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# AN AFFINE ALGEBRAIC TYPE OF THE PLÜCKER-MILNOR FORMULA ON $\mathbf{C}^2$

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*Dedicated to the memory of my friend Nobuo Sasakura*

Abstract. The classical Plücker formulas are the relations between the order, class and genus of a given smooth projective algebraic plane curve which were first established by J. Plücker (1834) and later generalized by M. Noether (1875) for projective algebraic plane curves which may have ordinary singularities. A century later, these relations have been extended to the general case of projective algebraic curves by M. Rosenlicht (1952) and to the local case of analytical curves by Milnor (1968). In this note we shall establish such a relation for the affine case of algebraic plane curves.

## 1. Introduction

**1.1.** Let $P : \mathbf{C}^2 \to \mathbf{C}$ be a polynomial of degree $d$. It is well-known that there exists a finite set $\Delta \subset \mathbf{C}$ such that $P : \mathbf{C}^2 \setminus P^{-1}(\Delta) \to \mathbf{C} \setminus \Delta$ is a locally trivial $\mathbf{C}^\infty$-fibration (Thom's theorem). We call the smallest set with this property the *bifurcation set of* $P$ and denote by $B_P$. Beside the set $P(\Sigma)$ of the critical values of $P$ on the set of its critical points $\Sigma \subset \mathbf{C}^2$, the bifurcation set $B_P$ may also contain the set $B_\infty$ of the so called *"critical values at infinity"*. Intuitively, $B_\infty$ is the set of $\tau \in \mathbf{C}$ such that the restriction of $P$ on any small neighbourhood of $P^{-1}(\tau)$ (outside any compact of $\mathbf{C}^2$) is not a trivial fibration. We say that $\tau \in \mathbf{C}$ (resp. $P^{-1}(\tau)$) is a *regular value* (resp. *generic fiber*) of $P$ if $\tau \in \mathbf{C} \setminus B_P$. Otherwise, $\tau \in B_P$ (resp. $P^{-1}(\tau)$) is called an *irregular value* (resp. *non-generic fiber*) of $P$.

**1.2.** We consider the canonical projective compactification $\mathbf{C}^2 \subset \mathbf{P}^2$. Denote the homogeneous coordinates of $\mathbf{P}^2$ by $X, Y, Z$ so that $x = X/Z$,

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$y = Y/Z$ are the original affine coordinates of $\mathbf{C}^2$. Let $L_\infty = \{Z = 0\}$ be the line at infinity in $\mathbf{P}^2$. We write

$$P(x, y) = P_0 + P_1(x, y) + ... + P_d(x, y),$$

where $P_i(x, y)$ is a homogeneous polynomial of degree $i$ for $i = 0, ..., d$. Let $C_\tau$ be the projective closure of the fiber $P^{-1}(\tau)$:

$$C_\tau = \{(X, Y, Z) \in \mathbf{P}^2 : F(X, Y, Z) - \tau Z^d = 0\},$$

where $F(X, Y, Z)$ is the homogeneous polynomial defined by:

$$F(X, Y, Z) = P(X/Z, Y/Z)Z^d = P_0 Z^d + P_1(X, Y)Z^{d-1} + ... + P_d(X, Y).$$

The intersection of $C_\tau$ and the line at infinity $C_\tau \cap L_\infty$ is independent of $\tau \in \mathbf{C}$ and it is the base point locus of the family $\{C_\tau : \tau \in \mathbf{C}\}$. Obviously we have $C_\tau \cap L_\infty = \{Z = P_d(X, Y) = 0\}$. Let $C_\tau \cap L_\infty = \{A_i = (\alpha_i, \beta_i, 0) \in \mathbf{P}^2 : i = 1, \dots, k\}$ be the points at infinity of $P$. We know that $\tau$ is a regular value at infinity ($\tau \in \mathbf{C} \setminus B_\infty$) if and only if for any $i = 1, .., k$ the family $\{(C_t, A_i) : t \in \mathbf{C}\}$ is topologically trivial near $t = \tau$ (cf. [HL]). This is the case if $P(x, y) - \tau$ is reduced and the local Milnor number $\mu_t$ of the family $\{(C_t, A_i) : t \in \mathbf{C}\}$ is constant in a neighbourhood of $\tau \in \mathbf{C}$ for any $i = 1, ..., k$.

