A UNIQUE RANGE SET OF P-ADIC MEROMORPHIC FUNCTIONS WITH 10 ELEMENTS

PEI-CHU HU AND CHUNG-CHUN YANG

Abstract. In this paper, we will exhibit a unique range set for p-adic meromorphic functions with 10 elements.

1. Introduction


For a non-constant meromorphic function $f$ on $\mathbb{C}$ and a set $S \subset \mathbb{C} \cup \{\infty\}$ we define

$$E_f(S) = \bigcup_{a \in S} \{mz \mid f(z) = a \text{ with multiplicity } m\}.$$

A set $S \subset \mathbb{C} \cup \{\infty\}$ is called an unique range set for meromorphic functions (URSM) if for any pair of non-constant meromorphic functions $f$ and $g$ on $\mathbb{C}$, the condition $E_f(S) = E_g(S)$ implies $f = g$. A set $S \subset \mathbb{C} \cup \{\infty\}$ is called an unique range set for entire functions (URSE) if for any pair of non-constant entire functions $f$ and $g$ on $\mathbb{C}$, the condition $E_f(S) = E_g(S)$ implies $f = g$. Gross and Yang [4] showed that the set

Received June 28, 1997


The work of the first author was partially supported by a Post-doctoral Grant of China and the second author was partially supported by UGC Grant of Hong Kong.
\[ S = \{ z \in \mathbb{C} \mid z + e^z = 0 \} \]

is a URSE. Recently, URSE and also URSM with finitely many elements have been found by Yi [13, 14], Li and Yang [10, 11], Mues and Reinders [12], Frank and Reinders [3], Hu and Yang [5]. Li and Yang [10] introduced the notation

\[ \lambda_M = \inf \{ \# S \mid S \text{ is a URSM } \}, \]
\[ \lambda_E = \inf \{ \# S \mid S \text{ is a URSE } \}, \]

where \#S is the cardinality of the set S. The best lower and upper bounds known so far are

\[ 5 \leq \lambda_E \leq 7, \quad 6 \leq \lambda_M \leq 11. \]

For \( p \)-adic meromorphic (or entire) function \( f \) on \( \mathbb{C}_p \), we can similarly define \( E_f(S) \) for a set \( S \subseteq \mathbb{C}_p \cup \{ \infty \} \), and introduce the notation \( \lambda_M \) and \( \lambda_E \). In [6] we obtained \( \lambda_E \leq 4 \) for \( p \)-adic entire functions and \( \lambda_M \leq 12 \) for \( p \)-adic meromorphic functions. W. Cherry ask us whether the Frank-Reinders’ method gives a \( p \)-adic URSM with 10 elements by using the \( -\log r \) term in their second main theorem. In this paper, we will give a confirmed answer to Cherry’s question, i.e., \( \lambda_M \leq 10 \) for \( p \)-adic meromorphic functions.

2. Nevanlinna Theory of \( p \)-adic Meromorphic Functions

Let \( p \) be a prime number, let \( \mathbb{Q}_p \) be the field of \( p \)-adic numbers, and let \( \mathbb{C}_p \) be the \( p \)-adic completion of the algebraic closure of \( \mathbb{Q}_p \). The absolute value \( | \cdot |_p \) in \( \mathbb{C}_p \) is normalized so that \( | p |_p = p^{-1} \). We further use the notion \( ord_p \) for the additive valuation on \( \mathbb{C}_p \).

