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ON SEGRE BOUND FOR THE REGULARITY INDEX OF FAT POINTS IN $\mathbf{P}^2$

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ABSTRACT. In this paper we reprove Segre bound for the regularity index of fat points in $\mathbf{P}^2$ by a simple and natural method which may be used to study fat points in $\mathbf{P}^n$, $n\geq 3$.

1. INTRODUCTION

Let $X = \{P_1, \dots, P_s\}$ be distinct points in the projective space $\mathbf{P}^n = \mathbf{P}^n(k)$, k an algebraically closed field, and let $m_1, \dots, m_s$ be positive integers. A form f (or a hypersurface) of the polynomial ring $R := k[X_0, \dots, X_n]$ is said to have multiplicity m_i at P_i if all derivatives of f of order $\leq m_i$ vanish at P_i.

If $\wp_1, \dots, \wp_s$ are the prime ideals in R corresponding to the points $P_1, \dots, P_s$, we will denote by $m_1P_1 + \cdots + m_sP_s$ the zero-scheme defined by the ideal $\wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$ of all forms of R vanishing at P_i with mutiplicity $\geq m_i$, $i = 1, \dots, s$. We will call $Z := m_1P_1 + \cdots + m_sP_s$ a set of fat points in $\mathbf{P}^n$.

Let consider the graded ring $A := R/(\wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s})$, which is the homogeneous coordinate ring of Z. It is well known that $A = \bigoplus_{t\geq 0} A_t$ is a one-dimensional Cohen-Macaulay graded ring whose multiplicity is

$$e = \sum_{i=1}^{s} \binom{m_i + n - 1}{n}.$$

Furthermore, the Hilbert function $H_A(t) := \dim_k A_t$ strictly increases until it reaches the mutiplicity, at which it stabilizes. The regularity index of A (or of the fat points Z) is defined to be the least integer t such that $H_A(t) = e$, and we will denote it by $r(Z)$ (or by $r(A)$).

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The regularity index $r(Z)$ gives upper bounds for the maximum degree of sygyzy modules of I_Z. It enables us to estimate the complexity of Z. The problem to find an upper bound for the regularity index has been dealt with by many authors and several different results have been obtained.

For almost all sets X of s points in $\mathbf{P}^2$ B. Segre [S] found the upper bound:

$$r(Z) \leq \max\Big\{m_1 + m_2 - 1, \Big[\frac{1}{2}\sum_{i=1}^{s} m_i\Big]\Big\} \quad \text{if} \quad m_1 \geq \cdots \geq m_s.$$

Such a set of points is always in general position. A set X of points in $\mathbf{P}^n$ is said to be *in general position* if no $n+1$ points of X lie on a hyperplane of $\mathbf{P}^n$. For fat points in general position in $\mathbf{P}^2$, E. Davis and G. Geramita [DG] first gave the bound $r(Z) \leq [\frac{sm}{2}]$ for the case $m_1 = \cdots = m_s = m$. Then M. V. Catalisano [C1] successfully proved the bound:

$$r(Z) \leq \max\Big\{m_1 + m_2 - 1, \Big[\frac{1}{2}\sum_{i=1}^{s} m_i\Big]\Big\} \quad \text{if} \quad m_1 \geq \cdots \geq m_s.$$

She also showed that this bound is attained for fat points lying on an irreducible conic [C2].

The above result was then generalized to fat points in general position in $\mathbf{P}^n$ by M.V. Catalisano, N.V. Trung and G. Valla, see [CTV]. They showed that

$$r(Z) \leq \max\Big\{m_1 + m_2 - 1, \Big[\frac{1}{n}\Big(\sum_{i=1}^{s} m_i + n - 2\Big)\Big]\Big\} \quad \text{if} \quad m_1 \geq \cdots \geq m_s.$$

and that this bound is sharp for fat points lying on a rational normal curve. We will call it the Segre bound for the regularity index of fat points in general position. See [G], [TV] for other bounds for fat points with stronger properties.

