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# ON CONTINUITY PROPERTIES OF THE SOLUTION MAP IN QUADRATIC PROGRAMMING

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Abstract. We study in detail the lower semicontinuity and the upper semicontinuity properties of the set-valued map $(D,A,c,b) \mapsto \mathrm{sol}\,(D,A,c,b)$, where $\mathrm{sol}(D,A,c,b)$ denotes the solution set of the quadratic programming problem

$$\text{Minimize } f(x) := c^T x + \tfrac{1}{2} x^T Dx \quad \text{subject to } Ax \geq b,\ x \geq 0.$$

In particular, a complete characterization for the lower semicontinuity of the map $\mathrm{sol}(\cdot)$ is obtained.

## 1. Introduction

Let there be given a matrix $A \in \mathbf{R}^{m\times n}$ and a matrix $D$ from the subspace $\mathbf{R}^{n\times n}_S$ of $\mathbf{R}^{n\times n}$ formed by symmetric square matrices of the order $n$. Let $c \in \mathbf{R}^n$ and $b \in \mathbf{R}^m$. Consider the following quadratic programming problem $(P)$:

$$\begin{aligned} &\text{Minimize} \quad f(x) := c^T x + \frac{1}{2} x^T Dx, \\ &\text{subject to} \quad Ax \geq b, \quad x \geq 0. \end{aligned}$$

(The superscript $^T$ stands for matrix transposition.) Let $\Delta(A,b)$, $\mathrm{sol}\,(D,A,c,b)$, and $S(D,A,c,b)$ denote the constraint set, the solution set, and the Karush-Kuhn-Tucker points set of $(P)$, respectively. That is,

$$\begin{aligned} &\Delta(A,b) = \{x \in \mathbf{R}^n : Ax \geq b,\ x \geq 0\}, \\ &\mathrm{sol}\,(D,A,c,b) = \{x \in \Delta(A,b) : f(x) \leq f(y) \quad \text{for every } y \in \Delta(A,b)\}, \end{aligned}$$

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and $x \in S(D, A, c, b)$ iff there exists $\lambda \in \mathbf{R}^m$ such that

$$Dx - A^T\lambda + c \geq 0, \quad Ax - b \geq 0, \tag{1.1}$$

$$x \geq 0, \quad \lambda \geq 0, \tag{1.2}$$

$$x^T(Dx - A^T\lambda + c) + \lambda^T(Ax - b) = 0. \tag{1.3}$$

In [14], [15] we have studied the upper semicontinuity of the set-valued map

$$(D, A, c, b) \mapsto S(D, A, c, b). \tag{1.4}$$

In this paper we will examine in detail the upper semicontinuity and the lower semicontinuity properties of the solution map

$$(D, A, c, b) \mapsto \mathrm{sol}\,(D, A, c, b). \tag{1.5}$$

If $D$ is positive semidefinite then $f(x)$ is a convex function,

$$\mathrm{sol}\,(D, A, c, b) = S(D, A, c, b),$$

and $(P)$ is a convex quadratic programming problem. For convex quadratic programming problems, continuity and/or differentiability properties of the solution map have been discussed, for example, in [2], [3], [4], [6], [7], [8], [13].

When $D$ is not assumed to be positive semidefinite, the objective function $f(x)$ may be nonconvex, and it may happen that $\mathrm{sol}\,(D, A, c, b) \neq S(D, A, c, b)$. For this general situation, results on characterizing continuity and differentiability properties of the set-valued map (1.5) seem to be very limited. We only know a sufficient condition for upper semicontinuity of the solution map given in [10].

Section 2 of this paper is devoted to the study of the upper semicontinuity of the map (1.5). In Section 3 we obtain a complete characterization of the lower semicontinuity property of the solution map, our main result.

In what follows, the scalar product and the Euclidean norm in a finite dimensional Euclidean space are denoted by $\langle \cdot T \cdot \rangle$ and $||\cdot||$, respectively. Vectors in Euclidean spaces are interpreted as columns-vectors. The notation $x \geq y$ (resp., $x > y$) means that every component of $x$ is greater or equal (resp., greater) the corresponding component of $y$. For each $A \in \mathbf{R}^{m\times n}$,

$$||A|| := \max\{||Ax|| : x \in \mathbf{R}^n, \ ||x|| = 1\}.$$

For a matrix $D \in \mathbf{R}_S^{n\times n}$ the norm $||D||$ is defined similarly. Finally, let $E$ denote the unit matrix in $\mathbf{R}_S^{n\times n}$, and let the symbol $|X|$ stand for the number of elements in $X$ if $X$ is a finite set.

## 2. Upper semicontinuity of the solution map

The following is an adaptation of Definition 1.4.1 of [1] to the set-valued map (1.5).

**Definition 2.1.** The solution map defined in (1.5) is said to be upper semicontinuous at $(D, A, c, b)$ if for any open set $\Omega \subset \mathbf{R}^n$ containing $\mathrm{sol}(D, A, c, b)$, there exists $\delta > 0$ such that $\mathrm{sol}(D', A', c', b') \subset \Omega$ for every $(D', A', c', b') \in \mathbf{R}_S^{n\times n} \times \mathbf{R}^{m\times n} \times \mathbf{R}^n \times \mathbf{R}^m$ satisfying

$$\max\{||D' - D||, ||A' - A||, ||c' - c||, |b' - b||\} < \delta.$$

For the inequality system

$$Ax \geq b, \quad x \geq 0, \tag{2.1}$$

the notion of regularity from [11] (p.755) can be stated equivalently as follows.

**Definition 2.2.** The system (2.1) is called regular if there exists $x_0 \in \mathbf{R}^n$ such that

$$Ax_0 > b, \quad x_0 \geq 0.$$

The next result is due to Nhan ([10], Theorem 3.4).