Recall that in a metric space whose metric comes from a Non-Archimedean norm, a sequence is Cauchy if and only if the difference between adjacent terms approaches zero; and if the metric space is complete, an infinite sum converges if and only if its general term approaches zero. So if we consider expressions of the form

\[ f(Z) = \sum_{n=0}^{\infty} a_n Z^n, \quad (a_n \in \mathbb{C}_p), \]

we can give a value \( \sum_{n=0}^{\infty} a_n z^n \) to \( f(z) \) whenever an \( z \) substituted for \( Z \) for which

\[ | a_n z^n |_p \to 0. \]
Define the “radius $\rho$ of convergence” by

$$\frac{1}{\rho} = \lim_{n \to \infty} \sup \left| a_n \right|^{1/n}.$$  

Then the series converges if $|z|_p < \rho$ and diverges if $|z|_p > \rho$. Also the function $f(z)$ is said to be $p$-adic analytic on $B(\rho)$, where

$$B(\rho) = \{ z \in \mathbb{C}_p \ | \ |z|_p < \rho \}.$$  

If $\rho = \infty$, the function $f(z)$ also is said to be $p$-adic entire on $\mathbb{C}_p$.

Consider non-constant $p$-adic analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \ (a_n \in \mathbb{C}_p)$$

on $B(\rho)$ ($0 < \rho \leq \infty$). The essence of the Wiman-Valiron method is the analysis of the behaviour of the function by means of the maximum term:

$$\mu(r, f) = \max_{n \geq 0} |a_n|_p r^n \quad (0 < r = |z|_p < \rho)$$

together with the central index:

$$\nu(r, f) = \max\{n \ | \ |a_n|_p r^n = \mu(r, f)\}.$$  

Define

$$\nu(0, f) = \lim_{r \to 0} \nu(r, f).$$

**Lemma 2.1** ([6]). The central index $\nu(r, f)$ increases as $r \to \rho$, and satisfies the formula:

$$\log \mu(r, f) = \log |a_{\nu(0, f)}|_p + \int_{0}^{r} \frac{\nu(t, f) - \nu(0, f)}{t} dt + \nu(0, f) \log r. \quad (0 < r < \rho)$$

The following technical lemma can be found in [2]:

**Lemma 2.2** (Weierstrass Preparation Theorem). There exists an unique monic polynomial $P$ of degree $\nu(r, f)$ and a $p$-adic analytic function $g$ on $B[r]$ such that $f = gP$, where

$$B[r] = \{ z \in \mathbb{C}_p \ | \ |z|_p \leq r \}. $$
Furthermore, $g$ does not have any zero inside $B[r]$, and $P$ has exactly $\nu(r, f)$ zeros, counting multiplicity, on $B[r]$.

Let $n(r, 1/f)$ denote the number of zeros (counting multiplicity) of $f$ with absolute value $\leq r$ and define the **valence function** of $f$ for $0$ by

$$N(r, 1/f) = \int_0^r \frac{n(t, 1/f) - n(0, 1/f)}{t} dt + n(0, 1/f) \log r \quad (0 < r < \rho).$$

Lemma 2.2 shows that

$$n(r, 1/f) = \nu(r, f).$$

Then Lemma 2.1 imply the **Jensen formula**:

$$(1) \quad N(r, 1/f) = \log \mu(r, f) - \log |a_n(0, \frac{1}{f})|_p.$$  

We also denote the number of distinct zeros of $f$ on $B[r]$ by $\pi(r, 1/f)$ and define

$$\overline{N}(r, 1/f) = \int_0^r \frac{\pi(t, 1/f) - \pi(0, 1/f)}{t} dt + \pi(0, 1/f) \log r \quad (0 < r < \rho).$$

For each $n$ we draw the graph $\gamma_n(t)$ which depicts $ord_p(a_n z^n)$ as a function of $t = ord_p(z)$. Then $\gamma_n(t)$ is a straight line with slope $n$. Let $\gamma(t, f)$ denote the boundary of the intersection of all of the half-planes lying under the lines $\gamma_n(t)$. This line is what we call the Newton polygon of the function $f(z)$ (see [9]). The points $t$ at which $\gamma(t, f)$ has vertices are called the **critical points** of $f(z)$. A finite segment $[\alpha, \beta]$ contains only finitely many critical points. It is clear that if $t$ is a critical point, then $ord_p(a_n) + nt$ attains its minimum at least at two values of $n$. Obviously, we have

$$\mu(r, f) = p^{-\gamma(t,f)},$$

where $r = p^{-t}$. A basic property of the Newton polygon is that, if $t = ord_p(z)$ is not a critical point, then

$$|f(z)|_p = p^{-\gamma(t,f)},$$
which implies
\[ |f(z)|_p = \mu(r, f). \]
Further, we note that if \( h \) is another \( p \)-adic analytic function on \( B(\rho) \), then
\[ (2) \quad \mu(r, fh) = \mu(r, f)\mu(r, h). \]

