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ON $(k+1)$-DIMENSIONAL SPACE-LIKE RULED SURFACE IN THE MINKOWSKI SPACE

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Abstract. In this paper, we introduce the $(k+1)$-dimensional space-like ruled surfaces in Minkowski space R_1^n and obtain interesting results related to asymptotic and tangential bundle of these spaces. Further, we give derivatives equations of these space-like ruled surfaces.

1. Introduction

We shall assume throughout the paper that all manifolds, maps, vector fields, etc... are differentiable of class C^∞. Let R^n be the n-dimensional vector space. The following symmetric, bilinear and non-degenerate metric tensor is called the Lorentz metric on R^n:

$$\langle X, Y\rangle = \sum_{i=1}^{n-1} x_i y_i - x_n y_n, \quad X = (x_1, x_2, \ldots, x_n), \quad Y = (y_1, y_2, \ldots, y_n).$$

R^n together with the Lorentz metric is called the n-dimensional Minkowski space, denoted by R_1^n. Let M be a surface on the n-dimensional Minkowski space R_1^n. If the induced metric on M is positive defined, then M is called the space-like surface. A curve α in R_1^n is space-like curve if $\langle \dot\alpha, \dot\alpha\rangle > 0$, where $\dot\alpha$ is the velocity vector of α. Further, the basic definitions and theorems related to the Minkowski space R_1^n have been found in [3]. The generalized ruled surface in n-dimensional Euclidean space has been studied by H. Frank and O. Giering [1], M. Juzza [2] and C. Thas [4], [5].

The aim of this paper is to define the $(k+1)$-dimensional generalized ruled surface in Minkowski space R_1^n and to obtain the derivatives equation of this space.

2. Space-like ruled surfaces

Let $\{e_1(t), e_2(t), \ldots, e_k(t)\}$ be an orthonormal vector field, which is defined at each point $\alpha(t)$ of a space-like curve of a n-dimensional Minkowski

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space R_1^n. This system spannes at the point $\alpha(t) \in R_1^n$ a k-dimensional subspace of a tangent space $T_{R_1^n}(\alpha(t))$. This subspace is denoted by $E_k(t)$ and is given by

$$E_k(t) = \{e_1(t), e_2(t), \ldots, e_k(t)\}.$$

If the subspace $E_k(t)$ moves along the curve α we obtain a $(k+1)$-dimensional surface in R_1^n. This surface is called a $(k+1)$-dimensional generalized space-like ruled surface of the n-dimensional Minkowski space R_1^n and is denoted by M. The subspace $E_k(t)$ and the space-like curve α are called the generating space and the base curve respectively. For this ruled surface we can give the following parametrization:

$$\phi(t, u_1, u_2, \ldots, u_k) = \alpha(t) + \sum_{i=1}^{k} u_i e_i(t). \tag{2.1}$$

If we take the derivate of ϕ with respect to t and u_i, $1 \leq i \leq k$, we get

$$\begin{aligned} \phi_t &= \dot{\alpha}(t) + \sum_{i=1}^{k} u_i \dot{e}_i(t), \\ \phi_{u_i} &= e_i(t), \quad 1 \leq i \leq k. \end{aligned}$$

Throughout the paper we assume that the system

$$\Big\{\dot{\alpha}(t) + \sum_{i=1}^{k} u_i \dot{e}_i(t), e_1(t), e_2(t), \ldots, e_k(t)\Big\} \tag{2.2}$$

is linear independent and that the subspace $E_k(t)$ is a space-like subspace. The vector subspace

$$Sp\Big\{e_1, e_2, \ldots, e_k, \dot{e}_1, \dot{e}_2, \ldots, \dot{e}_k\Big\}$$

is called the asymptotic bundle of M with respect to $E_k(t)$ and it is denoted by $A(t)$. We have

$$\dim(A(t)) = k + m, \quad 0 \leq m \leq k. \tag{2.3}$$

There exists an orthonormal basis of $A(t)$ which we denote as follows

$$\Big\{e_1(t), e_2(t), \ldots, e_k(t), a_{k+1}(t), a_{k+2}(t), \ldots, a_{k+m}(t)\Big\}.$$

Now there are two possibilities for the asymptotic bundle $A(t)$:
(i) $A(t)$ is a space-like subspace of R_1^n
(ii) $A(t)$ is a time-like subspace of R_1^n.