**Theorem 2.1.** *Assume that:*
($a_1$) *sol*$(D, A, 0, 0) = \{0\}$,
($a_2$) *the system* (2.1) *is regular.*
*Then, for any* $c \in \mathbf{R}^n$, *the map* $\mathrm{sol}\,(\cdot)$ *is upper semicontinuous at* $(D, A, c, b)$.

**Corollary 2.1.** *If the system* (2.1) *is regular and if the set* $\Delta(A, b)$ *is bounded, then* $\mathrm{sol}\,(\cdot)$ *is upper semicontinuous at* $(D, A, c, b)$.

*Proof.* Since the system (2.1) is regular, $\Delta(A, b)$ is nonempty. Besides, since

$$\Delta(A, b) + \Delta(A, 0) \subset \Delta(A, b),$$

$\Delta(A, 0)$ is a cone and $\Delta(A, b)$ is bounded, one has $\Delta(A, 0) = \{0\}$. Hence $\mathrm{sol}\,(D, A, 0, 0) = \{0\}$, and the desired property follows from Theorem 2.1.

*Remark 2.1.* Condition $(a_1)$ is equivalent to that $x^T Dx > 0$ for every $x \in \Delta(A,0) \setminus \{0\}$, i.e. the quadratic form $x^T Dx$ is strictly copositive on the cone $\Delta(A,0)$.

The next statement is a complement to Theorem 2.1.

**Theorem 2.2.** *Assume that:*

$(b_1)$ $S(D,A,0,0) = \{0\}$,

$(b_2)$ *the system* $Ax \geq 0$, $x \geq 0$ *is regular.*

*Then, for any* $(c,b) \in \mathbf{R}^n \times \mathbf{R}^m$, *the map* $\mathrm{sol}\,(\cdot)$ *is upper semicontinuous at* $(D,A,c,b)$.

*Proof.* Suppose that the assertion of the theorem is false. Then there is a pair $(c,b) \in \mathbf{R}^n \times \mathbf{R}^m$ such that there exist an open set $\Omega$ containing $\mathrm{sol}\,(D,A,c,b)$, a sequence $\{(D_k,A_k,c_k,b_k)\}$ converging to $(D,A,c,b)$, and a sequence $\{x_k\}$ such that

$$x_k \in \mathrm{sol}\,(D_k,A_k,c_k,b_k) \setminus \Omega \quad \text{for every } k \in \mathbf{N}.$$

If the norms $||x_k||$ $(k \in \mathbf{N})$ are bounded, then, without loss of generality, we may assume that $x_k \to x_0$ for some $x_0 \in \mathbf{R}^n$. Fix any $x \in \Delta(A,b)$. By $(b_2)$ and Theorem 1 of [11], there exists a sequence $\xi_k \in \Delta(A_k,b_k)$ converging to $x$ as $k \to \infty$. Since $x_k \in \mathrm{sol}\,(D_k,A_k,c_k,b_k)$, we have

$$c_k^T x_k + \frac{1}{2} x_k^T D_k x_k \leq c_k^T \xi_k + \frac{1}{2} \xi_k^T D_k \xi_k.$$

Letting $k \to \infty$ we get

$$c^T x_0 + \frac{1}{2} x_0^T D x_0 \leq c^T x + \frac{1}{2} x^T D x,$$

which shows that $x_0 \in \mathrm{sol}\,(D,A,c,b) \subset \Omega$. We have arrived at a contradiction, because $x_k \notin \Omega$ for all $k$ and $\Omega$ is open.

Now assume that the norms $||x_k||$ $(k \in \mathbf{N})$ are unbounded. By taking a subsequence, if necessary, we may assume that $||x_k|| \to \infty$. According to the first-order necessary optimality condition for quadratic programs (see [9], p. 491) and since $x_k \in \mathrm{sol}\,(D_k,A_k,c_k,b_k)$, for each $k$ there exists $\lambda_k \in \mathbf{R}^m$ such that

$$D_k x_k - A_k^T \lambda_k + c_k \geq 0, \quad A_k x_k - b_k \geq 0, \tag{2.2}$$

$$x_k \geq 0, \quad \lambda_k \geq 0, \tag{2.3}$$

$$x_k^T (D_k x_k - A_k^T \lambda_k + c_k) + \lambda_k^T (A_k x_k - b_k) = 0. \tag{2.4}$$

Since $||(x_k, \lambda_k)|| = \left(||x_k||^2 + ||\lambda_k||^2\right)^{1/2} \to \infty$, we may assume, without loss of generality, that $||(x_k, \lambda_k)|| \neq 0$ for all $k$ and that the sequence of vectors

$$\frac{(x_k, \lambda_k)}{||(x_k, \lambda_k)||} = \left(\frac{x_k}{||(x_k, \lambda_k)||}, \frac{\lambda_k}{||(x_k, \lambda_k)||}\right)$$

converges to some $(\bar{x}, \bar{\lambda}) \in \mathbf{R}^n \times \mathbf{R}^m$ with $||(\bar{x}, \bar{\lambda})|| = 1$. Dividing both sides of (2.2) and of (2.3) by $||(x_k, \lambda_k)||$, dividing both sides of (2.4) by $||(x_k, \lambda_k)||^2$, and taking the limits as $k \to \infty$ we obtain

$$D\bar{x} - A^T\bar{\lambda} \geq 0, \quad A\bar{x} \geq 0, \tag{2.5}$$

$$\bar{x} \geq 0, \quad \bar{\lambda} \geq 0, \tag{2.6}$$

$$\bar{x}^T(D\bar{x} - A^T\bar{\lambda}) + \bar{\lambda}^T A\bar{x} = 0. \tag{2.7}$$

The system (2.5)–(2.7) proves that $\bar{x} \in S(D, A, 0, 0)$. By $(b_1)$, $\bar{x} = 0$. Hence

$$-A^T\bar{\lambda} \geq 0, \quad \bar{\lambda} \geq 0. \tag{2.8}$$

Combining (2.8) and $(b_2)$ yields $\bar{\lambda} = 0$ (see Lemma 2.1 in [15]), hence $||(\bar{x}, \bar{\lambda})|| = 0$, a contradiction. The proof is complete.