By a meromorphic function \( f \) on \( B(\rho) \) we will mean the quotient \( \frac{g}{h} \) of two \( p \)-adic analytic functions \( g \) and \( h \) such that \( g \) and \( h \) have not any common factors in the ring of \( p \)-adic analytic functions on \( B(\rho) \). Note that (2) hold and that greatest common divisors of any two \( p \)-adic analytic functions exist. We can uniquely extend \( \mu \) to meromorphic function \( f = \frac{g}{h} \) by defining
\[ \mu(r, f) = \frac{\mu(r, g)}{\mu(r, h)}. \]
Also set
\[ \gamma(t, f) = \gamma(t, g) - \gamma(t, h). \]
It is clear that, if \( t = \text{ord}_p(z) \) is not a critical point for \( f(z) \), i.e., \( t \) is not a critical point for either \( g(z) \) or \( h(z) \), then
\[ |f(z)|_p = p^{-\gamma(t, f)} = \mu(r, f). \]

Define the counting function \( n(r, f) \) and the valence function \( N(r, f) \) of \( f \) for poles respectively by
\[ n(r, f) = n\left(r, \frac{1}{h}\right), \quad N(r, f) = N\left(r, \frac{1}{h}\right). \]
Then applying (1) for \( g \) and \( h \), we obtain the Jensen formula:
\[ (3) \quad N\left(r, \frac{1}{f}\right) - N(r, f) = \log \mu(r, f) - C_f, \]
where \( C_f \) is a constant depending only on \( f \). Define
\[ m(r, f) = \log^+ \mu(r, f) = \max\{0, \log \mu(r, f)\}. \]
Finally, we define the characteristic function:
\[ T(r, f) = m(r, f) + N(r, f). \]
Here we exhibit some basic facts which will be used in the following sections.

**Lemma 2.3** (First Main Theorem, cf. [1, 9]). Let $f$ be a non-constant meromorphic function in $B(\rho)$. Then for every $a \in \mathbb{C}_\rho$ we have

$$m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1) \quad (r \to \rho).$$

**Lemma 2.4** (The Lemma of Logarithmic Derivative, cf. [1, 2, 9]). Let $f$ be a nonconstant meromorphic function in $B(\rho)$. Then

$$m\left(r, \frac{f'}{f}\right) = O(1) \quad (r \to \rho).$$

**Lemma 2.5** (Second Main Theorem, cf. [1, 2, 9]) Let $f$ be a non-constant meromorphic function in $B(\rho)$ and let $a_1, \ldots, a_q$ be distinct numbers of $\mathbb{C}_\rho$. Then

$$(q - 1)T(r, f) \leq N(r, f) + \sum_{j=1}^q N\left(r, \frac{1}{f-a_j}\right) - N_1(r, f) - \log r + O(1),$$

where

$$N_1(r, f) = 2N(r, f) - N(r, f') + N\left(r, \frac{1}{f'}\right).$$

Furthermore, we have

$$N(r, f) + \sum_{j=1}^q N\left(r, \frac{1}{f-a_j}\right) - N_1(r, f) \leq N(r, f) + \sum_{j=1}^q N\left(r, \frac{1}{f-a_j}\right) - N_0\left(r, \frac{1}{f'}\right),$$

where

$$N_0\left(r, \frac{1}{f'}\right) \text{ is the valence function of the zeros of } f' \text{ where } f \text{ does not take one of the values } a_1, \ldots, a_q, \text{ and where}$$

$$\Theta_f(a) = 1 - \lim_{r \to \infty} \sup \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$
3. Uniqueness of \( p \)-adic meromorphic functions

We recall the following useful facts:

**Lemma 3.1 ([2]).** If \( f \) is a \( p \)-adic entire function on \( \mathbb{C}_p \) that is never zero, then \( f \) is constant.

