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A UNIQUE RANGE SET OF P-ADIC MEROMORPHIC FUNCTIONS WITH 10 ELEMENTS

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Abstract. In this paper, we will exhibit a unique range set for p-adic meromorphic functions with 10 elements.

1. Introduction

Nevanlinna theory is so beautiful that one would naturally be interested in determining how such a theory would look in the p-adic case. H. H. Khoai [7], H. H. Khoai and M. V. Quang [9], and A. Boutabaa [1] proved p-adic analogues of two "main theorems" and defect relations of Nevanlinna theory. H. H. Khoai [8] and W. Cherry and Zh. Ye [2] began to study several variable p-adic Nevanlinna theory, and proved the defect relation of hyperplanes in general position. Hu and Yang [6] proved p-adic analogues of the defect relation for moving targets and the second main theorem for differential polynomials.

For a non-constant meromorphic function f on $\mathbb{C}$ and a set $S \subset \mathbb{C}\cup\{\infty\}$ we define

$$E_f(S) = \bigcup_{a\in S}\{mz \mid f(z) = a \text{ with multiplicity } m\}.$$

A set $S \subset \mathbb{C}\cup\{\infty\}$ is called an *unique range set for meromorphic functions* (URSM) if for any pair of non-constant meromorphic functions f and g on $\mathbb{C}$, the condition $E_f(S) = E_g(S)$ implies $f = g$. A set $S \subset \mathbb{C} \cup \{\infty\}$ is called an *unique range set for entire functions* (URSE) if for any pair of non-constant entire functions f and g on $\mathbb{C}$, the condition $E_f(S) = E_g(S)$ implies $f = g$. Gross and Yang [4] showed that the set

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$$S = \{z \in \mathbb{C} \mid z + e^z = 0\}$$

is a URSE. Recently, URSE and also URSM with finitely many elements have been found by Yi [13, 14], Li and Yang [10, 11], Mues and Reinders [12], Frank and Reinders [3], Hu and Yang [5]. Li and Yang [10] introduced the notation

$$\lambda_M = \inf\{\#S \mid S \text{ is a URSM }\},$$
$$\lambda_E = \inf\{\#S \mid S \text{ is a URSE }\},$$

where $\#S$ is the cardinality of the set S. The best lower and upper bounds known so far are

$$5 \le \lambda_E \le 7, \quad 6 \le \lambda_M \le 11.$$

For p-adic meromorphic (or entire) function f on $\mathbb{C}_p$, we can similarly define $E_f(S)$ for a set $S \subset \mathbb{C}_p \cup \{\infty\}$, and introduce the notation λ_M and λ_E. In [6] we obtained $\lambda_E \le 4$ for p-adic entire functions and $\lambda_M \le 12$ for p-adic meromorphic functions. W. Cherry ask us whether the Frank-Reinders' method gives a p-adic URSM with 10 elements by using the $-\log r$ term in their second main theorem. In this paper, we will give a confirmed answer to Cherry's question, i.e., $\lambda_M \le 10$ for p-adic meromorphic functions.

2. Nevanlinna theory of p-adic meromorphic functions

Let p be a prime number, let $\mathbf{Q}_p$ be the field of p-adic numbers, and let $\mathbb{C}_p$ be the p-adic completion of the algebraic closure of $\mathbf{Q}_p$. The absolute value $|\ |_p$ in $\mathbb{C}_p$ is normalized so that $|\ p\ |_p = p^{-1}$. We further use the notion ord_p for the additive valuation on $\mathbb{C}_p$.

Recall that in a metric space whose metric comes from a Non-Archimedean norm, a sequence is Cauchy if and only if the difference between adjacent terms approaches zero; and if the metric space is complete, an infinite sum converges if and only if its general term approaches zero. So if we consider expressions of the form

$$f(Z) = \sum_{n=0}^{\infty} a_n Z^n, \quad (a_n \in \mathbb{C}_p),$$

we can give a value $\sum\limits_{n=0}^{\infty} a_n z^n$ to $f(z)$ whenever an z substituted for Z for which

$$|\ a_n z^n\ |_p \to 0.$$

Define the "radius ρ of convergence" by

$$\frac{1}{\rho} = \lim_{n\to\infty} \sup \mid a_n \mid_p^{\frac{1}{n}} .$$

Then the series converges if $\mid z \mid_p < \rho$ and diverges if $\mid z \mid_p > \rho$. Also the function $f(z)$ is said to be p-adic analytic on $B(\rho)$, where

$$B(\rho) = \{z \in \mathbb{C}_p \mid \mid z \mid_p < \rho\}.$$

If $\rho = \infty$, the function $f(z)$ also is said to be p-adic entire on $\mathbb{C}_p$.

