ACTA MATHEMATICA VIETNAMICA
Volume 24, Number 1, 1999, pp. 27–38



# COMPLETENESS OF A COLLECTION GENERATED BY TRANSLATING A SET OF FUNCTIONS

NGUYEN QUANG DIEU

Abstract. Some sufficient conditions for completeness of a collections generated from translates and dilates of a single function in the functional spaces $L_p(\mathbb{R}^n)$, $C(\mathbb{R}^n)$, $C_0(\mathbb{R}^n)$ and in the subspace of all entire functions $C^\infty(\mathbb{R}^n)$ are given.

## 1. Introduction

The problem of completeness of a collection of functions generated from translates and dilates of a single function in a certain functional space is common both in Approximation Theory as well as in Harmonic Analysis.

It has a long history and can be traced back to a work of $N$. Wiener in the 1930s, giving a necessary and sufficient condition on a collection of functions generated from translations of a single function to be complete in $L_1(\mathbb{R})$ or $L_2(\mathbb{R})$ (see [11], pp. 98-100). It is worth noting that N. Wiener's theorems are still valid in the multivariate case and recently V. Volchkov has obtained some generalizations of N. Wiener's theorems in the space $L_p(G)$ where $G$ is a bounded domain in $\mathbb{R}^n$ (see [10]).

Motivated by Wiener's theorems, a natural question arises as to under what conditions the collection of functions

$$\Big\{f(\mathbf{x}+\mathbf{c}) : f \in \mathbf{A}; \mathbf{c} \in \mathbf{S}\Big\}, \tag{1}$$

where $\mathbf{A}$ is a given set of functions defined on $\mathbb{R}^n$ and $\mathbf{S}$ is a given subset in $\mathbb{R}^n$, is complete in various function spaces.

In the univariate case, it is possible to choose $\mathbf{A}$ to be a set of only a single function defined on $\mathbb{C}$ and $\mathbf{S}$ to be a sequence of distinct complex numbers such that the restriction on $\mathbb{R}$ of the collection

$$\Big\{f(\mathbf{x}+\mathbf{c}) : \mathbf{c} \in \mathbf{S}\Big\}$$

Received March 20, 1997; in revised form August 2, 1998.
*1991 Mathematics subject classification.* 41A, 42B.
*Key words and phrases.* Collection of functions, completeness of a collection.

is complete in a certain functional space. This problem was considered extensively by R. A. Zalik in a series of papers [12], [13], [14], [15].

It is particularly interesting when $\mathbf{A}$ is generated by dilating a single function. Then, the collection (1) consists of translates and dilates of this function. The completeness of such a collection occurs both in wavelet analysis and in neural network approximation.

When $\mathbf{A}$ is the set of all radial functions in $C(\mathbb{R}^n)$, i.e., continuous functions which depend only on the distance to the origin, and $\mathbf{S}$ is a subset of $\mathbb{R}^n$ the completeness of the collection (1) in $C(\mathbb{R}^n)$ was studied in [1] by M. L. Agranovski and E. T. Quinto. They gave a complete characterization on $\mathbf{S}$ so that the collection is complete in $C(\mathbb{R}^2)$.

Another effective approach was given by A. Pinkus and B. Wajnryb, [6, 7], in which they introduced the collection

$$\mathcal{P}_1 = \Big\{ g^k(\cdot - \mathbf{b}) : \mathbf{b} \in \mathbb{R}^n,\ k \in \mathbb{Z}_+ \Big\},$$

where $g$ is a fixed polynomial, and proved some necessary and sufficient conditions on the completeness of these families in $C(\mathbb{R}^n)$.

The aim of the present paper is to find sufficient conditions on the sets $\mathbf{A}$ and $\mathbf{S}$ such that the collection (1) is complete in a certain functional space. Here we say a collection of functions are complete in a certain functional space if the linear span of the elements of the collection is dense in this space. In our consideration, $\mathbf{A}$, $\mathbf{S}$ are always assumed to be countable. Moreover, the set $\mathbf{A}$ may consist of functions which can be extended to entire functions. In this case, $\mathbf{S}$ is a subset of $\mathbb{C}^n$. In what follows, we say a collection of functions of the form (1) is complete in a space of functions on $\mathbb{R}^n$ if its restriction on $\mathbb{R}^n$ is complete in this space.

