ACTA MATHEMATICA VIETNAMICA
Volume 24, Number 1, 1999, pp. 109–115

A REMARK ON VITUSHKIN'S COVERING

NGUYEN VAN CHAU

ABSTRACT. It is shown that every polynomial map of the complex plane $\mathbb{C}^2$ with exceptional value set homeomorphic to the complex line $\mathbb{C}$ must have a singularity. This implies the nonexistence of Vitushkin's covering in the polynomial case.

1. INTRODUCTION

In [V1] A. G. Vitushkin constructed an example of a real 4-dimensional manifold X with a 2-dimensional submanifold M and a branching covering $F : X \longrightarrow \mathbb{R}^4$ branched only along M such that $X - M$ is homeomorphic to $\mathbb{R}^4$, where M is homeomorphic to $\mathbb{R}^2$ and the restriction of F on M is an embedding. S. Y. Orevkov in [O3] realized Vitushkin's covering as a complex analytic mapping from a Stein manifold onto a ball in the complex plane $\mathbb{C}^2$ (see also [O2]). Such examples are very important for a better understanding of the geometrical nature of the famous Jacobian Conjecture, which was posed by O. H. Keller [K] in 1939 and which is still open even in the 2-dimensional case. This conjecture asserts that *every polynomial map of the complex affine space* $\mathbb{C}^n$ *with non-zero constant Jacobian must be bijective*. (See [BCW] for a nice survey on this conjecture).

The aim of this short paper is to show that a kind of Vitushkin's covering does not exist in the polynomial case. We will prove the following

Theorem 1.1. *Let f be a polynomial map of $\mathbb{C}^2$ with nonzero-constant Jacobian. Assume that there exists a curve Γ homeomorphic to the complex line $\mathbb{C}$ such that the map $f : \mathbb{C}^2 - f^{-1}(\Gamma) \longrightarrow \mathbb{C}^2 - \Gamma$ is an unbranched covering. Then f is bijective.*

This theorem shows that a counterexample to the Jacobian Conjecture, if exists, should have a structure which is more complicated than Vitushkin's covering.

Received December 12, 1997
1991 Mathematics subject classification. Primary 14H09.
Key words and phrases. Jacobian conjecture, exceptional value set, polynomial automorphism.

In fact, for each polynomial map f of $\mathbb{C}^2$ with finite fibers the standard results on the resolution of singularity yield the existence of a compact algebraic variety X containing $\mathbb{C}^2$ as an open Zaziski subset and a regular extension F of f from X onto $\mathbb{C}\,\mathbb{P}^1 \times \mathbb{C}\,\mathbb{P}^1$ (see for example [V1] and [O1]). The set $D := X - \mathbb{C}^2$ is an algebraic curve in X, every irreducible component of which is isomorphic to the line $\mathbb{C}\,\mathbb{P}^1$. Each pair of these components either has no common point or transversally intersects at an unique common point. But every triple of them never have a common point. Removing from X the inverse image $F^{-1}(\{\infty\} \times \mathbb{C}\,\mathbb{P}^1)$ and $F^{-1}(\mathbb{C}\,\mathbb{P}^1 \times \{\infty\})$ one obtains the complex manifold X^* containing $\mathbb{C}^2$ and the curve $M := X^* - \mathbb{C}^2$. The restriction of F on X^* is a proper map onto $\mathbb{C}^2$. Consider the case when f is not a proper map. In this case, M is not empty. Some irreducible components of the curve M are isomorphic to the complex line $\mathbb{C}$ and restrictions of F on them are not constant. The remained irreducible components are isomorphic to $\mathbb{C}\,\mathbb{P}^1$, on which F is constant. Therefore, the simplest configuration of the extension of f may be the case when F branches only along M, M is isomorphic to the line $\mathbb{C}$ and the restriction of F on M is an embedding into $\mathbb{C}^2$. This is the situation for which Vitushkin's covering is an analytical realization.