Consider non-constant p-adic analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (a_n \in \mathbb{C}_p)$$

on $B(\rho)$ $(0 < \rho \leq \infty)$. The essence of the Wiman-Valiron method is the analysis of the behaviour of the function by means of the *maximum term*:

$$\mu(r, f) = \max_{n\geq 0} \mid a_n \mid_p r^n \quad (0 < r = \mid z \mid_p < \rho)$$

together with the *central index*:

$$\nu(r, f) = \max_{n\geq 0}\{n \mid \mid a_n \mid_p r^n = \mu(r, f)\}.$$

Define

$$\nu(0, f) = \lim_{r\to 0} \nu(r, f).$$

Lemma 2.1 ([6]). *The central index $\nu(r, f)$ increases as $r \to \rho$, and satisfies the formula:*

$$\log\mu(r, f) = \log \mid a_{\nu(0,f)} \mid_p + \int_0^r \frac{\nu(t, f) - \nu(0, f)}{t} dt + \nu(0, f)\log r. \quad (0 < r < \rho)$$

The following technical lemma can be found in [2]:

Lemma 2.2 (Weierstrass Preparation Theorem). *There exists an unique monic polynomial P of degree $\nu(r, f)$ and a p-adic analytic function g on $B[r]$ such that $f = gP$, where*

$$B[r] = \{z \in \mathbb{C}_p \mid \mid z \mid_p \leq r\}.$$

Furthermore, g does not have any zero inside $B[r]$, and P has exactly $\nu(r, f)$ zeros, counting multiplicity, on $B[r]$.

Let $n(r, \frac{1}{f})$ denote the number of zeros (couting multiplicity) of f with absolute value $\leq r$ and define the *valence function* of f for 0 by

$$N\big(r, \frac{1}{f}\big) = \int_0^r \frac{n\big(t, \frac{1}{f}\big) - n\big(0, \frac{1}{f}\big)}{t} dt + n\big(0, \frac{1}{f}\big) \log r \quad (0 < r < \rho).$$

Lemma 2.2 shows that

$$n\Big(r, \frac{1}{f}\Big) = \nu(r, f).$$

Then Lemma 2.1 imply the *Jensen formula*:

$$N(r, \frac{1}{f}) = \log \mu(r, f) - \log \mid a_{n(0, \frac{1}{f})} \mid_p . \tag{1}$$

We also denote the number of distinct zeros of f on $B[r]$ by $\overline{n}\big(r, \dfrac{1}{f}\big)$ and define

$$\overline{N}\Big(r, \frac{1}{f}\Big) = \int_0^r \frac{\overline{n}\big(t, \frac{1}{f}\big) - \overline{n}\big(0, \frac{1}{f}\big)}{t} dt + \overline{n}\Big(0, \frac{1}{f}\Big) \log r \quad (0 < r < \rho).$$

For each n we draw the graph $\gamma_n(t)$ which depicts $ord_p(a_n z^n)$ as a function of $t = ord_p(z)$. Then $\gamma_n(t)$ is a straight line with slope n. Let $\gamma(t, f)$ denote the boundary of the intersection of all of the half-planes lying under the lines $\gamma_n(t)$. This line is what we call the Newton polygon of the function $f(z)$ (see [9]). The points t at which $\gamma(t, f)$ has vertices are called the *critical points* of $f(z)$. A finite segment $[\alpha, \beta]$ contains only finitely many critical points. It is clear that if t is a critical point, then $ord_p(a_n) + nt$ attains its minimum at least at two values of n. Obviously, we have

$$\mu(r, f) = p^{-\gamma(t, f)},$$

where $r = p^{-t}$. A basic property of the Newton polygon is that, if $t = ord_p(z)$ is not a critical point, then