**Lemma 3.2 ([6]).** Let \( f \) be a non-constant \( p \)-adic meromorphic functions on \( \mathbb{C}_p \). Take a positive integer \( n \), \( \{a_0, a_1, ..., a_n\} \subset \mathbb{C}_p \) with \( a_0 \neq 0 \) and set
\[
L[f] = a_0 f^n + a_1 f^{n-1} + \cdots + a_n.
\]
Then
\[
T(r, L[f]) = nT(r, f) + O(1).
\]

**Theorem 3.1.** Take integer \( n \geq 10 \) and let \( b \in \mathbb{C}_p - \{0, -1\} \). Then the polynomial \( P(z) \) defined by
\[
P(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} + b
\]
has only simple zeros, and if \( f \) and \( g \) are non-constant \( p \)-adic meromorphic functions on \( \mathbb{C}_p \) such that \( E_f(S) = E_g(S) \), then \( f \equiv g \), where
\[
S = \{ z \in \mathbb{C}_p \mid P(z) = 0 \}.
\]

**Proof.** Write \( S = \{r_1, r_2, ..., r_n\} \) and define
\[
Q(z) = \frac{(n-1)(n-2)}{2} z^2 - n(n-2)z + \frac{n(n-1)}{2}.
\]
By two main theorems, we have the estimate
\[
(n-2)T(r, g) \leq \sum_{k=1}^{n} N\left(r, \frac{1}{g - r_k}\right) - \log r + O(1)
\]
\[
= \sum_{k=1}^{n} N\left(r, \frac{1}{f - r_k}\right) - \log r + O(1)
\]
\[
\leq nT(r, f) - \log r + O(1).
\]
Similarly we can obtain the estimate
\[
(n-2)T(r, f) \leq nT(r, g) - \log r + O(1).
\]
Define
\[ h_1 = \frac{1}{b} f^{n-2} Q(f), \quad h_2 = \frac{h_3}{b} g^{n-2} Q(g), \quad h_3 = \frac{P(f)}{P(g)}. \]

Then we have
\[ h_1 + h_2 + h_3 = 1. \]

Write \( f = f_1 f_2 \) and \( g = g_1 g_2 \), where pairs \( f_1, f_2 \) and \( g_1, g_2 \) are \( p \)-adic entire functions on \( \mathbb{C}_p \) without common factors, respectively. Then
\[ h_3 = c \left( \frac{g_2}{f_2} \right)^n, \quad c = \frac{P(f) f_2^n}{P(g) g_2^n}. \]

Note that \( c \) is an \( p \)-adic entire function on \( \mathbb{C}_p \) which is never zero, and hence is constant. Thus we have
\[ \overline{N}(r, h_3) \leq \overline{N}(r, f), \quad \overline{N}(r, \frac{1}{h_3}) \leq \overline{N}(r, g). \]

In the following, we will prove \( h_3 \equiv 1 \).