The present paper is organized as follows. In Section 2 we introduce some notations and auxiliary lemmas which will be used later. Next, in the first theorem of Section 3 we deal with completeness of a set generated from translates and dilates of a single function in the spaces $L_p(\mathbb{R}^n)$ and $C_0(\mathbb{R}^n)$. The other theorems treat the same problem in the space $H(\mathbb{R}^m)$. Our methods are based on duality theorems in various functional spaces.

## 2. Notations and auxiliary facts

Throughout this paper we shall use the following notations. If $\mathbf{z} = (z_1, \ldots, z_n)$, $\mathbf{u} = (u_1, \ldots, u_n)$, $\mathbf{x} = (x_1, \ldots, x_n)$ are multicomplex numbers, $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n := \big\{ \mathbf{a} \in \mathbb{Z}^n : 0 \le a_i\ \forall 1 \le i \le n \big\}$, then we denote

$$|\mathbf{z}| := \sum_{i=1}^{n} |z_i|,$$
$$\langle \mathbf{z}, \mathbf{u} \rangle := \sum_{i=1}^{n} z_i u_i,$$
$$\mathbf{z}\,\mathbf{u} := (z_1 u_1, \ldots, z_n u_n),$$
$$\mathbf{x}^{\mathbf{k}} := x_1^{k_1} \cdots x_n^{k_n};$$
$$\mathbf{k}! := k_1! \cdots k_n!.$$

If $f$ is a function of $n$ variables and $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ then we let

$$f^{(\mathbf{k})} := \frac{\partial^{|\mathbf{k}|} f}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \cdot$$

The Cartesian of $n$ copies $\mathbf{S}$ is denoted by $\mathbf{S}^n$. $\mathbf{T}_r$ stands for the interval $[-r, r]$. The space of all continuous functions on $\mathbb{R}^n$ vanishing at infinity with the uniform norm is denoted by $C_0(\mathbb{R}^n)$. The space $C_c(\mathbb{R}^n)$ consists of all continuous functions with compact support on $\mathbb{R}^n$. $C(\mathbb{R}^n)$ denotes the space of all continuous functions on $\mathbb{R}^n$ with the topology of uniform convergence on compact subsets. $C^\infty(\mathbb{R}^n)$ stands for the space of all infinitely differentiable functions with topology of derivatives converging on compact subsets. $\mathcal{D}(\mathbb{R}^n)$ stands for the space of all rapidly decreasing functions on $\mathbb{R}^n$. The space $H(\mathbb{R}^n)$ consists of all functions on $\mathbb{R}^n$ which can be extended to an entire function on $\mathbf{C}^n$. Let $L_p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$ stand for the space of Lebesgue $p$-integrable functions. The Fourier transform of a function $f$ in $L_p(\mathbb{R}^n)$ is written as $\hat{f}$ in distributional sense. The support of $f$ is denoted by supp $f$.

We let for $(\sigma, \ldots, \sigma) \in \mathbb{Z}_+^n$, $1 \leq p \leq \infty$, $B_{\sigma,p}(\mathbb{R}^n)$ denote the space of functions $f \in L_p(\mathbb{R}^n)$ which can be extended to an entire function of exponential type $\leq \sigma$. When $p = \infty$ we denote this space by $B_\sigma(\mathbb{R}^n)$. It is well-known that if $f \in B_{\sigma,p}(\mathbb{R}^n)$ then $f \in B_{\sigma,q}(\mathbb{R}^n) \subset B_\sigma(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ and $f^{(\mathbf{k})} \in B_{\sigma,p}(\mathbb{R}^n)$, $1 \leq p < q < \infty$, $\forall \mathbf{k} \in \mathbb{Z}_+^n$. For any $f \in B_\sigma(\mathbb{R}^n)$ we let $\sigma_f := \inf\{r > 0 : \text{supp } \hat{f} \subset \mathbf{T}_r^n\}$. It follows from the Paley-Wiener-Schwartz theorem that $\sigma_f \leq \sigma$ for any $f \in B_{\sigma,p}$, $1 \leq p \leq \infty$.

