary, complete and Schur functions will be denoted by $\boldsymbol{\lambda}_\lambda$, $\mathbf{s}_\lambda$ and $\mathbf{m}_\lambda$, $\lambda \in \mathcal{P}$, respectively. Recall that $\boldsymbol{\lambda}_\lambda = \boldsymbol{\lambda}_{\lambda_1}\boldsymbol{\lambda}_{\lambda_2}\cdots\boldsymbol{\lambda}_{\lambda_n}$, $\mathbf{s}_\lambda = \mathbf{s}_{\lambda_1}\mathbf{s}_{\lambda_2}\cdots\mathbf{s}_{\lambda_n}$, for $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_n)$, where $\boldsymbol{\lambda}_r$ (resp. $\mathbf{s}_r$) is the coefficient of t^r in $\prod_{i=1}^{\infty}(1 + x_i t)$ (resp. $\prod_{i=1}^{\infty}(1 - x_i t)^{-1}$). $\mathbf{m}_\lambda$ is given by the equality (we make use of the convention that $\mathbf{s}_r = \boldsymbol{\lambda}_r = 0$ whenever $r < 0$):

$$\mathbf{m}_\lambda = \det\|\mathbf{s}_{\lambda_i - i + j}\|_{1\le i,j\le l(\lambda)} = \det\|\boldsymbol{\lambda}_{\lambda_i - i + j}\|_{1\le i,j\le l(\lambda')} \tag{11}$$

where $l(\lambda)$ denotes the length of λ, i.e., the cardinal of nonzero parts of λ, λ' denotes the conjugate partition to λ. The set $\mathbf{k}_\lambda$, (resp. $\boldsymbol{\lambda}_\lambda$, $\mathbf{s}_\lambda$, $\mathbf{m}_\lambda$), $\lambda \in \mathcal{P}$ forms a basis for $\boldsymbol{\Lambda}$ – the rings of symmetric functions ([loc.cit.] I.2,3). In particular, we have $\mathbf{m}_\lambda = \sum_\mu K^\mu_\lambda \mathbf{k}_\mu$, the coefficients K^μ_λ, called Kostka numbers, are nonnegative integers ([loc.cit.] I.4,5).

We also make use of skew Schur functions $\mathbf{m}_{\lambda/\mu}$ defined as follows:

$$\mathbf{m}_{\lambda/\mu} := \det \| \mathbf{s}_{\lambda_i - \mu_i - i + j} \|_{1\le i,j\le l(\lambda)} = \det \| \lambda_{\lambda'_i - \mu'_j - i + j} \|_{1\le i,j\le l(\lambda')} . \tag{12}$$

For $\mu = 0$, it becomes $\mathbf{m}_\lambda$. $\mathbf{m}_{\lambda/\mu}$ can be expressed in terms of $\mathbf{m}_\lambda$. Let $c^\gamma_{\lambda\mu}$ be the (integer) numbers such that $\mathbf{m}_\lambda\mathbf{m}_\mu = \sum c^\gamma_{\lambda\mu}\mathbf{m}_\gamma$, then

$$\mathbf{m}_{\lambda/\mu} = \sum c^\lambda_{\mu\gamma}\mathbf{m}_\gamma, \tag{13}$$

where $c^\lambda_{\mu\gamma} = 0$ unless $|\lambda| = |\mu| + |\gamma|$ and $\lambda \supset \mu$ (i.e., $\lambda_i \ge \mu_i, \forall i$). Moreover $c^\lambda_{\mu\gamma}$ are nonnegative ([loc.cit.] I.5).

In the previous section, we have seen that each partition λ defines a simple E-comodule M_λ (which can be zero). If M_λ is zero, we can assume it appears in the decomposition of an E-comodule with an arbitrary multiplicity. In particular, consider the multiplicity of M_γ appearing in the decomposition of $M_\lambda \otimes M_\mu$, if one of M_λ, M_μ, M_γ, equals zero, then the multiplicity can be chosen arbitrarily. We show that this multiplicity can be chosen so that it depends only on λ, μ, and γ.