$$\mid f(z) \mid_p = p^{-\gamma(t, f)},$$

which implies

$$| f(z) |_p = \mu(r, f).$$

Further, we note that if h is another p-adic analytic function on $B(\rho)$, then

$$\mu(r, fh) = \mu(r, f)\mu(r, h). \tag{2}$$

By a *meromorphic function* f on $B(\rho)$ we will mean the quotient $\dfrac{g}{h}$ of two p-adic analytic functions g and h such that g and h have not any common factors in the ring of p-adic analytic functions on $B(\rho)$. Note that (2) hold and that greatest common divisors of any two p-adic analytic functions exist. We can uniquely extend μ to meromorphic function $f = \dfrac{g}{h}$ by defining

$$\mu(r, f) = \frac{\mu(r, g)}{\mu(r, h)}.$$

Also set

$$\gamma(t, f) = \gamma(t, g) - \gamma(t, h).$$

It is clear that, if $t = ord_p(z)$ is not a critical point for $f(z)$, i.e., t is not a critical point for either $g(z)$ or $h(z)$, then

$$| f(z) |_p = p^{-\gamma(t,f)} = \mu(r, f).$$

Define the *counting function* $n(r, f)$ and the *valence function* $N(r, f)$ of f for poles respectively by

$$n(r, f) = n\Big(r, \frac{1}{h}\Big), \quad N(r, f) = N\Big(r, \frac{1}{h}\Big).$$

Then applying (1) for g and h, we obtain the *Jensen formula*:

$$N\Big(r, \frac{1}{f}\Big) - N(r, f) = \log \mu(r, f) - C_f, \tag{3}$$

where C_f is a constant depending only on f. Define

$$m(r, f) = \log^+ \mu(r, f) = \max\{0, \log \mu(r, f)\}.$$

Finally, we define the *characteristic function*:

$$T(r, f) = m(r, f) + N(r, f).$$

Here we exhibit some basic facts which will be used in the following sections.

Lemma 2.3 (First Main Theorem, cf. [1, 9]). *Let f be a non-constant meromorphic function in $B(\rho)$. Then for every $a \in \mathbb{C}_p$ we have*

$$m\Big(r, \frac{1}{f-a}\Big) + N\Big(r, \frac{1}{f-a}\Big) = T(r, f) + O(1) \quad (r \to \rho).$$

Lemma 2.4 (The Lemma of Logarithmic Derivative, cf. [1, 2, 9]). *Let f be a nonconstant meromorphic function in $B(\rho)$. Then*

$$m\Big(r, \frac{f'}{f}\Big) = O(1) \quad (r \to \rho).$$

Lemma 2.5 (Second Main Theorem, cf. [1, 2, 9]) *Let f be a non-constant meromorphic function in $B(\rho)$ and let $a_1, ..., a_q$ be distinct numbers of $\mathbb{C}_p$. Then*

$$(q-1)T(r, f) \leq N(r, f) + \sum_{j=1}^{q} N\Big(r, \frac{1}{f-a_j}\Big) - N_1(r, f) - \log r + O(1),$$

where

$$N_1(r, f) = 2N(r, f) - N(r, f') + N\Big(r, \frac{1}{f'}\Big).$$

Furthermore, we have

$$N(r, f) + \sum_{j=1}^{q} N\Big(r, \frac{1}{f-a_j}\Big) - N_1(r, f) \leq \overline{N}(r, f) + \sum_{j=1}^{q} \overline{N}\Big(r, \frac{1}{f-a_j}\Big) - N_0\Big(r, \frac{1}{f'}\Big),$$

$$\sum_{a \in \mathbb{C}_p \cup \{\infty\}} \Theta_f(a) \leq 2,$$

where $N_0\Big(r, \frac{1}{f'}\Big)$ is the valence function of the zeros of f' where f does not take one of the values $a_1, ..., a_q$, and where

$$\Theta_f(a) = 1 - \lim_{r \to \infty} \sup \frac{\overline{N}\Big(r, \frac{1}{f-a}\Big)}{T(r, f)}.$$

3. Uniqueness of p-adic meromorphic functions

We recall the following useful facts:

Lemma 3.1 ([2]). *If f is a p-adic entire function on $\mathbb{C}_p$ that is never zero, then f is constant.*