## 2. The affine Plücker formula

Recall that if $C \subset \mathbf{P}^2$ is a projective plane curve of degree $d$ then we have the classical Plücker formula established by Rosenlicht (cf. [R], [BK], [S]):

$$2\Sigma_{x \in C}\delta_x(C) = (d-1)(d-2) - 2g,$$

here $g$ is the genus of the Riemann surface of $C$ and $\delta_x(C)$ is the codimension of the local ring $O_{C,x}$ in its integral closure $\bar{O}_{C,x}$ (in the field of the germs at $x \in C$ of the rational functions on $C$):

$$\delta_x(C) = \dim_{\mathbf{C}} \bar{O}_{C,x}/O_{C,x}.$$

For the local case, if $(C, o) \subset (\mathbf{C}^2, o)$ is the germ at $o \in C$ of a local analytical plane curve given by $\{\ (x, y) \in \mathbf{C}^2 : f(x, y) = 0\ \}$, then we have the local Plücker formula established by Milnor (cf. [M]):

$$2\delta_o(C) = \mu_o(C) + r_o(C) - 1,$$

here $\mu_o(C) = \dim_{\mathbf{C}}\mathbf{C}\{x,y\}/(f_x, f_y)$ is the Milnor number of $(C,o)$ (it is the first Betti number of the generic fiber of the Milnor fibration of $f$) and $r_o(C)$ is the number of the local branchs (irreducible components) of $(C,o)$. Note that in this form the local Plücker formula is really a relation between the geometrical characters of the Milnor fibration.

**2.1.** We fix $\tau_0 \in \mathbf{C}$, a value of $P$ and consider the corresponding fiber $X_{\tau_0} = P^{-1}(\tau_0) \subset \mathbf{C}^2$ and its projective closure $C_{\tau_0} \subset \mathbf{P}^2$. By the definition of regular value, if $\tau_0 \notin B_P$, $X_{\tau_0}$ is smooth and $\{(C_t, A_i) : t \in \mathbf{C}\}$ is equisingular as a family of varieties for $t$ closed to $\tau_0$. Hence, in this case we can forget the subscript $\tau_0$ by writing $X, C, ...$ instead of $X_{\tau_0}, C_{\tau_0}, ...$ if $\tau_0$ is a regular value. Let $\tilde{C}_{\tau_0}$ be the normalisation of $C_{\tau_0}$. Topologically, $\tilde{C}_{\tau_0}$ is obtained from $X_{\tau_0}$ by adding a finite set $\tilde{\Sigma}_\infty(X_{\tau_0})$ of $r_\infty(X_{\tau_0})$ points. By definition, $r_\infty(X_{\tau_0})$ is the total number of all local branchs of the fiber $C_{\tau_0}$ on the line at infinity $L_\infty$:

$$r_\infty(X_{\tau_0}) =: r_\infty(C_{\tau_0}) =: \sum_{i=1}^{k} r_i(C_{\tau_0}),$$

where $r_i(C_{\tau_0})$ is the number of ocal branchs at $A_i$ of $C_{\tau_0}$. Therefore the mixed Hodge structure on the cohomology of $X_{\tau_0}$ can be obtained from the logarithmic de Rham complex of $\tilde{C}_{\tau_0}$ (cf. [St]). In particular, we have the Hodge fibration of $H^1(X, \mathbf{C})$ as:

$$\begin{aligned} &F^0H^1(X_{\tau_0}, \mathbf{C}) = H^1(X_{\tau_0}, \mathbf{C}),\\ &F^1H^1(X_{\tau_0}, \mathbf{C}) \cong H^0(\tilde{C}_{\tau_0}, \Omega^1_{\tilde{C}_{\tau_0}}(\log\tilde{\Sigma}_\infty(X_{\tau_0}))),\\ &F^2H^1(X_{\tau_0}, \mathbf{C}) = 0. \end{aligned}$$

Our main results are the following affine analogues of the Plücker-Minor formula

**2.2 Theorem.** *Let $X_{\tau_0}$ be an affine smooth fiber of $P(x,y)$, $b_1(X_{\tau_0}) = \dim_{\mathbf{C}}H^1(X_{\tau_0}, \mathbf{C})$, the first Betti number of $X_{\tau_0}$, and $\delta_\infty(X_{\tau_0}) = \dim_{\mathbf{C}}F^1H^1(X_{\tau_0}, \mathbf{C})$. Then*

$$2\delta_\infty(X_{\tau_0}) = b_1(X_{\tau_0}) + r_\infty(X_{\tau_0}) - 1$$

*Proof.* From the definition of $\Omega^1_{\tilde{C}_{\tau_0}}(\log\tilde{\Sigma}_\infty(X_{\tau_0}))$ it is easy to see that the following sequence is exact

$$0 \to H^0(\tilde{C}_{\tau_0}, \Omega^1_{\tilde{C}_{\tau_0}}) \to H^0(\tilde{C}_{\tau_0}, \Omega^1_{\tilde{C}_{\tau_0}}(\log\tilde{\Sigma}_\infty(X_{\tau_0}))) \to \mathbf{C}^{r_\infty(X_{\tau_0})-1} \to 0$$