Consider a fixed point P of M. If P is given by $P=\phi(t,u_1,u_2,\ldots,u_k)$ then a bases of the tangent space in P is given by

$$\Big\{\dot{\alpha}+\sum_{i=1}^{k}u_i\dot{e}_i,e_1,e_2,\ldots,e_k\Big\}.$$

We can define any point P of $E_k(t)$ by changing u_i, $1\le i\le k$ for a fixed value of t. The space

$$Sp\Big\{\dot{\alpha},e_1,e_2,\ldots,e_k,\dot{e}_1,\dot{e}_2,\ldots,\dot{e}_k\Big\} \tag{2.4}$$

includes the union of all the tangent spaces of $E_k(t)$ at a point P. This space is denoted by $T(t)$ and called the tangential bundle of M in $E_k(t)$. It can be easily seen that

$$k+m\le\dim T(t)\le k+m+1,\quad 0\le m\le k. \tag{2.5}$$

In what follow we study seperately the properties of the asymptotic bundle $A(t)$ and of the tangential bundle which depend on their dimension.

We assume that the asymptotic bundle is a space-like subspace. If $\dim(T(t))=k+m$, then $\big\{e_1,e_2,\ldots,e_k,a_{k+1},a_{k+2},\ldots,a_{k+m}\big\}$ is an orthonormal base of $A(t)$ as well as of $T(t)$. Consequently the tangential bundle is space-like subspace. If the dimension of $T(t)$ is equal to $k+m+1$ we find that

$$\dot{\alpha}\notin Sp\big\{e_1,e_2,\ldots,e_k,a_{k+1},a_{k+2},\ldots,a_{k+m}\big\}.$$

In this case,

$$\big\{e_1,e_2,\ldots,e_k,a_{k+1},a_{k+2},\ldots,a_{k+m},a_{k+m+1}\big\} \tag{2.6}$$

is an orthonormal bases of $T(t)$. Since α is a space-like curve we find again that $T(t)$ is a space-like subspace.

Therefore we can give the following result:

Lemma 2.1. *If the asymptotic bundle $A(t)$ of M is a space-like subspace, then the tangential bundle $T(t)$ is also a space-like subspace.*

Now we assume that the asymptotic bundle $A(t)$ is a time-like subspace. If the dimension of $T(t)$ is equal to $k+m$, we get $\{e_1, e_2, \dots, e_k, a_{k+1}, a_{k+2}, \dots, a_{k+m}\}$ as the tangential bundle of $A(t)$ as well as of $T(t)$. That means that $T(t)$ is a time-like subspace.

If the dimension of $T(t)$ is equal to $k+m+1$, then we find an orthnormal base of $T(t)$ by

$$\{e_1, e_2, \dots, e_k, a_{k+1}, a_{k+2}, \dots, a_{k+m}, a_{k+m+1}\}.$$

And hence $T(t)$ is a time-like subspace and we can give the result below:

Lemma 2.2. *If the asymptotic bundle $A(t)$ of M is a time-like subspace, then allways the tangential bundle $T(t)$ is a time-like subspace.*

Theorem 2.3. *Let M be a $(k+1)$-dimensional space-like ruled surface in R_1^n and $E_k(t)$ the generating space of M. We can find an intervall J, such that $t_0 \in J \subset I$ and that then exist a unique orthonormal bases $\{e_1(t_0), e_2(t_0), \dots, e_k(t_0)\}$ of $E_k(t)$ which satisfies:*

$$\langle \dot{\overline{e}}_j, \overline{e}_i \rangle = 0, \quad 1 \le i, j \le k.$$