For arbitrary fat points in $\mathbf{P}^2$, an upper bound for the regularity index was first given by W. Fulton [Fu]:

$$r(Z) \leq \sum_{i=1}^{s} m_i - 1.$$

It was afterwards extended by E. Davis and A. Geramita [DG] to arbitrary fat points in $\mathbf{P}^n$. They also showed that the bound is attained if and only if the fat points lie on a line. It is clear that this bound is rather large. Recently G. Fatabbi [Fa] has found the following generalized version of Segre bound:

Theorem 1. *Let $P_1, \dots, P_s$ be arbitrary points in $\mathbf{P}^2$ and $Z = m_1P_1 + \cdots + m_sP_s$ a set of fat points in $\mathbf{P}^2$. Then*

$$r(Z) \leq \max\Big\{h-1, \Big[\frac{1}{2}\sum_{i=1}^{s} m_i\Big]\Big\},$$

where $h := \max\Big\{\sum_{j=1}^{k} m_{i_j} | P_{i_1}, \dots, P_{i_k} \text{ are collinear}\Big\}$.

It is clear that this bound is better than Fulton's bound. In her paper Fatabbi used elaborate geometric methods, and the proof is rather long. It is impossible to extend her proof for fat points in $\mathbf{P}^n$, $n \geq 3$.

We will reprove the above theorem by a simple algebraic proof. This proof follows the method of [CTV], [TV]. An overview of this method has been given in [T]. We would like to point out that our proof may be used to study fat points in $\mathbf{P}^n$, $n \geq 3$, because it depends on properties of forms and ideals in the ring $k[X_0, X_1, X_2]$, which may be generalized to the ring $k[X_0, \dots, X_n]$.

2. Proof of Theorem 1

First we recall two lemmas in [CTV] which have been proved simply by linear algebra.

The first lemma allows us to use induction to estimate the regularity index of fat points.

Lemma 1 [CTV, Lemma 1]. *Let $P_1, \dots, P_s, P$ be distinct points in $\mathbf{P}^n$ and let $\wp$ be the defining ideal of P. If $m_1, \dots, m_s$ and a are positive integers, $J := \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$, and $I := J \cap \wp^a$, then*

$$r(R/I) = \max\big\{a-1, r(R/J), r(R/(J+\wp^a))\big\}.$$

Note that $A/(J + \wp^a)$ is an Artinian ring. For an Artinian ring B we define the regularity index $r(B)$ to be the least integer t such that $H_B(t) = 0$.

The second lemma gives a simple characterization for $r(R/(J + \wp^a))$.

Lemma 2 [CTV, Lemma 3]. *let Let $P_1, \dots, P_s$ be distinct points in $\mathbf{P}^n$ and $m_1, \dots, m_s$, a positive integers. Put $J := \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$ and $\wp := (X_1, \dots, X_n)$. Then $r(R/(J + \wp^a)) \le t$ if and only if $X_0^{t-i} M \in J + \wp^{i+1}$ for every monomial M of degree i in $X_1, \dots, X_n$, comme $i = 0, \dots, a-1$.*

To estimate $r(R/(J + \wp^a))$ we shall need the following lemma.

Lemma 3. *Let $P_1, \dots, P_s, P$ be distinct points in $\mathbf{P}^2$ and $m_1, \dots, m_s$ positive integers. Put $J := \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$, and*

$$l := \max\Big\{ \sum_{j=1}^{r} m_{i_j} \big| P_{i_1}, \dots, P_{i_r}, P \text{ are collinear} \Big\},$$

$$t(J) := \max\Big\{ l, \Big[\frac{1}{2} \Big(\sum_{i=1}^{s} m_i + 1 \Big) \Big] \Big\}.$$

Then we can find $t = t(J)$ lines, say $L_1, \dots, L_t$, avoiding P such that $L_1 \cdots L_t \in J$.