This exact sequence give us the equality

$$\dim_{\mathbf{C}} H^0(\tilde{C}_{\tau_0}, \Omega^1_{\tilde{C}_{\tau_0}}(\log\tilde{\Sigma}_\infty(X_{\tau_0}))) = \dim_{\mathbf{C}} H^0(\tilde{C}_{\tau_0}, \Omega^1_{\tilde{C}_{\tau_0}}) + r_\infty(X_{\tau_0}) - 1. \tag{1}$$

Note that by the Serre's theorem of duality we have

$$\dim_{\mathbf{C}} H^0(\tilde{C}_{\tau_0}, \Omega^1_{\tilde{C}_{\tau_0}}) = \dim_{\mathbf{C}} H^1(\tilde{C}_{\tau_0}, O_{\tilde{C}_{\tau_0}}) = g(\tilde{C}_{\tau_0}).$$

Because of the isomorphism of the mixed Hodge structure

$$F^1H^1(X_{\tau_0}, \mathbf{C}) \cong H^0(\tilde{C}_{\tau_0}, \Omega^1_{\tilde{C}_{\tau_0}}(\log\tilde{\Sigma}_\infty(X_{\tau_0}))),$$

we get

$$\delta_\infty(X_{\tau_0}) := \dim_{\mathbf{C}} F^1H^1(X_{\tau_0}, \mathbf{C}) = \dim_{\mathbf{C}} H^0(\tilde{C}_{\tau_0}, \Omega^1_{\tilde{C}_{\tau_0}}(\log\tilde{\Sigma}_\infty(X_{\tau_0}))),$$

and now (1) becomes

$$\delta_\infty(X_{\tau_0}) = g(\tilde{C}_{\tau_0}) + r_\infty(X_{\tau_0}) - 1. \tag{2}$$

On the other hand, because $\tilde{C}_{\tau_0}$ is obtained from $X_{\tau_0}$ by adding $r_\infty(X_{\tau_0})$ points as remarked in 2.1, the Betti number of $X_{\tau_0}$ can be defined by

$$b_1(X_{\tau_0}) = 2g(\tilde{C}_{\tau_0}) + r_\infty(X_{\tau_0}) - 1. \tag{3}$$

Finally from (2) and (3) we get

$$2\delta_\infty(X_{\tau_0}) = b_1(X_{\tau_0}) + r_\infty(X_{\tau_0}) - 1.$$

This is the affine Plücker-Milnor formula which we want to prove. □

**2.3 Theorem.** *For any fiber $X_{\tau_0}$ of $P(x, y)$ we have*

$$b_1(X_{\tau_0}) = (d-1)(d-2) - 2\delta_a(X_{\tau_0}).$$

*Here*

$$2\delta_a(X_{\tau_0}) = \sum_{M\in\mathbf{C}^2} \mu(X_{\tau_0}, M) + \sum_{i=1}^{k} \mu(C_{\tau_0}, A_i) - k + 1.$$

*Therefore, if $X_{\tau_0}$ is a smooth fiber as in* 2.2*,*

$$2\delta_\infty(X_{\tau_0}) = d^2 - 3d + k + r_\infty(X_{\tau_0}) - \sum_{i=1}^{k} \mu(C_{\tau_0}, A_i).$$

*Proof.* Let $\kappa(X_{\tau_0})$ be the topological Euler number of $X_{\tau_0}$. Then $b_1(X_{\tau_0}) = 1 - \kappa(X_{\tau_0})$. By the additivity of $\kappa(X_{\tau_0})$, to compute it we can use the Mayer-Vietoris sequence and obtain

$$\begin{aligned}\kappa(X_{\tau_0}) &= \kappa(C_{\tau_0}) - k \\ &= 3d - d^2 + \sum_{M\in\mathbf{C}^2} \mu(C_{\tau_0}, M) + \sum_{i=1}^{k} \mu(C_{\tau_0}, A_i) - k.\end{aligned}$$

This verifies the first assertion. The second assertion follows from 2.2 because $2\delta_a(X_{\tau_0}) = \sum_{i=1}^{k} \mu(C_{\tau_0}, A_i) - k + 1$. □