Note that the image $B_f := F(M)$ is a subset of the *exceptional value set* E_f - the set of all values $a \in \mathbb{C}^2$ for which the number of solutions of the equation $f(x, y) = a$, $(x, y) \in \mathbb{C}^2$, not counted with multiplity, is smaller than the geometrical degree of f. Here, by the geometrical degree of f we simply mean the number of solutions of the equation $f(x, y) = a$ for generic values $a \in \mathbb{C}^2$. The set E_f is an algebraic curve in $\mathbb{C}^2$ composed of the curve B_f and the critical value set of f . The mapping $f : \mathbb{C}^2 - f^{-1}(E_f) \longrightarrow \mathbb{C}^2 - E_f$ is an unbranched covering. So, B_f is just the set of all values $a \in \mathbb{C}^2$ at which the number of solutions of the equation $f(x, y) = a$, counted with multiplity, is not constant. In Vitushkin's analytical covering the image B_f is diffeomorphic to $\mathbb{R}^2$. Using these notions, our result can be reformulated as follows:

Every polynomial map of $\mathbb{C}^2$ *with the exceptional value set* E_f *homeomorphic to the complex line* $\mathbb{C}$ *must have a singularity.*

The proof of Theorem 1.1 will use Lins and Zaidenberg's theorem [LZ], a generalization of Abhyankar-Moh-Suzuki's theorem [AM] on the embeddings of the line into the plane, and Orevkov's estimation on geometrical degree of f in term of the regular extension of f [O1].

2. Proof of Theorem 1.1

Consider a polynomial map f of $\mathbb{C}^2$ with finite fibres, $f = (P, Q)$, where

P, $Q \in \mathbb{C}[x, y]$. Let $J(f) := P_x Q_y - P_y Q_x$ be the Jacobian of f and E_f the exceptional value set of f. Let $F : X \longrightarrow \mathbb{C}\,\mathbb{P}^1 \times \mathbb{C}\,\mathbb{P}^1$ be a regular extension of f and $D := X - \mathbb{C}^2$. Denote by D_{fc} the curve composed of all components l of D such that $F(l) \cap \mathbb{C}^2$ is not empty, by D_f the curve composed of all components $l \subset D_{fc}$ the restriction of F on which is not constant, by D_c the closure of $D_{fc} - D_f$ in X, and by D_∞ the curve composed of all components not belong to D_{fc}.

Lemma 2.1. *Suppose L is a line in $\mathbb{C}^2$ such that L intersects E_f at an unique point. Then, every irreducible component of the curve $f^{-1}(L)$ is homomorphic to one of $\mathbb{C}$ and $\mathbb{C}^* := \mathbb{C} - \{0\}$. Furthermore, if $f^{-1}(L \cap E_f) \neq \emptyset$, then there is at least an irreducible component of $f^{-1}(L)$ homomorphic to $\mathbb{C}$.*

Proof. Let a be the unique common point of L and E_f and V be an irreducible component of the curve $f^{-1}(L)$. Let r be the number of the irreducible branches of V located at a point of $F^{-1}(a)$ and r^∞ the number of the irreducible branches of V located at a point of D_∞. Since $L \cap E_f = \{a\}$, $V - f^{-1}(a)$ is smooth and can be viewed as a punctured Riemann surface of genus g with exact $r + r^\infty$ of punctures. The restriction of f on V determines a n-fold unbranched covering from $V - f^{-1}(a)$ onto $L - \{a\}$ with degree n not larger than the geometrical degree of f. In particular, the number r and r^∞ are always positive. Let $\chi(V - f^{-1}(a))$ and $\chi(L - \{a\})$ be the Euler-Pointcare characteristic of $V - f^{-1}(a)$ and $L - \{a\}$, respectively. Then, by Riemann-Hurwitz's relation

$$2 - 2g - r - r^\infty = \chi(V - f^{-1}(a)) = n\chi(L - \{a\}) = 0.$$

It follows that $g = 0$ and $r = r^\infty = 1$. Hence, there is only an irreducible branch in V such that it locates at a point z_a of $F^{-1}(a)$. Obviously, V is homeomorphic to $\mathbb{C}^*$ (and $\mathbb{C}$) if $z_a \in D_f$ (res. $z_a \in \mathbb{C}^2$).