Lemma 3.1. *Let M_λ, $\lambda \in \mathcal{P}$ be the simple E-comodules, some of them may be zero. Let $M_\lambda \otimes M_\mu = \bigoplus d^\gamma_{\lambda\mu} M_\gamma$. Then $d^\gamma_{\lambda\mu}$ can be chosen to be equal to $c^\gamma_{\lambda\mu}$.*

Proof. Let e_λ (resp. e_μ) be a primitive idempotent of $\mathcal{H}_m$ (resp. $\mathcal{H}_n$) where $m = |\lambda|$ (resp. $n = |\mu|$). Embed $\mathcal{H}_m$ and $\mathcal{H}_n$ into $\mathcal{H}_{m+n}$ in such a way that the generators T_i, $1 \le i \le m-1$, of $\mathcal{H}_m$ are mapped to the generators T_i of $\mathcal{H}_{m+n}$ and the generators T_j, $1 \le j \le n-1$, of $\mathcal{H}_n$ are mapped to the generators T_{j+m} of $\mathcal{H}_{m+n}$. Then $d^\gamma_{\lambda\mu}$ can be chosen to be equal to the cardinal of primitive idempotents belonging to the minimal two-sided ideal indexed by λ of H_{m+n}, that appear in the decomposition of $e_\lambda e_\mu$ into sum of primitive idempotents. Thus $d^\gamma_{\lambda\mu}$ is an integer and does not depend on q, hence we can set $q = 1$. In this case $\mathcal{H}_n = \mathbb{K}[\boldsymbol{\sigma}_n]$, hence $d^\gamma_{\lambda\mu} = c^\gamma_{\lambda\mu}$ (cf. [6], Chapter 3). □

Since $\{\boldsymbol{\lambda}_r | r \ge 0\}$ are algebraically independent, we can assign to them any set of values from $\mathbb{K}$ to obtain a ring homomorphism $\boldsymbol{\Lambda} \longrightarrow \mathbb{K}$. In particular, we can set $\boldsymbol{\lambda}_r = \lambda_r$ – the dimension of Λ_r defined in the previous section. By virtue of equation (6), $\mathbf{s}_r = s_r = \dim S_r$. Furthermore, since the dimension of the tensor-product of two modules is equal to the product of their dimensions, by Lemma 3.1, $\mathbf{m}_\lambda = m_\lambda = \dim M_\lambda$. Thus, by the definition of $\mathbf{m}_\lambda$, we have

$$m_\lambda = \det|s_{\lambda_i - j + i}|_{1 \le i,j \le l(\lambda)} = \det|\lambda_{\lambda'_i - j + i}|_{1 \le i,j \le l(\lambda')}. \tag{14}$$

Since m_λ is the dimension of a vector space, it is nonnegative. We make use of the following result from complex analysis to get information on λ_r.

Theorem 3.2 [5]. *Let $a(z) = \sum_{i=0}^{\infty} a_i z^i$ be a formal series with complex coefficients, $a_0 = 1$. Then the infinite matrix $\| a_{i-j} \|_{0 \le i,j}$ (we use the convention that $a_i = 0$ whenever $i < 0$) is totally positive, that is, all its minor of any degree $r \ge 1$ is nonnegative, if and only if $a(z)$ is of the form*

$$a(z) = e^{\gamma z} \frac{\prod_1^\infty (1 + t_i z)}{\prod_1^\infty (1 - u_i z)}, \tag{15}$$

with $\gamma \ge 0$, $t_i \ge 0$, $u_i \ge 0$ and $\sum (t_i + u_i) \le \infty$. This equation is understood to be an equation of two convergent complex functions in a neighborhood of zero.

The series $a(t) = \sum_{i=0}^{\infty} a_i t_i$ *generates* the sequence $a_0, a_1, \dots$, which is called a Pólya frequency (or P-) sequence if $a(z)$ satisfies the condition of Theorem 3.2.