**2.4 Corollary** (The case of the generic fibers). *Let $C$ be the projective closure of the affine generic fiber $X$ of $P(x,y)$. Denote by $\mu^\infty(P) = \sum_{M\in\mathbf{C}^2} \mu(P,M)$ (resp. $\lambda^\infty(P) = \sum_{i=1}^{k} \sum_{t\in B_\infty} (\mu(C_t, A_i) - \mu(C, A_i))$) the total (resp. total jumps at infinity) of the Milnor number of $P(x,y)$, and by $\mu(B_P) = \sum_{t\in B_P} \sum_{x\in\mathbf{P}^2} \mu(C_t, x)$ (resp. $\mu_\infty(X) = \sum_{i=1}^{k} \mu(C, A_i)$) the complete total (resp. the total at infinity) of the Milnor number of $P(x,y)$ (resp. of the generic fiber $X$). Denote by $\#B_P$ the number of the critical values at infinity of $P(x,y)$. Then*

(i) $$2\delta_\infty(X) = \mu^\infty(P) + \lambda^\infty(P) + r_\infty(X) - 1.$$

(ii) $$\mu^\infty(P) + \lambda^\infty(P) + \sum_{i=1}^{k} \mu(C, A_i) = (d-1)(d-2) + k - 1.$$

*In particular, $B_P = \emptyset$ (i.e $P(x,y)$ define a trivial fibration over $\mathbf{C}^2$) if and only if*

$$\sum_{i=1}^{k} \mu(C, A_i) = (d-1)(d-2) + k - 1$$

(iii) $$\mu(B_P) = (d-1)(d-2) + (\#B_\infty - 1)\mu_\infty(X) + k - 1.$$

*Moreover, if* $\mu_\infty(X) \neq 0$*, the number of the critical values at infinity of* $P(x,y)$ *can be defined by*

$$\#B_\infty = (\mu(B_P) + \mu_\infty(X) - d(d-3) - k + 1)/\mu_\infty(X).$$

*Proof.* (i) This follows from 2.2 by using a result of [HL] which says that $\mu^\infty(P) + \lambda^\infty(P)$ is the first Betti number of the affine generic fiber of $P(x,y)$.

(ii) As (2) in the proof of 2.2 we have

$$\delta_\infty(X) = g(\tilde{C}) + r_\infty(X) - 1.$$

Applying the classical (and then local) Plücker formula for $C$ (resp. for $(C,x)$) we get (cf. 2.0):

$$g(\tilde{C}) = \frac{1}{2}(d-1)(d-2) - \sum_{x \in L_\infty} \delta(C,x),$$

$$2\delta(C,x) = \mu(C,x) + r(C,x) - 1.$$

Thus

$$2\delta_\infty(X) = (d-1)(d-2) - \sum_{x \in L_\infty} (\mu(C,x) + r(C,x) - 1) + 2r_\infty(X) - 2.$$

In other words,

$$2\delta_\infty(X) = (d-1)(d-2) - \sum_{x \in L_\infty} (\mu(C,x) - r(C,x)) + k - 2.$$

Now apply (i) we get (ii). Note that by definition, $B_P = \emptyset$ if and only if $\mu^\infty(P) = \lambda^\infty(P) = 0$.

(iii) Remembering the definitions of $\mu^\infty(P)$ and $\lambda^\infty(P)$ and remarking that

$$\sum_{t \in B_\infty} \sum_{i=1}^{k} (\mu(C_t, A_i) - \mu(C, A_i)) =$$
$$\sum_{t \in B_\infty} \sum_{i=1}^{k} \mu(C_t, A_i) - \#B_\infty \sum_{i=1}^{k} \mu(C, A_i)$$

we can rewrite (ii) in the form

$$\sum_{t\in B_P}\sum_{x\in\mathbf{P}^2}\mu(C_t,x)=$$

$$(d-1)(d-2)+(\#B_\infty-1)\sum_{i=1}^{k}\mu(C,A_i)+k-1,$$

which means

$$\mu(B_P)=(d-1)(d-2)+(\#B_\infty-1)\mu_\infty(X)+k-1.$$

**2.5 Examples.** (i) If $P=x(xy-1)$ then

$$d=3,\quad k=2,$$

$$B_P=B_\infty=\{0\},$$

$$\mu(B_P)=3,\quad \mu_\infty(X)=0$$

(ii) If $P=y^2+x^3y(2x+y)$ then

$$d=5,\quad k=3,\quad \Sigma=\{(0,0)\},$$

$$B_\infty=\emptyset \text{ (cf. (4.19) in [LO])},\quad B_P=P(\Sigma)=\{0\},$$

$$\mu(B_P)=10,\quad \mu(P,(0,0))=6,\quad \mu_\infty(X)=4.$$

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