We shall collect below some auxiliary facts which will be used in the next section.

By repeating the same argument as in the proof of Theorem 5.1 [4] (p. 529) we obtain the following lemma concerning the convolution of a compactly supported continuous function with a bounded Borel measure.

**Lemma 2.1.** *Let $g$ be a function of $C_c(\mathbb{R}^n)$ and $\mu$ be a bounded Borel measure. Then the convolution*

$$(g * \mu)(\cdot) := \int_{\mathbb{R}^n} g(\cdot - \mathbf{u}) d\mu(\mathbf{u})$$

*belongs to $L_p(\mathbb{R}^n)$, $\forall 1 \leq p \leq \infty$. Further, for any $h \in L_\infty(\mathbb{R}^n)$ we have*

$$g * (h * \mu) = h * (g * \mu).$$

The following lemma is contained implicitly in the proof of Theorem 1 [15].

**Lemma 2.2.** *Let $f$ be a function on $\mathbb{R}$ and $a > 0$ such that $e^{a|\cdot|} f(\cdot) \in L_1(\mathbb{R})$. Suppose that $S = \{a_k\}_{k=1}^{\infty}$ is a sequence of distinct real numbers satisfying the conditions*

$$\sum_{k=1}^{\infty} \Big(1 - \Big|\frac{1 - e^{\pi\alpha_k/2a}}{1 + e^{\pi\alpha_k/2a}}\Big|\Big) = \infty,$$

*and $\hat{f}(s) = 0$ for any $s \in S$. Then $f = 0$ a.e. on $\mathbb{R}$.*

Using this lemma and the Fubini theorem we can prove inductively the following

**Lemma 2.3.** *Let $f$ be a function on $\mathbb{R}^n$ and $a$ is a positive number satisfying*

$$e^{a|\cdot|} f(\cdot) \in L_1(\mathbb{R}^n).$$

*Suppose that $S = \{\alpha_k\}_{k=1}^{\infty}$ is a sequence of distinct real numbers satisfying the conditions*

$$\sum_{k=1}^{\infty} \Big(1 - \Big|\frac{1 - e^{\pi\alpha_k/2a}}{1 + e^{\pi\alpha_k/2a}}\Big|\Big) = \infty, \tag{2}$$

*and $\hat{f}(\mathbf{s}) = 0$ for any $\mathbf{s} \in S^n$. Then $f = 0$ a.e. on $\mathbb{R}^n$.*

Finally, we take for granted the known fact that if $f \in L_1(\mathbb{R}^n)$ and $\mu$ is a bounded Borel measure on $\mathbb{R}^n$ then the convolution $f * \mu \in L_1(\mathbb{R}^n)$ and $(\widehat{f * \mu})(\cdot) \equiv \hat{f}(\cdot)\hat{\mu}(\cdot)$, where as usual

$$\hat{\mu}(\cdot) := \int_{\mathbb{R}^n} e^{-i\langle \mathbf{x}, \cdot\rangle} d\mu(\mathbf{x}).$$

This fact will be used to prove the first theorem in the next section.