The above observation also shows that the number of irreducible components of $f^{-1}(L)$ homeomorphic to $\mathbb{C}$ is equal to the number of irreducible branches of $f^{-1}(L)$ located at $f^{-1}(L \cap E_f)$. Thus, if $f^{-1}(L \cap E_f) \neq \emptyset$, then there is at least an irreducible component of $f^{-1}(L)$ homeomorphic to $\mathbb{C}$. □

Lemma 2.2. *Assume that $J(f) \in \mathbb{C}^*$ and there exists a line $L \subset \mathbb{C}^2$ such that an irreducible components of the curve $f^{-1}(L)$ is diffeomorphic to the line $\mathbb{C}$. Then f is bijective.*

Proof. Let V be an irreducible component of $f^{-1}(L)$ and assume that

V is diffeomorphic to $\mathbb{C}$. By Abhyankar-Moh-Suzuki's theorem [AM] on the embedding of the line into the plane the curve V is isomorphic to $\mathbb{C}$. Hence, we can choose a suitable affine coordinate (x, y) in $\mathbb{C}^2$ so that $f(x, y) = (P(x, y), Q(x, y))$ and the line $\{x = 0\}$ is an irreducible component of the curver $P = 0$. Since $J(f) \in \mathbb{C}^*$, we have that $P(x, y) = xP^*(x, y)$ and $Q(x, y) = a + bx + cy +$ *higher terms*, where $P^*(0, y) \neq 0$ and $c \neq 0$.

Observe that f is bijective if $\deg P = 1$ or $\deg Q = 1$. Then we need only to consider the case $\deg Q > 1$. For this case we will show that P^* is a non-zero constant. Then, $\deg P = 1$ and, of couse, f is bijective. Assume for the contrary that P^* is not constant. Recall that the Newton's diagram Γ_g of a polynomial $g(x, y) = \sum a_{mn} x^m y^n$ is the convex hull of the set $\{(m, n) : a_{mn} \neq 0\} \cup \{(0, 0)\}$. Let Γ_P and Γ_Q be the Newton's diagrams of P and Q, respectively. According to a result of Nakai and Baba [NB] (see also [AO]), the condition $J(f) \in \mathbb{C}^*$ ensures that the convex sets Γ_P and Γ_Q are similar, i.e. $\deg Q.\Gamma_P = \deg P.\Gamma_Q$. Drawing the diagrams Γ_P and Γ_Q one can see that Γ_P has an edge connecting the vertice $(0, 0)$ to another vertice in the cone $(1, 0) + \mathbb{R}^2_+$ and Γ_Q has an edge connecting the vertices $(0, 0)$ and $(0, 1)$. This implies that Γ_P and Γ_Q can not be similar which contradicts the previous assumption. $\square$.

Consider the regular extension F of f and the associated curves D, D_∞, D_{fc}, D_f and D_c. The followings facts are due to S. Yu. Ozevkov (see the Lemmas 2.2, 3.1, 4.2 and 5.2 in [O1]):

(i) We can construct a regular extension $F : X \longrightarrow \mathbb{C}\,\mathbb{P}^1 \times \mathbb{C}\,\mathbb{P}^1$ such that each connected component K of the curve D_{fc} is composed of an irreducible component $l := l_0$ of D_f and a finite number of irreducible components l_i of D_c, $i = 1, 2, ..., k$, for which $l \cap D_\infty = \{*\}$ and

$$l_i \cap l_j = \begin{cases} \{*\} & \text{for } |i - j| = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

For convenience, we denote $l \cap l_1 = \{e_l\}$. The curve D_∞ can be reduced to one point, denoted by ∞. For each connected component K of the curve D_{fc} the curve $D_c \cap K$ can be reduced to the corresponding point e_l. By this procedure of reduction we obtain a compact manifold X^* and a continuous extension $F^* : X^* \longrightarrow \mathbb{C}^2 \cup \{\infty\}$ of f which is analytical everywhere except at most at the points ∞ and e_l.