Proof. Because $E_k(t)$ is a space-like subspace of the Minkowski space R_1^n, we have for the base $\{e_i(t)\}$, $1 \le i \le k$,

$$\langle e_i, e_j \rangle = \delta_{ij}, \quad 1 \le i, j \le k.$$

Let a_{jh}, $1 \le j, h \le k$ be the functions which are defined as solutions of the system of differential equations

$$\dot{a}_{jh} + \sum_{i=1}^{k} a_{ji} \langle \dot{e}_i, e_h \rangle = 0 \tag{2.7}$$

and

$$\overline{e}_j = \sum_{i=1}^{k} a_{ji} e_i.$$

In this case

$$\dot{\overline{e}}_j = \sum_{i=1}^{k} \dot{a}_{ji} e_i + \sum_{i=1}^{k} a_{ji} \dot{e}_i,$$

and therefore we get

$$\big\langle \dot{\overline{e}}_j, e_h \big\rangle = \sum_{i=1}^{k} \dot{a}_{ji} \langle e_i, e_h \rangle + \sum_{i=1}^{k} a_{ji} \big\langle \dot{e}_i, e_h \big\rangle = \dot{a}_{jh} + \sum_{i=1}^{k} a_{ji} \big\langle \dot{e}_i, e_h \big\rangle = 0,$$

$$\big\langle \dot{\overline{e}}_j, \overline{e}_s \big\rangle = \big\langle \dot{\overline{e}}_j, \sum_{h=1}^{k} a_{sh} e_h \big\rangle = \sum_{h=1}^{k} a_{sh} \big\langle \dot{\overline{e}}_j, e_h \big\rangle = 0.$$

As conclusion we find

$$\big\langle \overline{e}_j, \overline{e}_i \big\rangle^{\cdot} = \big\langle \dot{\overline{e}}_j, \overline{e}_i \big\rangle + \big\langle \overline{e}_j, \dot{\overline{e}}_i \big\rangle = 0.$$

If we compute the values of the solutions of (2.7) for we get an orthonormal matrix $\big[a_{jh}(t_0)\big]$ and the base $\big\{\overline{e}_i(t_0)\big\}$, $1 \leq i \leq k$, is orthogonal too. Therefore, for each point t it will be orthogonal, that is the condition

$$\langle \overline{e}_j, \overline{e}_i \rangle = \Big\langle \sum_{i=1}^{k} a_{ji} e_i, \sum_{t=1}^{k} a_{st} e_t \Big\rangle = \sum_{i=1}^{k} a_{ji} a_{st} = \delta_{js}$$

is satiesfied. This yields an orthonormal base with

$$\big\langle \dot{\overline{e}}_j, \overline{e}_s \big\rangle = 0, \quad 1 \leq i, j \leq k.$$

Theorem 2.4. *Let M be a $(k+1)$-dimensional space-like ruled surface, $E_k(t)$ its generating space and $A(t)$ the asymptotic bundle of M. If $A(t)$ is a time-like subspace, then we can find an open interval J such that for the system $\big\{e_1(t), e_2(t), \dots, e_m(t)\big\}$ of an orthonormal bases of $E_k(t)$ the following relations hold:*

$$\big\langle \mathring{e}_i(t), \mathring{e}_j(t) \big\rangle = 0, \quad 1 \leq i, j \leq m,\ i \neq j,$$

$$\begin{aligned} \big\langle \mathring{e}_1(t), \mathring{e}_1(t) \big\rangle > \dots > \big\langle \mathring{e}_{s-1}(t), \mathring{e}_{s-1}(t) \big\rangle &> \big\langle \mathring{e}_{s+1}(t), \mathring{e}_{s+1}(t) \big\rangle \\ &> \dots > \big\langle \mathring{e}_m(t), \mathring{e}_m(t) \big\rangle > 0, \\ &\big\langle \mathring{e}_s(t), \mathring{e}_s(t) \big\rangle < 0, \quad 1 \leq s \leq m, \end{aligned}$$

where $\mathring{e}_i(t)$ is defined by:

$$\mathring{e}_i(t) = \dot{e}_i(t) - \sum_{i=1}^{m} \big\langle \dot{e}(t), e_s(t) \big\rangle e_s(t).$$