## 3. Completeness in functional spaces

Throughout this section $\mathbf{A}$ is always a countable set. If $f \in H(\mathbb{R}^n)$ then its extension to $\mathbb{C}^n$ is also denoted by $f$. For any $1 \le p \le \infty$ we shall adopt the following conventions

$$\mathbf{U}\,\mathbf{B}_p(\mathbb{R}^n) := \bigcup_{\sigma>0} B_{\sigma,p}(\mathbb{R}^n),$$
$$\mathbf{U}\,\mathbf{B}(\mathbb{R}^n) := \bigcup_{\sigma>0} B_{\sigma}(\mathbb{R}^n),$$
$$q := \frac{p}{p-1}\,\cdot$$

For any function $f$ on $\mathbb{R}^n$ we define the function $\tilde{f}$ by $\tilde{f}(\cdot) := f(-\,\cdot)$. If $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ then we let $\mathbf{x}' = (x_1, \dots, x_{n-1})$.

**Theorem 1.** a) *Let $\mathbf{A}$ be a subset of $\mathbf{U}\,\mathbf{B}_1(\mathbb{R}^n)$ satisfying*

$$\bigcap_{f\in\mathbf{A}} \left\{\mathbf{x} \in \mathbb{R}^n : \hat{f}(\mathbf{x}) = 0\right\} = \emptyset. \tag{6}$$

*Suppose that $\varepsilon$ is a positive number and $S$ is a sequence of distinct complex numbers satisfying*

$$\operatorname{Im}\alpha \ge \delta|\alpha|, \quad \forall\alpha \in S, \quad \sum_{\alpha\in S,\alpha\neq 0} \frac{1}{|\alpha|} = \infty. \tag{7}$$

*Then the collection*

$$\mathcal{C} = \bigcup_{f\in\mathbf{A}} \left\{ f\Big(\cdot + \frac{\mathbf{m}\pi}{\sigma_f + \varepsilon}\,, \cdot + \alpha\Big) : \mathbf{m} \in \mathbb{Z}^{n-1},\ \alpha \in S \right\}$$

*is complete in $L_p(\mathbb{R}^n)$ $1 \le p < \infty$.*

b) *Let $\mathbf{A}$ be a subset of $\mathbf{U}\,\mathbf{B}(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ satisfying condition* (6), *and $S$ be a sequence of distinct complex numbers satisfying condition* (7). *Then the collection $\mathcal{C}$ is complete in $C_0(\mathbb{R}^n)$.*

*Proof.* a) Assume that $\mathcal{C}$ is incomplete in $L_p(\mathbb{R}^n)$ for some $1 \le p < \infty$. In view of the Hahn-Banach theorem we deduce that there exists a nonzero function $g \in L_q(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f\Big(\mathbf{x}' + \frac{\mathbf{m}\pi}{\sigma_f + \varepsilon}\,, x_n + \alpha\Big) g(\mathbf{x}) d\mathbf{x} = 0, \tag{8}$$
$$\forall f \in \mathbf{A}, \quad \forall\mathbf{m} \in \mathbb{Z}^{n-1},\ \forall\alpha \in S.$$

Let us fix $f \in \mathbf{A}$ and define the following function

$$h_f(\mathbf{z}) := (f * \tilde{g})(\mathbf{z}) \quad \forall \mathbf{z} \in \mathbb{C}^n. \tag{9}$$

By virtue of Theorem 3.6.2 [5] we have $h_f \in \mathbf{U\,B}(\mathbb{R}^n)$, and (8) simply means that

$$h_f\Big(\frac{\mathbf{m}\pi}{\sigma_f + \varepsilon}, \alpha\Big) = 0, \quad \forall \mathbf{m} \in \mathbb{Z}^{n-1}, \ \forall \alpha \in S. \tag{10}$$

For any $\mathbf{m} \in \mathbb{Z}^{n-1}$, we let

$$h^*_{\mathbf{m},f}(z) := \begin{cases} \left\{ \dfrac{h_f\Big(\dfrac{\mathbf{m}\pi}{\sigma_f + \varepsilon}, z\Big) - h_f\Big(\dfrac{\mathbf{m}\pi}{\sigma_f + \varepsilon}, \alpha_0\Big)}{z - \alpha_0} \right\}^2 & z \neq \alpha_0 \\ {h'_f}^2\Big(\dfrac{\mathbf{m}\pi}{\sigma_f + \varepsilon}, \alpha_0\Big), & z = \alpha_0, \end{cases}$$