Assume, to the contrary, that \( h_3 \not\equiv 1 \). First we prove that \( h_1 \) cannot be expressed linearly by \( \{1, h_3\} \) and \( \{1, h_2\} \), respectively. Assume that we have a linear expression
\[ h_1 = a_1 h_3 + a_2, \quad a_1, a_2 \in \mathbb{C}_p. \]

Since \( h_1 \) is not constant, then \( a_1 \neq 0 \), and \( h_3 \) is not constant. If \( a_2 \neq 0 \), then the second main theorem implies
\[
nT(r, f) = T(r, h_1) + O(1) \\
\leq \overline{N}(r, \frac{1}{h_1}) + \overline{N}(r, h_1) + \overline{N}(r, \frac{1}{h_1 - a_2}) - \log r + O(1) \\
\leq \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{Q(f)}) + \overline{N}(r, f) + \overline{N}(r, \frac{1}{h_3}) - \log r + O(1) \\
\leq 4T(r, f) + \overline{N}(r, g) - \log r + O(1) \\
\leq 4T(r, f) + T(r, g) - \log r + O(1) \\
\leq \left( 4 + \frac{n}{n-2} \right) T(r, f) - \log r + O(1),
\]
which yields \( n < 5 + \frac{2}{n-2} \), a contradiction! If \( a_2 = 0 \), setting

\[
Q(z) = \frac{(n-1)(n-2)}{2}(z-s_1)(z-s_2),
\]

then by \( h_1 = a_1 c \left( \frac{g_2}{f_2} \right)^n \), we see

\[
N \left( r, \frac{1}{f} \right) \geq \frac{n}{2} N \left( r, \frac{1}{f} \right), \quad N \left( r, \frac{1}{f-s_j} \right) \geq n N \left( r, \frac{1}{f-s_j} \right), \quad j = 1,2.
\]

Then

\[
\Theta_f(s_j) = 1 - \lim_{r \to \infty} \sup \frac{N \left( r, \frac{1}{f-s_j} \right)}{T(r,f)} \geq 1 - \frac{1}{n} (j = 1,2), \quad \Theta_f(0) \geq 1 - \frac{2}{n},
\]

and again by the second main theorem,

\[
1 - \frac{2}{n} + 2(1 - \frac{1}{n}) \leq \Theta_f(0) + \sum_{j=1}^{2} \Theta_f(s_j) \leq 2.
\]

This is impossible since \( n \geq 10 \).

Assume that we have a linear expression

\[
h_1 = b_1 h_2 + b_2, \quad b_1, b_2 \in \mathbb{C}_p.
\]

Since \( h_1 \) is not constant, then \( b_1 \neq 0 \), and \( h_2 \) is not constant. If \( b_2 \neq 0 \), then the second main theorem implies

\[
nT(r, f) = T(r, h_1) + O(1)
\]

\[
\leq N \left( r, \frac{1}{h_1} \right) + N(r, h_1) + N \left( r, \frac{1}{h_1-b_2} \right) - \log r + O(1)
\]

\[
\leq N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{Q(f)} \right) + N(r, f) + N \left( r, \frac{1}{h_2} \right) - \log r + O(1)
\]

\[
\leq 4T(r, f) + N \left( r, \frac{1}{g} \right) + N \left( r, \frac{1}{Q(g)} \right) - \log r + O(1)
\]

\[
\leq 4T(r, f) + 3T(r, g) - \log r + O(1)
\]

\[
\leq (4 + \frac{3n}{n-2})T(r, f) - \log r + O(1),
\]
which yields $n < 7 + \frac{6}{n - 2}$, a contradiction! If $b_2 = 0$, then we have $(1 + \frac{1}{b_1})h_1 + h_3 = 1$ which is impossible. Thus we proved the claim. In consequence, $h_2$ and $h_3$ are not constant.

Define
\[ F = \frac{1}{P(f)}, \quad G = \frac{1}{P(g)}. \]