*Remark 2.2.* Since $\Delta(A, b) + \Delta(A, 0) \subset \Delta(A, b)$, then $(b_2)$ implies $(a_2)$ if $\Delta(A, b)$ is nonempty. However, $(b_1)$ does not imply $(a_1)$.

Observe that neither $(a_1)$ nor $(a_2)$ is a necessary condition for the upper semicontinuity of the solution map $\mathrm{sol}\,(\cdot)$ at a given point $(D, A, c, b)$.

**Example 2.1.** Let $n = m = 1$, $D = [0]$, $A = [1]$, $c = 1$, $b = 1$. It can be easily verified that $\mathrm{sol}\,(D, A, c, b) = \{1\}$ and the map $\mathrm{sol}\,(\cdot)$ is upper semicontinuous at $(D, A, c, b)$. Since $\mathrm{sol}\,(D, A, 0, 0) = \{x \in \mathbf{R} : x \geq 0\}$, $(a_1)$ fails to hold.

**Example 2.2.** Let $n = m = 1$, $A = [-1]$, $b = 0$. If $A' = [-1 + \alpha]$, $b' = \beta$, where $\alpha$ and $\beta$ are sufficiently small. Then

$$\Delta(A', b') = \left\{x \in \mathbf{R} : 0 \leq x \leq \frac{-\beta}{1 - \alpha}\right\} \cdot$$

It is easily seen that for arbitrarily chosen $D$ and $c$, the map $\mathrm{sol}\,(\cdot)$ is upper semicontinuous at $(D, A, c, b)$, while condition $(a_2)$ does not hold.

## 3. Lower semicontinuity of the solution map

Specializing the notion of lower semicontinuous set-valued map to the solution map (1.5) we have the following

**Definition 3.1.** The solution map $(D, A, c, b) \mapsto \mathrm{sol}\,(D, A, c, b)$ is said to be lower semicontinuous at $(D, A, c, b)$ if $\mathrm{sol}\,(D, A, c, b) \neq \emptyset$ and, for each open set $\Omega \subset \mathbf{R}^n$ satisfying $\mathrm{sol}\,(D, A, c, b) \cap \Omega \neq \emptyset$, there exists $\delta > 0$ such that $\mathrm{sol}\,(D', A', c', b') \cap \Omega \neq \emptyset$ for every $(D', A', c', b') \in \mathbf{R}^{n\times n}_S \times \mathbf{R}^{m\times n} \times \mathbf{R}^n \times \mathbf{R}^m$ satisfying

$$\max\{||D' - D||, ||A' - A||, ||c' - c||, |b' - b||\} < \delta.$$

The map $\mathrm{sol}\,(\cdot)$ is called continuous at $(D, A, c, b)$ if it is simultaneously upper semicontinuous and lower semicontinuous at that point.

The above definition agrees with the one of [16, pp. 450-451], but differs slightly from the one stated in [1, Definition 1.4.2, p. 39].

Our main result can be stated as follows.

**Theorem 3.1.** *The solution map* $\mathrm{sol}\,(\cdot)$ *of the problem* $(P)$ *is lower semicontinuous at* $(D, A, c, b)$ *if and only if the following three conditions are satisfied:*

(a) *the system* $Ax \geq b$, $x \geq 0$ *is regular,*

(b) $\mathrm{sol}\,(D, A, 0, 0) = \{0\}$,

(c) $|\mathrm{sol}\,(D, A, c, b)| = 1$.

For proving Theorem 3.1 we need some lemmas.

**Lemma 3.1.** *If* $\mathrm{sol}\,(\cdot)$ *is lower semicontinuous then the system* $Ax \geq b$, $x \geq 0$ *is regular.*

*Proof.* If the system $Ax \geq b$, $x \geq 0$ is irregular, then according to [12 Lemma 3, p. 439], there exists a sequence $(A_k, b_k) \in \mathbf{R}^{m\times n} \times \mathbf{R}^m$ tending to $(A, b)$ such that $\Delta(A_k, b_k) = \emptyset$ for each $k$. Therefore, $\mathrm{sol}\,(D, A_k, c, b_k) = \emptyset$ for each $k$, contrary to the assumed lower semicontinuity of the solution map.

**Lemma 3.2.** *If the set-valued map* $\mathrm{sol}\,(\cdot)$ *is lower semicontinuous at* $(D, A, c, b)$ *then* $\mathrm{sol}\,(D, A, 0, 0) = \{0\}$.

*Proof.* To the contrary, assume that $\mathrm{sol}\,(D, A, 0, 0) \neq \{0\}$. Then there is a nonzero vector $\bar{x} \in \mathbf{R}^n$ such that

$$A\bar{x} \geq 0, \quad \bar{x} \geq 0, \quad \bar{x}^T D\bar{x} \leq 0. \tag{3.1}$$

Since $\Delta(A, b) \neq \emptyset$, it follows from (3.1) and the inclusion $\Delta(A, b) + \Delta(A, 0) \subset \Delta(A, b)$ that $\Delta(A, b)$ is unbounded. For every $\varepsilon > 0$, we get from (3.1) that $\bar{x}^T(D - \varepsilon E)\bar{x} < 0$. Hence, for any $x \in \Delta(A, b)$,

$$f(x + t\bar{x}) = c^T(x + t\bar{x}) + \frac{1}{2}(x + t\bar{x})^T(D - \varepsilon E)(x + t\bar{x}) \to -\infty$$

as $t \to \infty$. Thus, $\mathrm{sol}\,(D - \varepsilon E, A, c, b) = \emptyset$. This contradicts our assumption that $\mathrm{sol}\,(\cdot)$ is lower semicontinuous at $(D, A, c, b)$.