where $\alpha_0$ is an arbitrary fixed element of $S$. It is easy to see that $h^*_{m,f}(z) \in \mathbf{U\,B}_1(\mathbb{R})$ and vanishes on the set $S \setminus \{\alpha_0\}$. According to Theorem 3.1.3 [5], there exists a function $\psi_{\mathbf{m},f} \in C(\mathbb{R})$ with a compact support in $[-\sigma_{\mathbf{m},f}, \sigma_{\mathbf{m},f}]$ such that

$$h^*_{\mathbf{m},f}(z) = \int\limits_{\mathbb{R}} \psi_{\mathbf{m},f}(x) e^{ixz} dx, \quad \forall z \in \mathbb{C}. \tag{11}$$

We define

$$A_{\mathbf{m},f} = \Big\{ x \in \mathbb{R} : \psi_{\mathbf{m},f}(x - \sigma_{\mathbf{m},f}) \neq 0 \Big\}.$$

It follows from (10) that

$$h^*_{\mathbf{m},f}(z) = 0, \quad \forall z \in S \setminus \{\alpha_0\}.$$

Now we wish to prove that $h^*_{\mathbf{m},f} \equiv 0$. Otherwise, $A_{\mathbf{m},f}$ is a set with nonzero measure.

By introducing a new variable $y := x + \sigma_{\mathbf{m},f}$, from (11) we obtain

$$\int\limits_{\mathbb{R}^n} \psi_{\mathbf{m},f}(y - \sigma_{\mathbf{m},f}) e^{iy\alpha} dy = 0, \quad \forall \alpha \in S \setminus \{\alpha_0\}. \tag{12}$$

Consider the set of functions $\{\psi_{\mathbf{m},f}(\cdot - \sigma_{\mathbf{m},f})e^{i\cdot\alpha} : \alpha \in S \setminus \{\alpha_0\}\}$. By the assumption (7) and Theorem 1 [16] we know that this set is complete in $L_2(A_{\mathbf{m},f})$. On the other hand, we can write (12) in the form

$$\int\limits_{A_{\mathbf{m},f}} \chi A_{\mathbf{m},f}(y)\psi_f(y-\sigma_f)e^{iy\alpha}dy = 0, \quad \forall\alpha \in S \setminus \{\alpha_0\},$$

where $\chi_{A_{\mathbf{m},f}}(\cdot)$ is the characteristic function of $A_{\mathbf{m},f}$. Clearly, $\chi_{A_{\mathbf{m},f}} \in L_2(A_{\mathbf{m},f})$. By virtue of the Hahn-Banach theorem we conclude that $\chi_{A_{\mathbf{m},f}} \equiv 0$, which is impossible. Thus, $h^*_{\mathbf{m},f} \equiv 0$. Therefore

$$h_f\Big(\frac{\mathbf{m}\pi}{\sigma_f+\varepsilon}\ ,\cdot\Big) \equiv 0, \quad \forall\mathbf{m} \in \mathbb{Z}^{n-1}.$$

For each $z \in \mathbb{C}$ we define

$$h_{f,z}(\cdot) := h_f(\cdot, z).$$

It follows that $h_{f,z}$ is a function of $B_{\sigma_f}(\mathbb{R}^{n-1})$ vanishing on the set $\Big\{\dfrac{\mathbf{m}\pi}{\sigma_f+\varepsilon}\ : m \in \mathbb{Z}^{n-1}\Big\}$. Thus, by using a sampling representation theorem (Theorem 1 [2]) we obtain $h_{f,z} \equiv 0, \forall z \in \mathbb{C}$. This implies that $h_f \equiv 0$. In particular, we get

$$h_f(\mathbf{x}) = (f * \tilde{g})(\mathbf{x}) = 0, \quad \forall\mathbf{x} \in \mathbb{R}^n. \tag{13}$$