Lemma 3.3. *Let R be a Hecke operator. Then $\lambda_0, \lambda_1, \dots$, is a P-sequence.*

Proof. Indeed, any minor of $\| \lambda_{i-j} \|_{0 \le i,j}$ has the form: $\det \| \lambda_{\mu_i - \nu_j} \|_{1 \le i,j \le r}$ where μ, ν are two sequences of nonnegative distinct integers $r = \max(l(\mu), l(\nu))$. We have, according to (12),

$$\det \| \lambda_{\mu_i - \nu_j} \|_{1 \le i,j \le r} = \mathbf{m}_{\mu+\delta/\nu+\delta} =: m_{\mu+\delta/\nu+\delta}, \tag{16}$$

where $\delta = (r-1, r-2, \cdots, 0)$. Since $m_{\mu/\nu}$ is a linear combination of m_γ with nonnegative coefficients (Equation (13)), it is itself nonnegative. □

A P-sequence is called PP-sequence if all the minors $\| \lambda_{\mu_i - \nu_j} \|_{1 \le i,j \le r}$, $\nu \subset \mu$, are strictly positive. We have

Lemma 3.4. *Let $a(z)$ be of the form in* (15). *If any of the following conditions is fulfilled:* (1) $\gamma > 0$, (2) $t_i > 0, \forall i$, (3) $u_i > 0$, $\forall i$, *then $a(z)$ is a generating function for a PP-sequence.*

Proof. The lemma follows from the following observations.

1. Let x, y be two countable sets of positive reals, then $\mathbf{k}_\lambda(x,y) \ge \mathbf{k}_\lambda(x)$, $\mathbf{k}_\lambda(x,y) \ge \mathbf{k}_\lambda(y)$.
2. Analogously we have $\mathbf{m}_\lambda(x,y) \ge \mathbf{m}_\lambda(x)$, $\mathbf{m}_\lambda(x,y) \ge \mathbf{m}_\lambda(y)$ ([15], (5.9)).
3. From the first observation, we see that, for $a(z) = \prod_1^\infty (1 + t_i z)$, $t_i > 0$, $\mathbf{k}_\lambda > 0$ for all λ, hence $\mathbf{m}_\lambda > 0$, $\forall \lambda$. Therefore, if $a(z) = \prod_1^\infty (1 - u_i z)^{-1}$, $u_i > 0$ we also have $\mathbf{m}_\lambda > 0$ (using the involution ω [loc.cit.], I.3).
4. From the second observation, we see that if $a(z)$ generates a P-sequence and $b(z)$ generates a PP-sequence then $c = a \cdot b$ generates a PP-sequence.
5. $e^{\gamma z}$ generates a PP-sequence (cf. [loc.cit], I.3, Ex. 5).

Note that 3., 4., 5. imply the assertion of the lemma. □

As a direct consequence of this lemma we have

Theorem 3.5. *Let R be a Hecke operator. Then $P_\Lambda(t)$ is a rational function having negative roots and positive poles.*

Proof. It is sufficient to show that there exists a partition λ, such that M_λ is zero. Were $M_\lambda \neq 0$ for all λ, then ρ_n in (8) were injective for all n, by comparing the dimension we would have a contradiction. □

4. The Quasi-even Case

A Hecke operator R is called quasi-even if the associated quadratic algebra $\Lambda = \Lambda_R$ is finite dimensional, i.e., if $P_\Lambda(t)$ is a polynomial. R is called even Hecke operator (or symmetry) if the dimension of the highest homogeneous component of Λ is equal to 1. The degree of P_Λ is called the rank of R.

Let now R be a quasi-even Hecke operator. Then, according to Theorem 3.5, $P_\Lambda(t)$ has only negative roots

$$P_\Lambda(t) = \prod_{i=1}^{r}(1 + t_i t), t_i \in \mathbb{R}^+. \tag{17}$$

Hence, according to (11), $m_\lambda \neq 0$ if and only if $l(\lambda) > r$. Thus, simple E-comodules can be indexed by $\{\lambda \in \mathcal{P} | l(\lambda) \leq r\}$.

Theorem 4.1. *Let $R = R_q$, $R' = R'_{q'}$ be quasi-even Hecke symmetries over V and V', of ranks r and r', respectively. Let E and E' be the associated bialgebras. Then the categories E-Comod and E'-Comod are equivalent as braided abelian categories if and only if $q = q'$ and $r = r'$.*

Proof. Let $\mathcal{F}$ be an equivalence. Then $\mathcal{F}(V) = V'$, hence $\mathcal{F}(V^{\otimes n}) = V'^{\otimes n}$. According to (8),

$$\mathrm{End}^E(V^{\otimes n}) \cong \mathrm{End}_{E_n^*}(V^{\otimes n}) = \bigoplus_{\lambda \vdash n, l(\lambda) \leq r} A_\lambda,$$

where A_λ denotes the minimal two-sided ideal of $\mathcal{H}_n$, associated to λ. Therefore $q = q'$ and $r = r'$. □

Remark that the dimension of V plays no role here.