If $1, F, G$ are linearly independent, then
\[ H = \frac{F''}{F'} - \frac{G''}{G'} = -\frac{W}{F'G'} \neq 0, \]

where $W$ is the Wronskian of $1, F, G$. Note that poles of $H$ can only occur where $F'$ or $G'$ has a zero. We write $N_0(r, \frac{1}{F'})$ for the valence function of the zeros of $F'$ where $F$ does not take one of the values $A_1 = 0$, $A_2 = \frac{1}{b}$ and $A_3 = \frac{1}{b + 1}$. $N_0(r, \frac{1}{G'})$ is defined analogously. Then
\[
N(r, H) \leq \sum_{j=1}^{3} \left\{ N_2 \left( r, \frac{1}{F - A_j} \right) - N \left( r, \frac{1}{F - A_j} \right) \\
+ N_2 \left( r, \frac{1}{G - A_j} \right) - N \left( r, \frac{1}{G - A_j} \right) \right\} \\
+ N_0 \left( r, \frac{1}{F'} \right) + N_0 \left( r, \frac{1}{G'} \right),
\]

where $N_k(r, f)$ is the valence function of $f$ which counts a pole according to its multiplicity if the multiplicity is less than or equal to $k$ and counts a pole $k$ times if its multiplicity is great than $k$. Note that $H$ has a zero at every point where $F$ and $G$ have a simple pole. It follows that
\[
\overline{N}(r, F) + \overline{N}(r, G) \leq N \left( r, \frac{1}{H} \right) + \frac{1}{2} \{ N(r, F) + N(r, G) \}.
\]

By the first main theorem and the lemma of logarithmic derivatives, we see
\[
\overline{N}(r, F) + \overline{N}(r, G) \leq N(r, H) + \frac{1}{2} \{ T(r, F) + T(r, G) \} + O(1).
\]
The second main theorem applied to $F$ and $G$ gives

$$2\{T(r, F) + T(r, G) + \log r\} \leq \sum_{j=1}^{3} \left\{ \mathcal{N}\left(r, \frac{1}{F - A_j}\right) + \mathcal{N}\left(r, \frac{1}{G - A_j}\right) \right\}$$

$$+ \mathcal{N}(r, F) + \mathcal{N}(r, G) - N_0\left(r, \frac{1}{F'}\right)$$

$$- N_0\left(r, \frac{1}{G'}\right) + O(1).$$

Hence we obtain

$$\frac{3}{2} \{T(r, F) + T(r, G)\} + 2\log r \leq \sum_{j=1}^{3} \left\{ N_2\left(r, \frac{1}{F - A_j}\right) + N_2\left(r, \frac{1}{G - A_j}\right) \right\} + O(1).$$

Since

$$P'(z) = \frac{n(n-1)(n-2)}{2} z^{n-3}(z - 1)^2,$$

we have $P(1) = 1 + b$ with multiplicity 3 and $P(0) = b$ with multiplicity $n - 2$. Therefore we can write

$$P(z) - b - 1 = (z - 1)^3 Q_1(z), \quad Q_1(1) \neq 0,$$

$$P(z) - b = z^{n-2} Q(z), \quad Q(0) \neq 0,$$

where $Q_1(z)$ is a polynomial of degree $n - 3$, having only simple zeros. For every $a \in \mathbb{C}_p - \{b, b + 1\}$, $P(z) - a$ has only simple zeros. In particular, $P(z)$ has only simple zeros and thus $S$ has exactly $n$ elements. From the first main theorem we conclude that

$$N_2\left(r, \frac{1}{F - A_1}\right) = N_2\left(r, P(f)\right) = 2\mathcal{N}(r, f)$$

$$\leq 2T(r, f) + O(1),$$

$$N_2\left(r, \frac{1}{F - A_2}\right) = N_2\left(r, \frac{1}{P(f) - b}\right) \leq N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{Q(f)}\right)$$

$$\leq 4T(r, f) + O(1),$$

$$N_2\left(r, \frac{1}{F - A_3}\right) = N_2\left(r, \frac{1}{P(f) - b - 1}\right)$$

$$\leq 2\mathcal{N}\left(r, \frac{1}{f - 1}\right) + N_2\left(r, \frac{1}{Q_1(f)}\right)$$