Lemma 3.2 ([6]). *Let f be a non-constant p-adic meromorphic functions on $\mathbb{C}_p$. Take a positive integer n, $\{a_0, a_1, ..., a_n\} \subset \mathbb{C}_p$ with $a_0 \neq 0$ and set*

$$L[f] = a_0 f^n + a_1 f^{n-1} + \cdots + a_n.$$

Then

$$T(r, L[f]) = nT(r, f) + O(1).$$

Theorem 3.1. *Take integer $n \geq 10$ and let $b \in \mathbb{C}_p - \{0, -1\}$. Then the polynomial $P(z)$ defined by*

$$P(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2) z^{n-1} + \frac{n(n-1)}{2} z^{n-2} + b$$

has only simple zeros, and if f and g are non-constant p-adic meromorphic functions on $\mathbb{C}_p$ such that $E_f(S) = E_g(S)$, then $f \equiv g$, where

$$S = \{z \in \mathbb{C}_p \mid P(z) = 0\}.$$

Proof. Write $S = \{r_1, r_2, ..., r_n\}$ and define

$$Q(z) = \frac{(n-1)(n-2)}{2} z^2 - n(n-2) z + \frac{n(n-1)}{2}.$$

By two main theorems, we have the estimate

$$\begin{aligned}(n-2)T(r, g) &\leq \sum_{k=1}^{n} \overline{N}\Big(r, \frac{1}{g - r_k}\Big) - \log r + O(1) \\ &= \sum_{k=1}^{n} \overline{N}\Big(r, \frac{1}{f - r_k}\Big) - \log r + O(1) \\ &\leq nT(r, f) - \log r + O(1).\end{aligned}$$

Similarly we can obtain the estimate

$$(n-2)T(r, f) \leq nT(r, g) - \log r + O(1).$$

Define

$$h_1 = -\frac{1}{b} f^{n-2} Q(f), \quad h_2 = \frac{h_3}{b} g^{n-2} Q(g), \quad h_3 = \frac{P(f)}{P(g)}.$$

Then we have

$$h_1 + h_2 + h_3 = 1.$$

Write $f = \dfrac{f_1}{f_2}$ and $g = \dfrac{g_1}{g_2}$, where pairs f_1, f_2 and g_1, g_2 are p-adic entire functions on $\mathbb{C}_p$ without common factors, respectively. Then

$$h_3 = c\Big(\frac{g_2}{f_2}\Big)^n, \quad c = \frac{P(f) f_2^n}{P(g) g_2^n}.$$

Note that c is an p-adic entire function on $\mathbb{C}_p$ which is never zero, and hence is constant. Thus we have

$$\overline{N}(r, h_3) \le \overline{N}(r, f), \quad \overline{N}\Big(r, \frac{1}{h_3}\Big) \le \overline{N}(r, g).$$

In the following, we will prove $h_3 \equiv 1$.

Assume, to the contrary, that $h_3 \not\equiv 1$. First we prove that h_1 can not be expessed linearly by $\{1, h_3\}$ and $\{1, h_2\}$, respectively. Assume that we have a linear expression

$$h_1 = a_1 h_3 + a_2, \quad a_1, a_2 \in \mathbb{C}_p.$$

Since h_1 is not constant, then $a_1 \neq 0$, and h_3 is not constant. If $a_2 \neq 0$, then the second main theorem implies

$$\begin{aligned}
nT(r, f) &= T(r, h_1) + O(1) \\
&\le \overline{N}\Big(r, \frac{1}{h_1}\Big) + \overline{N}(r, h_1) + \overline{N}\Big(r, \frac{1}{h_1 - a_2}\Big) - \log r + O(1) \\
&\le \overline{N}\Big(r, \frac{1}{f}\Big) + \overline{N}\Big(r, \frac{1}{Q(f)}\Big) + \overline{N}(r, f) + \overline{N}\Big(r, \frac{1}{h_3}\Big) - \log r + O(1) \\
&\le 4T(r, f) + \overline{N}(r, g) - \log r + O(1) \\
&\le 4T(r, f) + T(r, g) - \log r + O(1) \\
&\le \Big(4 + \frac{n}{n-2}\Big) T(r, f) - \log r + O(1),
\end{aligned}$$