**Lemma 3.3.**

(i) *If* $\mathrm{sol}\,(D, A, 0, 0) = \{0\}$ *then for any* $(c, b) \in \mathbf{R}^n \times \mathbf{R}^m$, $\mathrm{sol}\,(D, A, c, b)$ *is a compact set.*

(ii) *If* $\mathrm{sol}\,(D, A, 0, 0) = \{0\}$ *and if* $\Delta(A, b)$ *is nonempty, then* $\mathrm{sol}\,(D, A, c, b)$ *is nonempty for every* $c \in \mathbf{R}^n$*.*

*Proof.* (i) Suppose that $\mathrm{sol}\,(D, A, 0, 0) = \{0\}$, but $\mathrm{sol}\,(D, A, c, b)$ is unbounded for some $(c, b)$. Then there is a sequence $\{x_k\} \subset \mathrm{sol}\,(D, A, c, b)$ such that $||x_k|| \to \infty$ as $k \to \infty$. Fixing any $x \in \Delta(A, b)$, one has

$$c^T x_k + \frac{1}{2}x_k^T D x_k \leq c^T x + \frac{1}{2}x^T D x, \tag{3.2}$$

$$A x_k \geq b, \quad x_k \geq 0. \tag{3.3}$$

Without loss of generality we may assume that the sequence $||x_k||^{-1}x_k$ converges to some $\bar{x}$ with $||\bar{x}|| = 1$. Using (3.2) and (3.3) it is easy to show that $\bar{x}^T D\bar{x} \leq 0$, $A\bar{x} \geq 0$, $\bar{x} \geq 0$. This contradicts the fact that $\mathrm{sol}\,(D, A, 0, 0) = \{0\}$. We have thus proved that $\Delta := \mathrm{sol}\,(D, A, c, b)$ is a bounded set. Fix any $\bar{x} \in \Delta$. Since $\Delta = \{x \in \Delta(A, b) : f(x) = f(\bar{x})\}$, $\Delta$ is closed. Hence $\Delta$ is a compact set.

(ii) Let $\mathrm{sol}\,(D, A, 0, 0) = \{0\}$, $\Delta(A, b) \neq \emptyset$, and let $c \in \mathbf{R}^n$ be given arbitrarily. If the quadratic form $f(x) = c^T x + \frac{1}{2}x^T D x$ is bounded below on the polyhedron $\Delta(A, b)$, then by the Frank-Wolfe theorem (see [6, Theorem 2.8.1]), the solution set $\mathrm{sol}\,(D, A, c, b)$ is nonempty. Now assume that there exists a sequence $x_k \in \Delta(A, b)$ such that $f(x_k) \to -\infty$ as $k \to \infty$. By taking a subsequence, if necessary, we may assume that

$$c_k^T x_k + \frac{1}{2}x_k^T D x_k \leq 0 \tag{3.4}$$

for all $k$, $||x_k|| \to \infty$ and that $||x_k||^{-1}x_k$ converges to some $\bar{x}$ as $k \to \infty$. It is a simple matter to show that $\bar{x} \in \Delta(A, 0)$. Dividing both sides of

(3.4) by $||x_k||^2$ and letting $k \to \infty$ one gets $\bar{x}^T D\bar{x} \le 0$. As $||\bar{x}|| = 1$, $\mathrm{sol}\,(D, A, 0, 0) \ne \{0\}$, which is impossible.

**Lemma 3.4.** *If the map* $\mathrm{sol}\,(\cdot)$ *is lower semicontinuous at* $(D, A, c, b)$ *then the set* $\mathrm{sol}\,(D, A, c, b)$ *is finite.*

*Proof.* We define $N = n + m$,

$$M = \begin{pmatrix} D & -A^T \\ A & 0 \end{pmatrix}, \quad q = \begin{pmatrix} c \\ -b \end{pmatrix},$$

and consider the problem of finding vectors $z = \begin{pmatrix} x \\ \lambda \end{pmatrix} \in \mathbf{R}^N$ satisfying

$$Mz + q \ge 0, \quad z \ge 0, \quad z^T(Mz + q) = 0. \tag{3.5}$$

For a nonempty subset $\alpha \subset \{1, 2, \dots, N\}$, the principal submatrix $\left(m_{ij}\right)_{i,j\in\alpha}$ of $M = \left(m_{ij}\right)_{1\le i,j\le N}$ is denoted by $M_\alpha$. For a vector $p \in \mathbf{R}^N$, the column-vector with the components $\left(p_i\right)_{i\in\alpha}$ is denoted by $p_\alpha$. Let $z = (z_1, z_2, \dots, z_N)^T$ be a nonzero solution of the linear complementary problem (3.5). Let $J = \{j : z_j = 0\}$, $I = \{i : z_i > 0\}$. Since $z_J = 0$ and $(Mz + q)_I = 0$, then $M_I z_I = -q_I$. Therefore, if $\det M_I \ne 0$ then $z$ is defined uniquely via $q$ by the formulas

$$z_J = 0, \quad z_I = -M_I^{-1}(q_I). \tag{3.6}$$

Given any nonempty subset $I \subset \{1, \dots, n\}$ we define

$$\mathcal{Q}_I := \{q \in \mathbf{R}^N : -q_I = M_I z_I \text{ for some } z \in \mathbf{R}^N\}.$$

If $\det M_I = 0$ then $\mathcal{Q}_I$ is a proper subspace of $\mathbf{R}^N$. In particular, $\mathcal{Q}_I$ is nowhere dense in $\mathbf{R}^N$. By Baire's Lemma ([5, p.15]), the union $\mathcal{Q} := \cup\{\mathcal{Q}_I : I \subset \{1, 2, \dots, N\},\ I \ne \emptyset,\ \det M_I = 0\}$ is nowhere dense. Hence, there exists a sequence $q^{(k)} = \begin{pmatrix} c^{(k)} \\ -b^{(k)} \end{pmatrix}$ converging to $q = \begin{pmatrix} c \\ -b \end{pmatrix}$ such that $q^{(k)} \notin \mathcal{Q}$ for all $k$.