Proof. The case $t = 1$ is trivial because either $J = \wp_1$ or $J = \wp_1 \cap \wp_2$. If $t > 1$ we will show that there exists a line L avoiding P such as if we define

$$J' := \Big(\bigcap_{P_i \notin L} \wp_i^{m_i} \Big) \cap \Big(\bigcap_{P_i \in L} \wp_i^{m_i - 1} \Big),$$

then $t(J') \le t(J) - 1$. We distinguish two cases.

Case 1. $l \ge \Big[\frac{1}{2} \Big(\sum_{i=1}^{s} m_i + 1 \Big) \Big]$. If there exist three lines, say d_1, d_2, d_3, passing through P such that $\sum_{P_i \in d_j} m_i = l$, then

$$\Big[\frac{1}{2} \Big(\sum_{i=1}^{s} m_i + 1 \Big) \Big] \ge \Big[\frac{3l + 1}{2} \Big] > l,$$

a contradiction. Therefore, there are at most two lines, say d_1, d_2, passing through P such that $\sum_{P_i \in d_1} m_i = \sum_{P_i \in d_2} m_i = l$. Choose L to be a line avoiding P passing through a point $P_i \in d_1$ and, if d_2 exists, a point $P_j \in d_2$.

Case 2. $l < \Big[\frac{1}{2} \Big(\sum_{i=1}^{s} m_i + 1 \Big) \Big]$. There exists a point $P_j \notin d_1$. Choose L to be a line avoiding P and passing through a point $P_i \in d_1$ and P_j.

Clearly, $t(J') \leq t(J) - 1$ for such a line L. Let $t' = t(J')$. By induction we may assume that there exist lines, say $L_1, \ldots, L_{t'}$, avoiding P such that $L_1 \cdots L_{t'} \in J'$. Since $L \in \prod_{P_i \in L} \wp_i$, we get

$$L_1 \cdots L_{t'} L \in J' \prod_{P_i \in L} \wp_i \subseteq J.$$

Lemma 4. *Let $P_1, \ldots, P_s$, P be distinct points in $\mathbf{P}^2$, and $m_1 \geq \cdots \geq m_s \geq a$ positive integers. Let $\wp$ be the defining prime ideal of P and $J := \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$. Then*

$$r(R/(J + \wp^a)) \leq \max\Big\{l + a - 1, \Big[\frac{1}{2}\Big(\sum_{i=1}^{s} m_i + a\Big)\Big]\Big\},$$

where $l := \max\Big\{\sum_{j=1}^{r} m_{i_j} \big| P_{i_1}, \ldots, P_{i_r}, P \text{ are collinear}\Big\}$.

Proof. Let

$$t := \max\Big\{l, \Big[\frac{1}{2}\Big(\sum_{i=1}^{s} m_i + a\Big)\Big]\Big\},$$

and let us assume $P = (1, 0, 0)$, so that $\wp = (X_1, X_2)$.

Case 1. All of $P_1, \ldots, P_s, P$ lie on a line. In this case, we get $t = l = \sum_{i=1}^{s} m_i$. By Lemma 3 we can find l lines, say $L_1, \ldots, L_l$, avoiding P such that $L_1 \cdots L_l \in J$. For every $j = 1, \ldots, l$ we can write $L_j = X_0 + H_j$ for a suitable linear form $H_j \in \wp$. Then $X_0^l \in J + \wp$. Let i be any integer, $0 \leq i \leq a - 1$, and let M be a monomial of degree i in X_1, X_2. Then $X_0^l M \in J + \wp^{i+1}$. This implies $X_0^{l+a-1-i} M \in J + \wp^{i+1}$. By Lemma 2 we get

$$r(R/(J + \wp^a)) \leq l + a - 1.$$

Case 2. All of $P_1, \ldots, P_s, P$ don't lie on a line. In this case we can find a point P_k, $2 \leq k \leq s$, such that P, P_1, P_k are not collinear. After a suitable change of coordinates we may further assume that $P = (1, 0, 0)$, $P_1 = (0, 1, 0)$, $P_k = (0, 0, 1)$. Let i be any integer, $0 \leq i \leq a - 1$, and let $M = X_1^{c_1} X_2^{c_2}$, $c_1 + c_2 = i$, be a monomial of degree i in X_1, X_2. Set $m_1' = m_1 - i + c_1$, $m_k' = m_k - i + c_2$, $m_i' = m_i$ if $i \neq 1, k$. Let