$$\leq (n - 1)T(r, f) + O(1).$$
It follows that
\[ \sum_{j=1}^{3} N_{2j}(r, \frac{1}{F - A_j}) \leq (n + 5)T(r, f) + O(1) = (1 + \frac{5}{n})T(r, F) + O(1), \]
and the same inequality holds with \( f \) and \( F \) replaced by \( g \) and \( G \). Thus we would get \( \frac{3}{2} < 1 + \frac{5}{n} \), and hence \( n < 10 \) which is a contradiction to our assumptions. It follows that \( 1, F, G \) are linearly dependent. Then there exists \((c_1, c_2, c_3) \in \mathbb{C}_p^3 - \{0\}\) such that
\[ c_1 + c_2 F + c_3 G = 0, \]
and hence
\[ -bc_1 h_1 + c_3 h_3 = -bc_1 - c_2. \]
This is impossible.

Therefore we must have \( h_3 = 1 \), i.e. \( P(f) = P(g) \). Set \( h = \frac{f}{g} \). We see
\[ \frac{(n - 1)(n - 2)}{2} (h^n - 1)g^2 - n(n - 2)(h^{n-1} - 1)g + \frac{n(n - 1)}{2} (h^{n-2} - 1) = 0. \]
If \( h \) is constant, (4) implies \( h^n - 1 = 0 \) and \( h^{n-1} - 1 = 0 \). It follows that \( h = 1 \) and hence \( f = g \).

It remains to consider the case that \( h \) is not constant. We write (4) in the form
\[ ((h^n - 1)g - \frac{n}{n - 1}(h^{n-1} - 1))^2 = -\frac{n}{(n - 1)^2(n - 2)^2} \varphi(h), \]
where \( \varphi \) is defined by
\[ \varphi(z) = (n - 1)^2(z^n - 1)(z^{n-2} - 1) - n(n - 2)(z^{n-1} - 1)^2. \]
An elementary calculation gives
\[ \varphi^{(k)}(1) = 0 \quad (0 \leq k \leq 3), \quad \varphi^{(4)}(1) = 2n(n - 1)^2(n - 2) \neq 0. \]
Hence we can write
\[ \varphi(z) = (z - 1)^4(z - t_1)(z - t_2) \cdots (z - t_{2n-6}), \]
where \( t_1, \ldots, t_{2n-6} \in \mathbb{C}_p - \{1\} \). Now assume that
\[
\varphi(z) = \varphi'(z) = 0,
\]
for some \( z \in \mathbb{C}_p \). A simple calculation shows that \( z \) satisfies the following equation
\[
(n - 1)(n - 2)(z^n - 1) - 2n(n - 2)(z^{n-1} - 1) + n(n - 1)(z^{n-2} - 1) = 0.
\]
Hence \( \varphi \) has at least \((2n - 6) - (n - 1) = n - 5\) simple zeros in \( \mathbb{C}_p - \{1\} \), w.l.o.g., assume that \( t_1, \ldots, t_{n-5} \) are simple zeros of \( \varphi \). From (5) we see that
\[
\Theta_h(t_j) \geq \frac{1}{2} \quad (1 \leq j \leq n - 5).
\]
Thus the second main theorem yields
\[
2 \geq \sum_{j=1}^{n-5} \Theta_h(t_j) \geq \frac{n - 5}{2},
\]
and hence \( n \leq 9 \) in contradiction to our assumption \( n \geq 10 \). This complete the proof of the theorem. \( \square \)

References

5. P. C. Hu and C. C. Yang, Uniqueness of meromorphic functions on \( \mathbb{C}^m \), Complex Variables 30 (1996), 235-270.
8. Ha Huy Khoai, Heights for p-adic holomorphic functions of several variables, Max-Planck Institut Für Mathematik 89-83 (1989).
11. P. Li and C. C. Yang, *Some further results on the unique range sets of meromorphic functions*, Preprint.

**Department of Mathematics**  
**Shandong University**  
**Jinan 250100, Shandong P. R. China**

**Department of Mathematics**  
**The Hong Kong University of Science & Technology**  
**Clear Water Bay, Kowloon, Hong Kong**