Now, fix any $x \in \Delta = \mathrm{sol}\,(D, A, c, b)$ and let $\varepsilon > 0$ be given arbitrarily. Since $\mathrm{sol}\,(\cdot)$ is lower semicontinuous at $(D, A, c, b)$, there exists $\delta_\varepsilon > 0$ such that

$$x \in \mathrm{sol}\,(D, A, c', b') + \varepsilon B_{\mathbf{R}^n}$$

for all $(c', b')$ satisfying $\max\{||c' - c||, ||b' - b||\} < \delta_\varepsilon$, where $B_{\mathbf{R}^n}$ denotes the closed unit ball in $\mathbf{R}^n$. Hence, for each $k$ sufficiently large, there exists $x^{(k)} \in \mathrm{sol}\,(D, A, c^{(k)}, b^{(k)})$ such that

$$||x - x^{(k)}|| \leq \varepsilon. \tag{3.7}$$

Since $x^{(k)} \in \mathrm{sol}\,(D, A, c^{(k)}, b^{(k)})$, there exists $\lambda^{(k)}$ such that $z^{(k)} := \begin{pmatrix} x^{(k)} \\ \lambda^{(k)} \end{pmatrix}$ is a solution of the linear complementary problem

$$Mz + q^{(k)} \geq 0, \quad z \geq 0, \quad z^T(Mz + q^{(k)}) = 0.$$

(See [9, p. 491]). Let $J_k = \{j : z_j^{(k)} = 0\}$, $I_k = \{i : z_i^{(k)} > 0\}$. If $I_k = \emptyset$ then $z^{(k)} = 0$. If $I_k \neq \emptyset$, then $\det M_{I_k} \neq 0$ because $q^{(k)} \notin \mathcal{Q}$. Consequently,

$$z_{J_k}^{(k)} = 0, \quad z_{I_k}^{(k)} = -M_{I_k}^{-1}\left(q_{I_k}^{(k)}\right). \tag{3.8}$$

Since the set $\{1, 2, \ldots, N\}$ has only $2^N$ subsets, then one can find a subset $I \subset \{1, 2, \ldots, N\}$ and a subsequence $\{k_i\} \subset \{k\}$ such that $I_{k_i} = I$ for all $k_i$. Let $\mathcal{Z}$ denote the set of all $z \in \mathbf{R}^N$ such that there exists a nonempty subset $I \subset \{1, \ldots, N\}$ with the property that $\det M_I \neq 0$, $z_I = -M_I^{-1}(q_I)$ and $z_J = 0$, where $J := \{1, \ldots, N\} \setminus I$. It is clear that $\mathcal{Z}$ is finite. It follows from (3.8) that the sequence $z_{I_{k_i}}^{(k_i)}$ is convergent and that the limit belongs to the finite set $\tilde{\mathcal{Z}} := \mathcal{Z} \cup \{0\}$. For every $z = \begin{pmatrix} \xi \\ \lambda \end{pmatrix}$ we put $pr_1(z) = \xi$. Since $pr_1(z^{(k_i)}) = x^{(k_i)}$, from what has been said it follows that the sequence $\{x^{(k_i)}\}$ has a limit $\bar{\xi}$ in the finite set $\tilde{\mathcal{X}} := \{pr_1(z) : z \in \tilde{\mathcal{Z}}\}$. By (3.7), $x \in \tilde{\mathcal{X}} + \varepsilon B_{\mathbf{R}^n}$. Since $\varepsilon > 0$ can be arbitrarily small, $x$ must belong to $\tilde{\mathcal{X}}$. We have thus shown that $\Delta = \mathrm{sol}\,(D, A, c, b) \subset \tilde{\mathcal{X}}$. Hence $\Delta$ is a finite set, and the lemma follows.

**Lemma 3.5.** *The set* $\mathcal{G} := \{(D, A) : \mathrm{sol}\,(D, A, 0, 0) = \{0\}\}$ *is open in* $\mathbf{R}_S^{n\times n} \times \mathbf{R}^{m\times n}$.

*Proof.* Assume to the contrary that there is a sequence $\{(D_k, A_k)\}$ converging to $(D, A) \in \mathcal{G}$ such that $\mathrm{sol}\,(D_k, A_k, 0, 0) \neq \{0\}$ for all $k$. Then for each $k$ there exists a vector $x_k$ such that $||x_k|| = 1$ and

$$A_k x_k \geq 0, \quad x_k \geq 0, \quad x_k^T D_k x_k \leq 0. \tag{3.9}$$

Without loss of generality we may assume that $\{x_k\}$ converges to some $x_0$ with $||x_0|| = 1$. Taking the limits in (3.9) as $k \to \infty$, we obtain

$$Ax_0 \geq 0, \quad x_0 \geq 0, \quad x_0^T D x_0 \leq 0.$$

This contradicts the assumption that $\mathrm{sol}\,(D, A, 0, 0) = \{0\}$.