$$t' = \max\Big\{l', \Big[\frac{1}{2}\Big(\sum_{i=1}^{s} m_i' + 1\Big)\Big]\Big\},$$

where $l' := \max\Big\{\sum\limits_{j=1}^{r} m'_{i_j} \big| P_{i_1}, \dots, P_{i_r}, P \text{ are collinear}\Big\}$. By Lemma 3 we can find t' lines, say $L_1, \dots, L_{t'}$, avoiding P such that $L_1 \cdots L_{t'} \in J' := \wp_1^{m'_1} \cap \cdots \cap \wp_s^{m'_s}$. Since $M = X_1^{c_1} X_2^{c_2} \in \wp_1^{i-c_1} \cap \wp_k^{i-c_2}$, we get $ML_1 \cdots L_{t'} \in J$. For every $j = 1, \dots, t'$ we can write $L_j = X_0 + H_j$ for a suitable linear form $H_j \in \wp$. Then $X_0^{t'} M \in J + \wp^{i+1}$.

If $t' = l'$, then $t' + i \le l' + a - 1 \le l + a - 1$. If $t' = \Big[\frac{1}{2}\Big(\sum\limits_{i=1}^{s} m'_i + 1\Big)\Big]$, then

$$t' + i = \Big[\frac{1}{2}\Big(\sum_{i=1}^{s} m_i - 2i + c_1 + c_2 + 1\Big)\Big] + i \le \Big[\frac{1}{2}\Big(\sum_{i=1}^{s} m_i + a\Big)\Big].$$

Hence $t' + i \le t$. Therefore, $X_0^{t-i} M \in J + \wp^{i+1}$. By Lemma 2 we get

$$r(R/(J + \wp^a)) \le t.$$

Now we are going to prove Theorem 1.

Proof of Theorem 1. The case $s = 1$ is trivial. For $s > 1$ we will assume that $m_1 \ge \cdots \ge m_s$. Let $J := \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}}$, $I = J \cap \wp_s^{m_s}$. By Lemma 1

$$r(R/I) = \max\big\{m_s - 1, r(R/J), r(R/(J + \wp_s^{m_s}))\big\}.$$

By induction we may assume that

$$r(R/J) \le \max\Big\{h' - 1, \Big[\frac{1}{2}\sum_{i=1}^{s-1} m_i\Big]\Big\},$$

where

$$h' := \max\Big\{\sum_{j=1}^{r} m'_{i_j} \big| P_{i_1}, \dots, P_{i_r} \text{ are collinear, } P_{i_j} \in \{P_1, \dots, P_{s-1}\}\Big\}.$$

By Lemma 4

$$r(R/(J + \wp_s^{m_s})) \le \max\Big\{l + m_s - 1, \Big[\frac{1}{2}\sum_{i=1}^{s} m_i\Big]\Big\},$$

where $l := \max\left\{ \sum_{j=1}^{r} m_{i_j} \,\middle|\, P_{i_1}, \dots, P_{i_r}, P_s \text{ are collinear}\right\}$. Since $h \le h'$ and $l + m_s \le h$ we get

$$\begin{aligned} r(R/I) &\le \max\left\{ m_s - 1, h' - 1, l + m_s - 1, \left[\frac{1}{2}\sum_{i=1}^{s} m_i\right]\right\} \\ &\le \max\left\{ h - 1, \left[\frac{1}{2}\sum_{i=1}^{s} m_i\right]\right\}. \end{aligned}$$

Remark. The above proof for Theorem 1 is much shorter than Fatabbi's proof. From our proof one can also see that in order to get a similar bound in $\mathbf{P}^n$, $n \ge 3$, one has to find an appropriate bound for $r(R/(J + \wp_s^{m_s}))$. This is more or less a purely algebraic problem.

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