Proof. Let

$$e(t)=\sum_{i=1}^{m}\gamma_i(t)e_i(t),\quad \|e(t)\|=1 \tag{2.8}$$

be the constant unit vector and

$$\overset{\circ}{e}(t)=\dot{e}(t)-\sum_{s=1}^{m}\big\langle \dot{e}(t),e_s(t)\big\rangle e_s(t) \tag{2.9}$$

an arbitrary space-like vector. With (2.8) and (2.9) we find

$$\overset{\circ}{e}_i(t)=\dot{e}_i(t)-\sum_{s=1}^{m}\big\langle \dot{e}_i(t),e_s(t)\big\rangle e_s(t) \tag{2.10}$$

and

$$\overset{\circ}{e}(t)=\sum_{i=1}^{m}\gamma_i(t)\overset{\circ}{e}_i(t). \tag{2.11}$$

From this equations we also get

$$e^2(t)=\sum_{i,j=1}^{m}\gamma_i(t)\gamma_j(t)\big\langle \overset{\circ}{e}_i(t),\overset{\circ}{e}_j(t)\big\rangle,\quad t\in J. \tag{2.12}$$

Since the $A(t)$ is a time-like subspace we obtain from (2.10) a bases $\{e_1,e_2,\dots,e_k,\overset{\circ}{e}_1,\overset{\circ}{e}_2,\dots,\overset{\circ}{e}_m\}$ of the asymptotic bundle $A(t)$. Because $A(t)$ is a time-like subspace, one of the vectors $\overset{\circ}{e}_1,\overset{\circ}{e}_2,\dots,\overset{\circ}{e}_m$ is a time-like vector. Let $\overset{\circ}{e}_s$, $1\le s\le m$ be these time-like vectors. Every generating space $E_k(t)$ determines in $S_1^{n-1}\subset R_1^n$ a $S^{k-1}(t)$ unit subsphere. Suppose that for all $t\in J$, the functions $\overset{\circ}{e}{}^2(t)$ has an extremum on $S^{k-1}(t)$. In this case $\big\langle \overset{\circ}{e}_i,\overset{\circ}{e}_i\big\rangle=\varepsilon_i$, $1\le i\le m$, γ_i and $\varepsilon_i\lambda^2$ and with the help of the Lagrange product we get

$$F(t,\gamma_i)=\overset{\circ}{e}{}^2(t,\gamma_i)-\varepsilon_i\lambda^2\big[e^2(t,\gamma_i)-1\big]. \tag{2.13}$$

If we replace (2.8) and (2.12) in this last equation and take the partial derivate of F according to γ_i, then we get

$$F_{\gamma_i}(t)=\sum_{j=1}^{m}\gamma_j\big\langle \overset{\circ}{e}_i,\overset{\circ}{e}_j\big\rangle-\varepsilon_i\lambda^2\gamma_i(t)=0,\quad 1\le i\le m. \tag{2.14}$$

For this homogenous linear system of equations, in $\gamma_1, \gamma_2, \dots, \gamma_m$, we can find at least a value $\lambda^2 \in R$ such that its coefficident matrix is symmetric and singular. Therefore, there exists for all $t \in J$ a nontrivial solution $(\gamma_1, \gamma_2, \dots, \gamma_m)$. Suppose now that for all $t_0 \in I_m \subset J$ the base vector $e_m(t_0)$ of the generating space $E_k(t)$ is a solution of $e(t) = \sum_{i=1}^{m} \gamma_i(t) e_i(t)$ and that in this base vector, $\overset{\circ}{e}{}^2(t, \gamma_i)$ has an absolute minimum on $S^{k-1}(t_0)$. Hence

$$\gamma_1(t_0) = \dots = \gamma_{m-1}(t_0) = 0, \quad \gamma_m(t_0) = 1,$$

and we get

$$\begin{aligned} &\langle \overset{\circ}{e}_1, \overset{\circ}{e}_m \rangle = \dots = \langle \overset{\circ}{e}_s, \overset{\circ}{e}_m \rangle = \dots = \langle \overset{\circ}{e}_{m-1}, \overset{\circ}{e}_m \rangle = 0, \\ &\langle \overset{\circ}{e}_m, \overset{\circ}{e}_m \rangle = \lambda_m^2(t_0) = 0. \end{aligned} \tag{2.15}$$