For each subset $\alpha \subset \{1, \dots, m\}$ with the complement $\bar{\alpha}$ and $\beta \subset \{1, \dots, n\}$ with the complement $\bar{\beta}$ let

$$F(\alpha, \beta) := \{x \in \mathbf{R}^n : (Ax)_\alpha > b_\alpha,\ (Ax)_{\bar{\alpha}} = b_{\bar{\alpha}},\ x_\beta > 0,\ x_{\bar{\beta}} = 0\}.$$

Obviously,

$$\Delta(A, b) = \bigcup_{(\alpha,\beta)} F(\alpha, \beta).$$

Besides, for every $x \in \Delta(A, b)$ there exists a unique pair $(\alpha, \beta)$ such that $x \in F(\alpha, \beta)$. In addition, if $(\alpha, \beta) \neq (\alpha', \beta')$ then $F(\alpha, \beta) \cap F(\alpha', \beta') = \emptyset$.

**Lemma 3.6.** *If the solution set* $\mathrm{sol}\,(D, A, c, b)$ *is finite, then for any* $\alpha \subset \{1, \dots, m\}$ *and for any* $\beta \subset \{1, \dots, n\}$ *we have*

$$|\mathrm{sol}\,(D, A, c, b) \cap F(\alpha, \beta)| \leq 1. \tag{3.10}$$

*Proof.* For every $x \in \Delta(A, b)$ we put

$$I(x) = \{i : (Ax)_i = b_i\}, \quad J(x) = \{j : x_j = 0\}.$$

The cone

$$\mathcal{F}_x = \{v \in \mathbf{R}^n : (Av)_i \geq 0,\ v_j \geq 0 \text{ for all } i \in I(x),\ j \in J(x)\}$$

is the tangent cone of $\Delta(A, b)$ at $x$. By Theorem 2.8.4 of [6], a point $x \in \Delta(A, b)$ is a local minimum of $(P)$ if and only if

(i) $(Dx + c)^T v \geq 0$ for every $v \in \mathcal{F}_x$,

(ii) If $v \in \mathcal{F}_x$ and $(Dx + c)^T v = 0$, then $v^T D v \geq 0$.

We now suppose that for some $\alpha \subset \{1, \dots, m\}$ and $\beta \subset \{1, \dots, n\}$ the set $\mathrm{sol}\,(D, A, c, b) \cap F(\alpha, \beta)$ contain two distinct elements $\bar{x}$, $\bar{y}$. Since $\bar{x}, \bar{y} \in F(\alpha, \beta)$, $I(\bar{x}) = I(\bar{y})$ and $J(\bar{x}) = J(\bar{y})$, hence $\mathcal{F}_{\bar{x}} = \mathcal{F}_{\bar{y}}$. For any $t \in [0, 1]$, since $F(\alpha, \beta)$ is convex, $x_t := t\bar{x} + (1-t)\bar{y} \in F(\alpha, \beta)$. Therefore, $I(\bar{x}) = I(\bar{y}) = I(x_t)$, $J(\bar{x}) = J(\bar{y}) = J(x_t)$, and $\mathcal{F}_{\bar{x}} = \mathcal{F}_{\bar{y}} = \mathcal{F}_{x_t}$.

Fix any $t \in (0,1)$. Let $v := \bar{y} - \bar{x}$. Since $v \in \mathcal{F}_{\bar{x}}$ and $-v \in \mathcal{F}_{\bar{y}}$, $v$ and $-v$ belong to $\mathcal{F}_{x_t}$. Since $\bar{x}$ and $\bar{y}$ are solutions of $(P)$,

$$(D\bar{x} + c)^T v = (D\bar{y} + c)^T v = 0. \tag{3.11}$$

This implies that

$$(Dx_t + c)^T v = t(D\bar{x} + c)^T v + (1-t)(D\bar{y} + c)v = 0. \tag{3.12}$$

As $\bar{x}$ is a solution of $(P)$, (3.11) implies that $v^T Dv \geq 0$. Noting that $\bar{y} - x_t = \tau v$ for some $\tau > 0$, one deduces from the last inequality and (3.12) that

$$\begin{aligned} f(\bar{y}) - f(x_t) &= (Dx_t + c)^t(\tau v) + \frac{1}{2}(\tau v)^T D(\tau v) \\ &= \tau^2 v^T Dv \geq 0. \end{aligned}$$

Then we get $x_t \in \operatorname{sol}(D, A, c, b)$ for all $t \in (0,1)$, which contradicts the finiteness of $\operatorname{sol}(D, A, c, b)$.

**Lemma 3.7.** *If the map* $\operatorname{sol}(\cdot)$ *is lower semicontinuous at* $(D, A, c, b)$ *then*

$$|\operatorname{sol}(D, A, c, b)| = 1.$$

*Proof.* On the contrary, suppose that in $\operatorname{sol}(D, A, c, b)$ we can find two distinct vectors $\bar{x}$, $\bar{y}$. Let $J(\bar{x}) = \{j : \bar{x}_j = 0\}$, $J(\bar{y}) = \{j : \bar{y}_j = 0\}$.