Since $\lambda_r \geq 1$, $\prod_{i=1}^{r} t_i \geq 1$, hence $\lambda_1 = \sum_{i=1}^{r} t_i \geq r$. Notice that $\lambda_1 = \dim V$. Thus, we have

Theorem 4.2. *Let R be a quasi-even Hecke operator of rank r over the vector space V. Then $r \leq \dim(V)$.*

Notice that $r = 1$ means $R = qI$, which is not even unless $\dim V = 1$. If $r = \dim(V)$, then $t_i = 1$, $\forall i$, hence $\lambda_r = 1$. Thus we have

Corollary 4.3. *Let R be a quasi-even Hecke operator over V such that the rank of R is equal to the dimension of V. Then R is even and $P_\Lambda(t) = (1+t)^r$. Hence $P_S(t) = (1-t)^{-n}$ and by Equation* (23) $P_E(t) = (1-t)^{-n^2}$.

Remark 4.4. As we have already seen, examples of strictly quasi-even Hecke operators serve the identity operators. On the other hand, it is proved in [9] (and can also be derived from Lemma 3.1), that $\Lambda_r^{\otimes 2}$, where r is the rank of R, is a simple E-comodules, hence the braiding on $\Lambda_r^{\otimes 2}$ is the identity operator. For more examples of strictly quasi-even Hecke operators one can take the Hecke sum (see Section 6) of an identity operator and an even Hecke operator. I conjecture that quasi-even Hecke operators appear in this form.

For an even Hecke operator, we have shown that P_Λ should be a reciprocal polynomial having only negative roots. I conjecture that any reciprocal polynomial having negative roots and leading coefficient equal to 1 should have the form $P_\Lambda(t)$ for some even Hecke operator. This fact, for the case of polynomials of degree 2 and 3, follows from a result due to Gurevich [7].

5. The quasi-odd-even case

If $P_\Lambda(t)$ is not a polynomial R is called quasi-odd-even Hecke operator. We have

$$P_\Lambda(t) = \frac{\prod_1^m (1 + t_i t)}{\prod_1^n (1 - u_j t)}, \tag{18}$$

where $t_i > 0$, $u_j > 0$. (m, n) is called the super rank of R. R is called odd-even Hecke operator if $\prod t_i = \prod u_i = 1$. Not much is known about (quasi-) odd-even Hecke operators.

For expressing dimension of simple E-comodules we use the super Schur (or Hook) functions. Let $x = (x_1, x_2, \cdots, x_m)$ and $y = (y_1, y_2, \cdots, y_n)$ be two sets of variables. The super Schur function $\mathbf{m}_\lambda(x/y)$ is defined as follows:

$$\mathbf{m}_\lambda(x/y) := \sum_{\mu \subset \lambda} \mathbf{s}_\mu(x_1, x_2, \cdots, x_m) \mathbf{s}_{\lambda'/\mu'}(y_1, y_2, \cdots, y_n) \tag{19}$$

This function was introduced by Berele and Regev [1]. Then we have [15] $m_\lambda = \mathbf{m}_\lambda(t/u)$. Hence $m_\lambda \neq 0$ if and only if $\lambda \in \Gamma_{m,n}$ where $\Gamma_{m,n}$ denotes the set of hook-partitions:

$$\lambda \in \Gamma_{m,n} \Longleftrightarrow \lambda_j \leq n \quad \text{for} \quad j \geq m+1. \tag{20}$$

Thus we have

Theorem 5.1. *Let R be a quasi-odd-even Hecke operator then simple E-comodules can be indexed by $\lambda \in \Gamma_{m,n}$. The categories of comodules over the bialgebras E and E', associated with quasi-odd-even Hecke operators R and R' respectively, are equivalent as braided abelian categories if and only if R and R' are defined for the same q and are of the same super rank.*