In a similar way we can do this for $e_{m-1}(t_0)$ and find

$$\gamma_1(t_0) = \dots = \gamma_m(t_0) = 0, \quad \gamma_{m-1}(t_0) = 1,$$

and

$$\begin{aligned} &\langle \overset{\circ}{e}_1, \overset{\circ}{e}_{m-1} \rangle = \dots = \langle \overset{\circ}{e}_s, \overset{\circ}{e}_{m-1} \rangle = \dots = \langle \overset{\circ}{e}_m, \overset{\circ}{e}_{m-1} \rangle = 0, \\ &\langle \overset{\circ}{e}_{m-1}, \overset{\circ}{e}_{m-1} \rangle = \lambda_{m-1}^2(t_0) > 0. \end{aligned}$$

Because $\lambda_m^2(t_0)$ is the absolute minimum on $S^{k-1}(t)$ of $\overset{\circ}{e}{}^2(t, \gamma_i)$ in an intervall $I_m \subset J$ of the covering of I, we get

$$\langle \overset{\circ}{e}_{m-1}, \overset{\circ}{e}_{m-1} \rangle > \langle \overset{\circ}{e}_m, \overset{\circ}{e}_m \rangle > 0.$$

The above method can be applied to all space-like base vectors $e_i(t_0)$ such that on a covering of the intervall $J \subset I$ we get

$$\langle \overset{\circ}{e}_1, \overset{\circ}{e}_1 \rangle > \dots > \langle \overset{\circ}{e}_{s-1}, \overset{\circ}{e}_{s-1} \rangle > \langle \overset{\circ}{e}_{s+1}, \overset{\circ}{e}_{s+1} \rangle > \dots > \langle \overset{\circ}{e}_m, \overset{\circ}{e}_m \rangle > 0.$$

Now let $e_s(t_0)$ in $I_s \subset J \subset I$ a solution vector of $e(t)$ and in the base vector $e_s(t_0)$ the functions $\overset{\circ}{e}{}^2(t, \gamma_i)$ has an absolute minimum on $S^{k-1}(t)$. In this case we find $t_0 \in I_s$ such that

$$\begin{gathered} \gamma_1(t_0) = \dots = \gamma_{s-1}(t_0) = \gamma_{s+1}(t_0) = \dots = \gamma_m(t_0) = 0, \\ \gamma_s(t_0) = 1. \end{gathered} \tag{2.16}$$

This last equation and (2.12) yield

$$\left\langle \overset{\circ}{e}_1, \overset{\circ}{e}_s \right\rangle = \cdots = \left\langle \overset{\circ}{e}_{s-1}, \overset{\circ}{e}_s \right\rangle = \cdots = \left\langle \overset{\circ}{e}_m, \overset{\circ}{e}_s \right\rangle = 0,$$
$$\left\langle \overset{\circ}{e}_s, \overset{\circ}{e}_s \right\rangle = -\lambda_s^2(t_0) = 0.$$

Consequently, we have, in a covering of the interval $J \subset I$,

$$\left\langle \overset{\circ}{e}_i, \overset{\circ}{e}_j \right\rangle = 0, \quad 1 \leq i, j \leq k, \quad i \neq j$$

and

$$\left\langle \overset{\circ}{e}_1, \overset{\circ}{e}_1 \right\rangle > \cdots > \left\langle \overset{\circ}{e}_{s-1}, \overset{\circ}{e}_{s-1} \right\rangle > \left\langle \overset{\circ}{e}_{s+1}, \overset{\circ}{e}_{s+1} \right\rangle > \cdots > \left\langle \overset{\circ}{e}_m, \overset{\circ}{e}_m \right\rangle > 0,$$

$$\left\langle \overset{\circ}{e}_s, \overset{\circ}{e}_s \right\rangle < 0.$$

This completes the proof.