which yields $n < 5 + \frac{2}{n-2}$, a contradiction! If $a_2 = 0$, setting

$$Q(z) = \frac{(n-1)(n-2)}{2}(z - s_1)(z - s_2),$$

then by $h_1 = a_1 c\left(\frac{g_2}{f_2}\right)^n$, we see

$$N\Big(r, \frac{1}{f}\Big) \geq \frac{n}{2}\overline{N}\Big(r, \frac{1}{f}\Big), \quad N\Big(r, \frac{1}{f - s_j}\Big) \geq n\overline{N}\Big(r, \frac{1}{f - s_j}\Big), \quad j = 1, 2.$$

Then

$$\Theta_f(s_j) = 1 - \limsup_{r\to\infty} \frac{\overline{N}\Big(r, \frac{1}{f-s_j}\Big)}{T(r, f)} \geq 1 - \frac{1}{n} \ (j = 1, 2), \quad \Theta_f(0) \geq 1 - \frac{2}{n},$$

and again by the second main theorem,

$$1 - \frac{2}{n} + 2(1 - \frac{1}{n}) \leq \Theta_f(0) + \sum_{j=1}^{2} \Theta_f(s_j) \leq 2.$$

This is impossible since $n \geq 10$.

Assume that we have a linear expression

$$h_1 = b_1 h_2 + b_2, \quad b_1, b_2 \in \mathbb{C}_p.$$

Since h_1 is not constant, then $b_1 \neq 0$, and h_2 is not constant. If $b_2 \neq 0$, then the second main theorem implies

$$\begin{aligned}
nT(r, f) &= T(r, h_1) + O(1) \\
&\leq \overline{N}\Big(r, \frac{1}{h_1}\Big) + \overline{N}(r, h_1) + \overline{N}\Big(r, \frac{1}{h_1 - b_2}\Big) - \log r + O(1) \\
&\leq \overline{N}\Big(r, \frac{1}{f}\Big) + \overline{N}\Big(r, \frac{1}{Q(f)}\Big) + \overline{N}(r, f) + \overline{N}\Big(r, \frac{1}{h_2}\Big) - \log r + O(1) \\
&\leq 4T(r, f) + \overline{N}\Big(r, \frac{1}{g}\Big) + \overline{N}\Big(r, \frac{1}{Q(g)}\Big) - \log r + O(1) \\
&\leq 4T(r, f) + 3T(r, g) - \log r + O(1) \\
&\leq (4 + \frac{3n}{n-2})T(r, f) - \log r + O(1),
\end{aligned}$$

which yields $n < 7 + \dfrac{6}{n-2}$, a contradiction! If $b_2 = 0$, then we have $(1 + \dfrac{1}{b_1})h_1 + h_3 = 1$ which is impossible. Thus we proved the claim. In consequence, h_2 and h_3 are not constant.

Define

$$F = \frac{1}{P(f)}, \quad G = \frac{1}{P(g)}.$$

If $1, F, G$ are linearly independent, then

$$H = \frac{F''}{F'} - \frac{G''}{G'} = -\frac{W}{F'G'} \not\equiv 0,$$

where W is the Wronskian of $1, F, G$. Note that poles of H can only occur where F' or G' has a zero. We write $N_0\Big(r, \dfrac{1}{F'}\Big)$ for the valence function of the zeros of F' where F does not take one of the values $A_1 = 0,\ A_2 = \dfrac{1}{b}$ and $A_3 = \dfrac{1}{b+1}$. $N_0(r, \dfrac{1}{G'})$ is defined analogously. Then

$$\begin{aligned} N(r,H) \le \sum_{j=1}^{3} \Big\{ & N_{2)}\Big(r, \frac{1}{F - A_j}\Big) - \overline{N}\Big(r, \frac{1}{F - A_j}\Big) \\ & + N_{2)}\Big(r, \frac{1}{G - A_j}\Big) - \overline{N}\Big(r, \frac{1}{G - A_j}\Big)\Big\} \\ & + N_0\Big(r, \frac{1}{F'}\Big) + N_0\Big(r, \frac{1}{G'}\Big), \end{aligned}$$

where $N_{k)}(r, f)$ is the valence function of f which counts a pole according to its multiplicity if the multiplicity is less than or equal to k and counts a pole k times if its multiplicity is great than k. Note that H has a zero at every point where F and G have a simple pole. It follows that

$$\overline{N}(r,F) + \overline{N}(r,G) \le N\Big(r, \frac{1}{H}\Big) + \frac{1}{2}\{N(r,F) + N(r,G)\}.$$