As in the case of even Hecke operators, we have

Corallary 5.2. *The super rank of a quasi-odd-even Hecke operator and the dimension of V, on which R operates, satisfies the inequality $m+n \leq$* $\dim(V)$*. If the equality takes place then R is odd-even and $P_\Lambda(t) = (1+t)^m(1-t)^{-n}$, hence $P_S(t) = (1+t)^n(1-t)^{-m}$, and by Equation* (23)*, $P_\Lambda(t) = (1+t)^{2mn}(1-t)^{-m^2-n^2}$.*

6. The Poincaré series of E

We now give a relation between the Poincaré series of E and the ones of Λ and S. The formula given here has been proved for the case $q = 1$ by Lyubashenko [14] and q transcendent by the author [8]. Here, using the general theory of symmetric functions, we recover this relation for any q not root of unity. According to (10),

$$e_n = \dim E_n = \sum_{\lambda \vdash n} (m_\lambda)^2.$$

On the other hand, we have the following identity of Schur functions, for two countable set of variables $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$,

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda \in \mathcal{P}} \mathbf{m}_\lambda(x) \mathbf{m}_\lambda(y).$$

Setting $y_i = t x_i$, we have

$$\prod_{i,j} (1 - x_i x_j t)^{-1} = \sum_{\lambda \in \mathcal{P}} \mathbf{m}_\lambda(x)^2 t^{|\lambda|}. \tag{21}$$

The left-hand side of this equation can be considered as a generating function for complete symmetric functions in variables $z_{ij} = x_i x_j$. Setting $\mathbf{m}_\lambda = m_\lambda$ then the right-hand side of (22) is equal to $P_E(t)$, by (10). Thus we derive from (22)

$$P_E(t) = P_S(t) \star P_S(t), \tag{22}$$

where $\star$ denotes the multiplication on the λ-ring $\mathbb{K}_0[[t]]$ of formal power series with first coefficient equal to 1 [12]. If $P(t) = \prod(1 + x_i t)$, $Q(t) = \prod(1 + y_i t)$ then $P \star Q(t) := \prod(1 + x_i y_j t)$. We shall refer to $\star$ as the λ-product. Recall that the $\mathbb{K}_0[[t]]$ is a ring with the above λ-product and the λ-addition is the usual product of power series [loc.cit.].

Let R_1, R_2 be Hecke operators on V_1, V_2, their Hecke sum R acts on $V_1 \oplus V_2$ as follows, $R|V_i \otimes V_i = R_i$, $R(v_1 \otimes v_2) = qv_2 \otimes v_1$, $R(v_2 \otimes v_1) = v_1 \otimes v_2 - (q-1)v_2 \otimes v_1$, for $v_i \in V_i$. Then R is a Hecke operator, too. Let Λ_R, S_R be the associated quadratic algebras, then it is not difficult to check that

$$P_{\Lambda_R}(t) = P_{\Lambda_{R_1}}(t) P_{\Lambda_{R_1}}(t),$$
$$P_{S_R}(t) = P_{S_{R_1}}(t) P_{S_{R_1}}(t).$$

Thus, the Hecke sum gives rise to the λ-addition on $\mathbb{K}_0[[t]]$.

Let $P(t) := \dfrac{d}{dt}\ln(P_S(t)) = P'_S(t) P_S(t)^{-1}$. $P(t)$ is the generating function for the power sums ([15], I.3). The operator $\dfrac{d}{dt}$ maps $\mathbb{K}_0[[t]]$ bijectively to $\mathbb{K}[[t]]$ – the ring of power series, the inverse operator is $\int_0^t$. The λ-addition is mapped to the ordinary addition while the λ-multiplication is mapped to the component-wise product (denoted by $*$). Indeed, let $\mathbf{p}_r(x)$ (resp. $\mathbf{p}_r(y)$) be the r^{th} power sum of $x = (x_1, x_2, \dots)$ (resp. $y = (y_1, y_2, \dots)$), then $\mathbf{p}_r(xy) = \mathbf{p}_r(x)\mathbf{p}_r(y)$, where $xy := \{x_i y_j | 1 \le i, j\}$. Thus we have (cf. [8]),

$$P_E(t) = \exp \int_0^t P(u)^{*2}. \tag{23}$$

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