Theorem 2.5. *Let M be a $(k+1)$-dimensional space-like ruled surface and $A(t)$ the asymptotic bundle of M. Let $A(t)$ be a space-like subspace and $\{e_1(t), e_2(t), \ldots, e_k(t)\}$ an orthonormal bases of $E_k(t)$. We can find an open interval J such that for the system $\{e_1(t), e_2(t), \ldots, e_m(t)\}$ the following relations hold:*

$$\left\langle \overset{\circ}{e}_i(t), \overset{\circ}{e}_j(t) \right\rangle = 0, \quad 1 \leq i, j \leq m, \quad i \neq j,$$
$$\left\langle \overset{\circ}{e}_1(t), \overset{\circ}{e}_1(t) \right\rangle > \left\langle \overset{\circ}{e}_2(t), \overset{\circ}{e}_2(t) \right\rangle > \cdots > \left\langle \overset{\circ}{e}_m(t), \overset{\circ}{e}_m(t) \right\rangle > 0,$$

where $\overset{\circ}{e}_i(t)$ is given by

$$\overset{\circ}{e}_i(t) = \dot{e}_i(t) - \sum_{s=1}^{m} \left\langle \dot{e}(t), e_s(t) \right\rangle e_s(t).$$

Proof. Let

$$e(t) = \sum_{i=1}^{m} \gamma_i(t) e_i(t), \quad \|e(t)\| = 1 \tag{2.17}$$

be the constant unit vector and

$$\mathring{e}(t) = \dot{e}(t) - \sum_{s=1}^{m} \langle \dot{e}(t), e_s(t) \rangle e_s(t) \tag{2.18}$$

an arbitrary space-like vector. With (2.17) and (2.18) we find

$$\mathring{e}_i(t) = \dot{e}_i(t) - \sum_{s=1}^{m} \langle \dot{e}_i(t), e_s(t) \rangle e_s(t) \tag{2.19}$$

and

$$\mathring{e}(t) = \sum_{i=1}^{m} \gamma_i(t) \mathring{e}_i(t). \tag{2.20}$$

From this equations we also get

$$e^2(t) = \sum_{i,j=1}^{0\ m} \gamma_i(t)\gamma_j(t) \langle \mathring{e}_i(t), \mathring{e}_j(t) \rangle, \quad t \in J. \tag{2.21}$$

Since the $A(t)$ is a space-like subspace we obtain from (2.19) a bases $\{e_1, e_2, \ldots, e_k, \mathring{e}_1, \mathring{e}_2, \ldots, \mathring{e}_m\}$ of $A(t)$, the asymptotic bundle. Each generating space $E_k(t)$ determines a unit subsphere $S^{k-1}(t)$ on $S_1^{n-1} \subset R_1^n$. Let the functions $\mathring{e}^2(t)$ have an extremum on $S^{k-1}(t)$ for all $t \in J$. In this case, with γ_i, $1 \leq i \leq m$, and λ^2 and with help of the Lagrange product we obtain the following functions

$$F(t, \gamma_i) = \mathring{e}^2(t, \gamma_i) - \lambda^2 \left[e^2(t, \gamma_i) - 1 \right]. \tag{2.22}$$

If we use (2.17) and (2.21) and take the partial derivate of F according to γ_i, $1 \leq i \leq m$, we get

$$F_{\gamma_i}(t) = \sum_{j=1}^{m} \gamma_j \langle \mathring{e}_i, \mathring{e}_j \rangle - \lambda^2 \gamma_i(t) = 0, \quad 1 \leq i \leq m.$$

Now, following the proof of Theorem 2.4 we can complete the proof.

Because of Theorem 2.5 and Theorem 2.6 we can give the following corollary:

Corolary 2.6. *For the asymptotic bundle*

$$A(t) = Sp\{\alpha, e_1, e_2, \dots, e_k, e_1, e_2, \dots, e_k\}$$

we can find an orthonormal bases in the following form:

$$\{e_1, e_2, \dots, e_k, \overset{\circ}{e}_1, \overset{\circ}{e}_2, \dots, \overset{\circ}{e}_m\}, \quad 0 \le m \le k. \tag{2.23}$$

Theorem 2.7. *Let M be a $(k+1)$-dimensional space-like ruled surface in R_1^n with generating space $E_k(t)$ and asymptotic bundle $A(t)$. We can choose an orthonormal bases $\{e_1(t), e_2(t), \dots, e_k(t)\}$ of $E_k(t)$ such that the following relations are held:*

$$\dot{e}_i = \sum_{j=1}^{k} \alpha_{ij} e_j + \kappa_i a_{k+1}, \quad 1 \le i \le m,$$

$$\dot{e}_s = \sum_{j=1}^{k} \alpha_{sj} e_j, \quad m+1 \le s \le k,$$

where $\alpha_{ij} = -\alpha_{ji}$ and $\kappa_1 > \kappa_2 > \dots > \kappa_m > 0$.