(ii) Let $\pi : X \longrightarrow X^*$ be the natural projection. Then, for each $l \subset D_f$ the local degree $\deg_x F^*$ of F^* at x is a constant $\mu_l F^*$ for almost $x \in \pi(l)$, except at most at the point e_l and a finite number of exceptional points. These exceptional points are either singular points of the curve

$\mathrm{Det}DF^* = 0$ or the points at which the restriction of F^* on $\pi(l)$ is not a local embedding.

Lemma 2.3 ([O1, Lemma 4.2]). *If $J(f) \in C^*$, then*

$$\deg F^* - 1 = \sum_{l \subset D_f} \Big(\mu_l F^* - \sum_{x \in \pi(l) - \{\infty\}} (\mu_l F^* - \deg_x F^*) \Big),$$

where $\deg F^*$ denotes the geometrical degree of the proper map F^*.

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let f and Γ be as in the statement of the theorem. Assume to the contrary that f is not bijective. Then, by definition, Γ is just the exceptional value set E_f of f, $\Gamma = E_f$. Since Γ is homeomorphic to the line $\mathbb{C}$, by Lins and Zaidenberg's theorem [LZ] the curve Γ is isomorphic to a quasi-homogenous irreducible curve given by a parameterization $\xi \to (\xi^n, \xi^m)$ with $\gcd(m, n) = 1$. Thus, changing affine coordinates in $\mathbb{C}^2$ we can assume that the curve E_f is a quasi-homogenous irreducible curve. It follows that there always exists a line L such that L intersects E_f at an unique point. Let L be such a line and $L \cap E_f = \{a\}$.

If $f^{-1}(a) \neq \emptyset$, by Lemma 2.1 the curve $f^{-1}(L)$ has an irreducible component diffeomorphic to $\mathbb{C}$. Therefore, by Lemma 2.2 we get a contradiction. Thus, the proof is complete if we can show that

$$f^{-1}(a) \neq \emptyset.$$

Consider the extension $F^* : X^* \longrightarrow \mathbb{C}^2 \cup \{\infty\}$ of f. By definition, $f^{-1}(a) \neq \emptyset$ if and only if

$$\sum_{l \subset D_f} \sum_{x \in \pi(l), F^*(x) = a} \deg_x F^* \leq \deg F^* - 1.$$

By the equality in Lemma 2.3, the preceding inequality holds if

$$\mu_l(a) := \sum_{x \in \pi(l), F^*(x) = a} \deg_x F^* \leq \mu_l F^* + \sum_{x \in \pi(l) - \{\infty\}} (\deg_x F^* - \mu_l F^*)$$

for every irreducible component $l \subset D_f$.

We shall prove now the last inequality.

Let l be an irreducible component of the curve D_f, $l \subset D_f$. Note that by the assumption of the theorem the curve E_f is just the image $F^*\pi(D_f)$

and has only one singular point a. Therefore, as shown in (ii), the degree $\deg_x F^*$ is locally constant on $\pi(l)$ except at most at the point e_l and the points in $F^{*-1}(a) \cap \pi(l)$. Furthermore, since l is homeomorphic to $\mathbb{C}\,\mathbb{P}^1$, $\pi(l) - \{\infty\}$ is homeomorphic to $\mathbb{C}$ and the restriction F_l^* of F^* from $\pi(l) - \{\infty\}$ onto E_f is a proper map of a finite degree. Denote by d_l the degree of the map F_l^*.

Case $e_l \in F^{*-1}(a)$: In this case, the mapping $F_l^* : \pi(l) - \{\infty\} - F^{*-1}(a) \longrightarrow E_f - \{a\}$ is a d_l-fold unbranched covering. This implies that $F^{*-1}(a) \cap \pi(l)$ consists of an unique point, denoted by a_l. Then we obtain

$$\mu_l(a) = \deg_{a_l} F^* = \mu_l F^* + \sum_{x \in \pi(l) - \{\infty\}} (def_x F^* - \mu_l F^*).$$