By the first main theorem and the lemma of logarithmic derivatives, we see

$$\overline{N}(r,F) + \overline{N}(r,G) \le N(r,H) + \frac{1}{2}\{T(r,F) + T(r,G)\} + O(1).$$

The second main theorem applied to F and G gives

$$
\begin{aligned}
2\{T(r,F)+T(r,G)+\log r\} \le \sum_{j=1}^{3}\Big\{\overline{N}\Big(r,\frac{1}{F-A_j}\Big)+\overline{N}\Big(r,\frac{1}{G-A_j}\Big)\Big\}& \\
+\overline{N}(r,F)+\overline{N}(r,G)-N_0\Big(r,\frac{1}{F'}\Big)& \\
-N_0\Big(r,\frac{1}{G'}\Big)+O(1).&
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
\frac{3}{2}\{T(r,F)+T(r,G)\}+2\log r \le \sum_{j=1}^{3}\Big\{N_{2)}\Big(r,\frac{1}{F-A_j}\Big)& \\
+N_{2)}\Big(r,\frac{1}{G-A_j}\Big)\Big\}+O(1).&
\end{aligned}
$$

Since

$$
P'(z)=\frac{n(n-1)(n-2)}{2}z^{n-3}(z-1)^2,
$$

we have $P(1)=1+b$ with multiplicity 3 and $P(0)=b$ with multiplicity $n-2$. Therefore we can write

$$
\begin{aligned}
P(z)-b-1&=(z-1)^3Q_1(z), \quad Q_1(1)\ne 0,\\
P(z)-b&=z^{n-2}Q(z), \quad Q(0)\ne 0,
\end{aligned}
$$

where $Q_1(z)$ is a polynomial of degree $n-3$, having only simple zeros. For every $a\in \mathbb{C}_p-\{b,b+1\}$, $P(z)-a$ has only simple zeros. In particular, $P(z)$ has only simple zeros and thus S has exactly n elements. From the first main theorem we conclude that

$$
\begin{aligned}
N_{2)}\Big(r,\frac{1}{F-A_1}\Big)&=N_{2)}(r,P(f))=2\overline{N}(r,f)\\
&\le 2T(r,f)+O(1),\\
N_{2)}\Big(r,\frac{1}{F-A_2}\Big)&=N_{2)}\Big(r,\frac{1}{P(f)-b}\Big)\le 2\overline{N}\Big(r,\frac{1}{f}\Big)+N_{2)}\Big(r,\frac{1}{Q(f)}\Big)\\
&\le 4T(r,f)+O(1),\\
N_{2)}\Big(r,\frac{1}{F-A_3}\Big)&=N_{2)}\Big(r,\frac{1}{P(f)-b-1}\Big)\\
&\le 2\overline{N}\Big(r,\frac{1}{f-1}\Big)+N_{2)}\Big(r,\frac{1}{Q_1(f)}\Big)\\
&\le (n-1)T(r,f)+O(1).
\end{aligned}
$$

It follows that

$$\sum_{j=1}^{3} N_{2)}\Big(r, \frac{1}{F-A_j}\Big) \leq (n+5)T(r,f)+O(1) = (1+\frac{5}{n})T(r,F)+O(1),$$

and the same inequality holds with f and F replaced by g and G. Thus we would get $\frac{3}{2} < 1+\frac{5}{n}$, and hence $n < 10$ which is a contradiction to our assumptions. It follows that $1, F, G$ are linearly dependent. Then there exists $(c_1, c_2, c_3) \in \mathbb{C}_p^3 - \{0\}$ such that

$$c_1 + c_2 F + c_3 G = 0,$$

and hence

$$-bc_1 h_1 + c_3 h_3 = -bc_1 - c_2.$$

This is impossible.