Proof. Because of Corollary 2.6 we can find an orthonormal bases of $A(t)$ in the form:

$$\{e_1, e_2, \dots, e_k, \overset{\circ}{e}_1, \overset{\circ}{e}_2, \dots, \overset{\circ}{e}_m\}, \quad 0 \le m \le k.$$

If we define

$$a_{k+1} = \frac{\overset{\circ}{e}_i}{\|\overset{\circ}{e}_i\|}, \quad 1 \le i \le m, \tag{2.24}$$

we can find an orthonormal bases of the asymptotic bundle $A(t)$ in the following form:

$$\{e_1, e_2, \dots, e_k, a_{k+1}, a_{k+2}, \dots, a_{k+m}, a_{k+m+1}\}.$$

Moreover, we can write

$$\dot{e}_i = \sum_{j=1}^{k} \alpha_{ij} e_j + \sum_{v=1}^{m} \sigma_{iv} a_{k+v}, \quad 1 \le i \le m, \tag{2.25}$$

because $\dot{e}_i \in Sp\{e_1, e_2, \ldots, e_k, a_{k+1}, a_{k+2}, \ldots, a_{k+m}\}$. Since $\langle e_i, e_j\rangle = \delta_{ij}$, $1 \leq i, j \leq k$, we get

$$\langle \dot{e}_i, e_j\rangle = -\langle e_i, \dot{e}_j\rangle. \tag{2.26}$$

Therefore we see that $\alpha_{ij} = -\alpha_{ji}$ using (2.25) and (2.26).

From the relations (2.25) we evalute σ_{iv}. Two cases could be appeared:

(i) Let $A(t)$ be a time-like subspace. Then

$$\sigma_{iv} = \varepsilon_v\langle \dot{e}_i, a_{k+v}\rangle, \quad \varepsilon_v = \langle a_{k+v}, a_{k+v}\rangle = \pm 1.$$

If we replace $\dot{e}_i(t)$ by its vector value we get

$$\sigma_{iv} = \varepsilon_v\langle \overset{\circ}{e}_i, a_{k+v}\rangle.$$

Using equation (2.24) in this last equation we obtain

$$\sigma_{iv} = \frac{\varepsilon_v}{\|\overset{\circ}{e}_v\|}\ \langle \overset{\circ}{e}_i, \overset{\circ}{e}_v\rangle.$$

Now we denote $\|\overset{\circ}{e}_v\|$ by κ_v. From Theorem 2.4 we get $\sigma_{ii} = \kappa_i$ and $\kappa_1 > \kappa_2 > \cdots > \kappa_m > 0$.
Therefore, the equation (2.25) yields

$$\dot{e}_i = \sum_{j=1}^{k} \alpha_{ij} e_j + \kappa_i a_{k+i}, \quad 1 \leq i \leq m$$

(ii) Let $A(t)$ be a space-like subspace. In this case,

$$\sigma_{iv} = \varepsilon_v\langle \dot{e}_i, a_{k+v}\rangle.$$

Therefore, if we follow the method of (i) we obtain $\kappa_1 > \kappa_2 > \cdots > \kappa_m > 0$ and

$$\dot{e}_i = \sum_{j=1}^{k} \alpha_{ij} e_j + \kappa_i a_{k+1}, \quad 1 \leq i \leq m.$$

This proves the first part of the theorem.

In both cases, if $A(t)$ is a time-like subspace or a space-like subspace, we get for $s \neq v$, that $\sigma_{sv} = 0$ if we write $m+1 \leq s \leq k$ in the equation (2.25). But this is sufficient for

$$\dot{e}_s(t) = \sum_{j=1}^{k} \alpha_{sj} e_j, \quad m+1 \leq s \leq k.$$

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