If $J(\bar{x}) \neq J(\bar{y})$, then there exists $j_0$ such that $\bar{x}_{j_0} = 0$ and $\bar{y}_{j_0} > 0$, or there exists $j_1$ such that $\bar{x}_{j_1} > 0$ and $\bar{y}_{j_1} = 0$. By symmetry, it is enough to consider the first case. As $\bar{y} \in \operatorname{sol}(D, A, c, b)$ and $y_{j_0} > 0$, there is an open neighborhood $U$ of $\bar{y}$ such that $f(y) \geq f(\bar{y})$ and $y_{j_0} > 0$ for every $y \in U$. Fix any $\varepsilon > 0$ and put $c(\varepsilon) = (c_i(\varepsilon))$, where

$$c_i(\varepsilon) = \begin{cases} c_i & \text{if } \ i \neq j_0 \\ c_i + \varepsilon & \text{if } \ i = j_0. \end{cases}$$

Let $f_\varepsilon(x) = f(x) + \varepsilon x_{j_0}$, where, as before, $f(x) = c^T x + \dfrac{1}{2} x^T Dx$. Consider the quadratic program

$$\text{Minimize } f_\varepsilon(x) \quad \text{subject to } x \in \Delta(A, b),$$

whose solution set is $\mathrm{sol}\,(D, A, c(\varepsilon), b)$. For every $y \in U$, we have

$$f_\varepsilon(y) = f(y) + \varepsilon y_{j_0} > f(y) \geq f(\bar{y})$$
$$= f(\bar{x}) = f_\varepsilon(\bar{x}).$$

Hence $y \notin \mathrm{sol}\,(D, A, c(\varepsilon), b)$, so

$$\mathrm{sol}\,(D, A, c(\varepsilon), b) \cap U = \emptyset. \tag{3.13}$$

Since $\varepsilon > 0$ can be arbitrarily small, (3.13) contradicts our assumption that $\mathrm{sol}\,(\cdot)$ is lower semicontinuous at $(D, A, c, b)$.

Now suppose that $J(\bar{x}) = J(\bar{y})$. Let $\alpha$ and $\alpha'$ be the index sets such that

$$\bar{x} \in F(\alpha, \beta), \quad \bar{y} \in F(\alpha', \beta),$$

where $\beta$ is the complement of $J(\bar{x}) = J(\bar{y})$ in $\{1, \ldots, n\}$. By Lemma 3.4, $\mathrm{sol}\,(D, A, c, b)$ is a finite set. Then, by Lemma 3.6, $\alpha \neq \alpha'$. Hence at least one of the sets $\alpha \setminus \alpha'$ and $\alpha' \setminus \alpha$ must be nonempty. By symmetry, it suffices to consider the first case. Let $i_0 \in \alpha \setminus \alpha'$. Then we have

$$(A\bar{x})_{i_0} > b_{i_0}, \quad (A\bar{y})_{i_0} = b_{i_0}. \tag{3.14}$$

As $\mathrm{sol}\,(D, A, c, b)$ is finite, one can find a neighborhood $W$ of $\bar{y}$ such that

$$\mathrm{sol}|, (D, A, c, b) \cap W = \{\bar{y}\}. \tag{3.15}$$

Fix any $\varepsilon > 0$ and put $b(\varepsilon) = (b_i(\varepsilon))$, where

$$b_i(\varepsilon) = \begin{cases} b_i & \text{if } \ i \neq i_0 \\ b_i + \varepsilon & \text{if } \ i = i_0. \end{cases}$$

By (3.14) there exists $\delta > 0$ such that $\bar{x} \in \Delta(A, b(\varepsilon))$ for every $\varepsilon \in (0, \delta)$. Since $\Delta(A, b(\varepsilon)) \subset \Delta(A, b)$, then

$$\inf_{x \in \Delta(A, b(\varepsilon))} f(x) \geq \inf_{x \in \Delta(A, b)} f(x) = f(\bar{x}).$$

Therefore, for every $\varepsilon \in (0, \delta)$, $\bar{x} \in \mathrm{sol}\,(D, A, c, b(\varepsilon))$. Moreover,

$$\mathrm{sol}\,(D, A, c, b(\varepsilon)) \subset \mathrm{sol}\,(D, A, c, b).$$

It is clear that $\bar{y} \notin \Delta(A, b(\varepsilon))$. Then we have $\mathrm{sol}\,(D, A, c, b(\varepsilon)) \subset \mathrm{sol}\,(D, A, c, b) \setminus \{\bar{y}\}$. Hence, by (3.15), $\mathrm{sol}\,(D, A, c, b(\varepsilon)) \cap W = \emptyset$ for every $\varepsilon \in (0, \delta)$. This contradicts the lower semicontinuity of $\mathrm{sol}\,(\cdot)$ at $(D, A, c, b)$. Lemma 3.7 is proved.

**Proof of Theorem 3.1.** If $\mathrm{sol}\,(\cdot)$ is lower semicontinuous at $(D, A, c, b)$ then from Lemmas 3.1, 3.2, and 3.7, we get (a), (b), and (c).

Conversely, assume that the conditions (a), (b) and (c) are fulfilled. Let $\Omega$ be an open set containing the unique solution $\bar{x} \in \mathrm{sol}\,(D, A, c, b)$. By (a) and by Lemma 3 of [12], there exists $\delta_1 > 0$ such that $\Delta(A', b') \neq \emptyset$ for every pair $(A', b')$ satisfying $\max\{||A' - A||, ||b' - b||\} < \delta_1$. By (b) and by Lemma 3.5, there exists $\delta_2 > 0$ such that $\mathrm{sol}\,(D', A', 0, 0) = \{0\}$ for every pair $(D', A')$ satisfying $\max\{||D' - D||, ||A' - A||\} \leq \delta_2$. For $\delta := \min\{\delta_1, \delta_2\}$, by the second assertion of Lemma 3.3 we have $\mathrm{sol}\,(D', A', c', b') \neq \emptyset$ for every $(D'A', c', b')$ satisfying

$$\max\{||D' - D||, ||A' - A||, ||c' - c||, ||b' - b||\} < \delta. \tag{3.16}$$

By (a) and (b), it follows from Theorem 2.1 that $\mathrm{sol}\,(\cdot)$ is upper semicontinuous at $(D, A, c, b)$. Hence $\mathrm{sol}\,(D', A', c', b') \subset \Omega$ for every $(D'A', c', b')$ satisfying (3.16) if $\delta > 0$ is small enough. For such a $\delta$ it follows as above that $\mathrm{sol}\,(D', b', c', b') \cap \Omega \neq \emptyset$ for every $(D', A', c', b')$ satisfying (3.16). This shows that $\mathrm{sol}\,(\cdot)$ is lower semicontinuous at $(D, A, c, b)$. The proof is complete.