Case $e_l \notin F^{*-1}(a)$: For convenience, denote $e := F^*(e_l)$. Considering the d_l-fold unbranched covering $F_l^* : \pi(l) - \{\infty\} - F^{*-1}(\{a; e\}) \longrightarrow E_f - \{a; e\}$, we can see that

$$d_l = \# F_l^{*-1}(a) + \# F_l^{*-1}(e) - 1.$$

On the other hand, since $e \neq a$, the curve E_f is smooth at e. Therefore, as shown in (ii), at each point $x \in F_l^{*-1}(e) - \{e_l\}$ the mapping F_l^* is a local embedding and $\deg_x F^* = \mu_l F^*$. This implies

$$d_l = \deg_{e_l} F_l^* + \# F_l^{*-1}(e) - 1.$$

Hence, we have that

$$\# F^{*-1}(a) = \deg_{e_l} F_l^*. \tag{1}$$

The following estimation on the degree of F^* at the point e_l can be easily verified:

$$\deg_{e_l} F_l^* . \mu_l F^* \leq \deg_{e_l} F^*. \tag{2}$$

Using (1) and (2) we obtain

$$\begin{aligned}
\mu_l F^* + &\sum_{x \in \pi(l) - \{\infty\}} (\deg_x F^* - \mu_l F^*) \\
&= \mu_l F^* + \sum_{x \in \pi(l), F^*(x) = a} (\deg_x F^* - \mu_l F^*) + \deg_{e_l} F^* - \mu_l F^* \\
&= \mu_l F^* + \mu_l(a) - \# F_l^{*-1}(a) . \mu_l F^* + \deg_{e_l} F^* - \mu_l F^* \\
&\geq \mu_l F^* + \mu_l(a) - \# F_l^{*-1}(a) . \mu_l F^* + \deg_{e_l} F_l^* . \mu_l F^* - \mu_l F^* \\
&\geq \mu_l(a).
\end{aligned}$$

This concludes the proof.

Acknowledgements

This paper was written when the author visited IMPA, Rio de Janeiro, Brazil, by a grant IMPA and TWAS. The author is grateful to these institutes for support and hospitality. The author would like to express his thanks to Professor J. Palis and Professor C. Gutierrez for their helps.

References

[AM] S. S. Abhyankar and T. T. Moh, *Embedding of the line in the plane*, J. Reine Angrew. Math. **276** (1975), 149-166.

[AO] H. Appelgate and H. Onishi, *The Jacobian conjecture in two variables*, J. Pure Appl. Algebra **37** (1985), 215-227.

[BCW] H. Bass, E. H. Connell and D. W. Right, *The Jacobian Conjecture: redution of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. **7** (1982), 287-330.

[LZ] V. Y. Lin and M. G. Zaidenberg, *An irreducible, simply connected algebraic curve in* $\mathbb{C}^2$ *is equivalent to a quasihomogeneous curve* (Russian), Dokl. Akad. Nauk SSSR **271** (1983), 1048–1052.

[NB] Y. Nakai and K. Baba, *A generalization of Magnu's theorem*, Osaka J. Math. **14** (1977), 403-409.

[O1] S. Y. Orevkov, *On three-sheeted polynomial mappings of* $\mathbb{C}^2$ (Russian), Izvestiya. Akad. Nauk SSSR Ser. Mat. **50** (1986), 1231–1240.

[O2] S. Y. Orevkov, *An example connection with the Jacobian Conjecture*, Mat. Zametki **47** (1990), 127-136.

[O3] S. Yu. Orevkov, *Rudolph diagram and analytical realization of Vitushkin's covering*, Mat. Zametki **60** (1996), 206-224.

[V1] A. G. Vitushkin, *Certain examples in connection with the problems about polynomial transformations of* $\mathbb{C}^n$, Izvestiya Akad. Nauk SSSR, Ser. Mat. **35** (1971), 269-279.

[V2] A. G. Vitushkin, *On polynomial transformations of* $\mathbb{C}^n$, Manifolds-Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973), pp. 415–417. Univ. Tokyo Press, Tokyo, 1975.

Institute of Mathematics Hanoi
P. O. Box 631, Boho 10000, Hanoi, Vietnam
E-mail: nvchau@ioit.ncst.ac.vn