Therefore we must have $h_3 = 1$, i.e. $P(f) = P(g)$. Set $h = \frac{f}{g}$. We see

(4)

$$\frac{(n-1)(n-2)}{2}(h^n-1)g^2 - n(n-2)(h^{n-1}-1)g + \frac{n(n-1)}{2}(h^{n-2}-1) = 0.$$

If h is constant, (4) implies $h^n - 1 = 0$ and $h^{n-1} - 1 = 0$. It follows that $h = 1$ and hence $f = g$.

It remains to consider the case that h is not constant. We write (4) in the form

$$((h^n-1)g - \frac{n}{n-1}(h^{n-1}-1))^2 = -\frac{n}{(n-1)^2(n-2)}\varphi(h), \tag{5}$$

where φ is defined by

$$\varphi(z) = (n-1)^2(z^n-1)(z^{n-2}-1) - n(n-2)(z^{n-1}-1)^2.$$

An elementary calculation gives

$$\varphi^{(k)}(1) = 0 \ (0 \leq k \leq 3), \quad \varphi^{(4)}(1) = 2n(n-1)^2(n-2) \neq 0.$$

Hence we can write

$$\varphi(z) = (z-1)^4(z-t_1)(z-t_2)\cdots(z-t_{2n-6}),$$

where $t_1, ..., t_{2n-6} \in \mathbb{C}_p - \{1\}$. Now assume that

$$\varphi(z) = \varphi'(z) = 0,$$

for some $z \in \mathbb{C}_p$. A simple calculation shows that z satisfies the following equation

$$(n-1)(n-2)(z^n - 1) - 2n(n-2)(z^{n-1} - 1) + n(n-1)(z^{n-2} - 1) = 0.$$

Hence φ has at least $(2n-6) - (n-1) = n-5$ simple zeros in $\mathbb{C}_p - \{1\}$, w.l.o.g., assume that $t_1, ..., t_{n-5}$ are simple zeros of φ. From (5) we see that

$$\Theta_h(t_j) \geq \frac{1}{2} \; (1 \leq j \leq n-5).$$

Thus the second main theorem yields

$$2 \geq \sum_{j=1}^{n-5} \Theta_h(t_j) \geq \frac{n-5}{2},$$

and hence $n \leq 9$ in contradiction to our assumption $n \geq 10$. This complete the proof of the theorem. □

References

1. A. Boutabaa, *Applications de la théorie de Nevanlinna p-adic*, Collect. Math. **42** (1991), 75-93.
2. W. Cherry and Z. Ye, *Non-Archimedean Nevanlinna theory in several variables and the Non-Archimedean Nevanlinna inverse problem*, preprint.
3. G. Frank and M. Reinders, *A unique range set for meromorphic functions with* 11 *elements*, Preprint.
4. F. Gross and C. C. Yang, *On preimage and range sets of meromorphic functions*, Proc. Japan Acad. **58** (1982), 17-20.
5. P. C. Hu and C. C. Yang, *Uniqueness of meromorphic functions on* $\mathbb{C}^m$, Complex Variables **30** (1996), 235-270.
6. P. C. Hu and C. C. Yang, *Value distribution theory of p-adic meromorphic functions*, Preprint.
7. Ha Huy Khoai, *On p-adic meromorphic functions*, Duke Math. J. **50** (1983), 695-711.
8. Ha Huy Khoai, *Heights for p-adic holomorphic functions of several variables*, Max-Planck Institut Für Mathematik 89-83 (1989).
9. Ha Huy Khoai and M. V. Quang, *On p-adic Nevanlinna theory*, Lecture Notes in Math. **1351** (1988), 146-158, Springer-Verlag.
10. P. Li and C. C. Yang, *On the unique range sets of meromorphic functions*, Proc. Amer. Math. Soc. (to appear).

11. P. Li and C. C. Yang, *Some further results on the unique range sets of meromorphic functions*, Preprint.
12. E. Mues and M. Reinders, *Meromorphic function sharing one value and unique range sets*, Kodai Math. J. **18** (1995), 515-522.
13. H. X. Yi, *On a question of Gross*, Sci. China, Ser. **A 38** (1995), 8-16.
14. H. X. Yi, *The unique range sets of entire or meromorphic functions*, Complex Variables **28** (1995), 13-21.

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