The following fact follows directly from Theorems 3.1 and 2.1.

**Corollary 3.1.** *If the map* $\mathrm{sol}\,(\cdot)$ *is lower semicontinuous at* $(D, A, c, b)$ *then it is upper semicontinuous at* $(D, A, c, b)$*, hence it is continuous at the point.*

Let us mention two other interesting consequences of Theorem 3.1.

**Corollary 3.2.** *If* $D$ *is a negative semidefinite matrix, then the map sol*$(\cdot)$ *is continuous at* $(D, A, c, b)$ *if and only if the following conditions are satisfied*

(i) *the system* $Ax \geq b,\ x \geq 0$ *is regular,*
(ii) $\Delta(A, b)$ *is a compact set, and*
(iii) $|\mathrm{sol}\,(D, A, c, b)| = 1$.

*Proof.* Assume that $\mathrm{sol}\,(\cdot)$ is lower semicontinuous at $(D, A, c, b)$. By Theorem 3.1, conditions (i) and (iii) are satisfied. Moreover,

$$\mathrm{sol}\,(D, A, 0, 0) = \{0\}. \tag{3.17}$$

We claim that $\Delta(A,0) = \{0\}$. Indeed, by assumption, $x^T Dx \leq 0$ for every $x \in \Delta(A,0)$. If there exists no $\bar{x} \in \Delta(A,0)$ with the property that $\bar{x}^T D\bar{x} < 0$ then $\mathrm{sol}\,(D,A,0,0) = \Delta(A,0)$, and (3.17) forces $\Delta(A,0) = \{0\}$. If $\bar{x}^T D\bar{x} < 0$ for some $\bar{x} \in \Delta(A,0)$, then it is obvious that $\mathrm{sol}\,(D,A,0,0) = \emptyset$, which is impossible. Property (ii) follows easily from the equality $\Delta(A,0) = \{0\}$.

Conversely, suppose that (i), (ii) and (iii) are satisfied. As $\Delta(A,b) \neq \emptyset$ by (i), (ii) implies $\Delta(A,0) = \{0\}$. Therefore, $\mathrm{sol}\,(D,A,0,0) = \{0\}$. Since the conditions (a), (b) and (c) in Theorem 3.1 are satisfied, $\mathrm{sol}\,(\cdot)$ is lower semicontinuous at $(D,A,c,b)$. The proof is complete.

**Corollary 3.3.** *If $D$ is a positive definite matrix, then the map sol$(\cdot)$ is continuous at $(D,A,c,b)$ if and only if the system $Ax \geq b$, $x \geq 0$ is regular.*

*Remarks.* We would like to observe that the proof of Theorem 3.1 can be shortened by using a device shown to us by Hoang Xuan Phu. The argument for proving Lemma 3.4 can be applied for studying the lower semicontinuity property of the set of the Karush-Kuhn-Tucker points, and of the set of the local solutions of $(P)$.

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## References

1. J.-P. Aubin and H. Frankowska, *Set-Valued Analysis,* Birkhäuser, Berlin, 1990.
2. A. Auslender and P. Coutat, *Sensitivity analysis for generalized linear-quadratic problems,* J. Optim. Theory Appl. **88** (1996), 541–559.
3. B. Bank, J. Guddat, D. Klatte, B. Kummer and K. Tammer, *Non-Linear Parametric Optimization,* Akademie-Verlag, Berlin, 1982.
4. M. J. Best and N. Chakravarti, *Stability of linearly constrained convex quadratic programs,* J. Optim. Theory Appl. **64** (1990), 43–53.
5. H. Brezis, *Analyse Fonctionnelle,* 2$^{e}$ tirage, Masson, Paris, 1987.
6. R. W. Cottle, J.-S. Pang and R. E. Stone, *The Linear Complementarity Problem,* Academic Press, New York, 1992.
7. J. W. Daniel, *Stability of the solution of definite quadratic programs,* Math. Programming. **5** (1973), 41–53.
8. J. Guddat, *Stability in convex quadratic parametric programming,* Mathematische Operationsforschung und Statistik **7** (1976), 223–245.
9. K. G. Murty, *Linear and Combinatorial Programming,* John Wiley, New York, 1976.
10. N. H. Nhan, *On the Stability of Linear Complementarity Problems and Quadratic Programming Problems,* Master thesis, University of Hue, Hue, 1995.

11. S. M. Robinson, *Stability theory for systems of inequalities, Part I: Linear systems,* SIAM J. Num. Anal. **12** (1975), 754–769.
12. S. M. Robinson, *A characterization of stability in linear programming,* Operations. Res. **25** (1977), 435–447.
13. S. M. Robinson, *Generalized equations and their solutions, Part I: Basic theory,* Math. Programming Study **10** (1979), 128–141.
14. N. N. Tam, *Further results on stability of the Karush-Kuhn-Tucker points in quadratic programming,* Preprint 96/21, Hanoi Institute of Mathematics, 1996.
15. N. N. Tam and N. D. Yen, *Stability properties of the Karush-Kuhn-Tucker points in some nonconvex quadratic programming problems,* Preprint 96/4, Hanoi Institute of Mathematics, 1996.
16. E. Zeidler, *Nonlinear Functional Analysis and Its Applications. I: Fixed-Point Theorems,* Springer-Verlag, New York, 1986.

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