As $f \in L_1(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$ and $\tilde{g} \in L_q(\mathbb{R}^n)$, from (13) and Lemma 3.1 [4] we deduce that $\operatorname{supp}\hat{\tilde{g}}$ is contained in the set

$$Z_f = \Big\{\mathbf{x} \in \mathbb{R}^n : \hat{f}(-\mathbf{x}) = 0\Big\}.$$

On the other hand, from (6) we infer that $\operatorname{supp}\hat{\tilde{g}}$ is empty and consequently $g = 0$ a.e., a contradiction. Thus the conclusion follows.

b) Suppose $\mathcal{C}$ is incomplete in $C_0(\mathbb{R}^n)$. Applying the Hahn-Banach theorem, we see that there exists a nonzero bounded measure $\mu$ such that

$$\int\limits_{\mathbb{R}^n} f\Big(\mathbf{x}' + \frac{\mathbf{m}\pi}{\sigma_f+\varepsilon}\ , x_n + \alpha\Big)d\mu(\mathbf{x}) = 0, \tag{14}$$
$$\forall f \in \mathbf{A}, \quad \forall\mathbf{m} \in \mathbb{Z}^{n-1}, \quad \forall\alpha \in S.$$

For any $f \in \mathbf{A}$, we denote

$$h_f^*(\mathbf{z}) := (\tilde{f} * \mu)(\mathbf{z}) \quad \forall \mathbf{z} \in \mathbb{C}^n.$$

By modifying the proof of theorem 3.6.2 in [5] it is not hard to show that $h_f^*$ is a function of $B_{\sigma_f}(\mathbb{R}^n)$. Furthermore, we derive from (14) that

$$h_f^*\Big(\frac{\mathbf{m}\pi}{\sigma_f + \varepsilon}\ , -\alpha\Big) = 0, \quad \forall \mathbf{m} \in \mathbb{Z}^{n-1}, \quad \forall \alpha \in S.$$

By repeating the same argument as in the preceding part we conclude that $h_f^* \equiv 0$. In particular,

$$(\tilde{f} * \mu)(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Now let $g$ be an arbitrary nonzero function of $C_c(\mathbb{R}^n)$. In view of Lemma 2.1 we obtain

$$\tilde{f} * (g * \mu) = g * (\tilde{f} * \mu) \equiv 0.$$

Put $g * \mu = h$. We have $\tilde{f} * h \equiv 0$. Since $\tilde{f} \in L_\infty(\mathbb{R}^n)$ and by Lemma 2.1 $h \in L_1(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$, it follows from Lemma 3.1 [4] that for any $f \in \mathbf{A}$, $\operatorname{supp} \hat{f}$ is contained in the set

$$\mathcal{E} = \Big\{\mathbf{x} \in \mathbb{R}^n : \hat{h}(\mathbf{x}) = 0\Big\} = \Big\{\mathbf{x} \in \mathbb{R}^n : \hat{g}(\mathbf{x})\hat{\mu}(\mathbf{x}) = 0\Big\}.$$

Hence $\bigcup\limits_{f \in \mathbf{A}} \operatorname{supp} \hat{f}$ is contained in $\mathcal{E}$.

On the other hand, from (6) we deduce

$$\bigcup_{f \in \mathbf{A}} \operatorname{supp} \hat{f} = \mathbb{R}^n.$$

Thus, we have $\mathcal{E} = \mathbb{R}^n$. Since $g$ is a nonzero function of $C_c(\mathbb{R}^n)$, we conclude that $\hat{g}(\mathbf{x})$ is an entire function and therefore vanishes on $\mathbb{R}^n$ at a set with zero measure. Hence $\hat{\mu} = 0$ a.e. on $\mathbb{R}^n$. This implies that $\mu = 0$, a contradiction. $\square$

To complete the present paper we shall prove a theorem on the closure span of a function rapidly decreasing in the space $C^\infty(\mathbb{R}^n)$.

**Theorem 2.** *Let $f$ be a nonzero function of $\mathcal{D}(\mathbb{R}^n)$ such that its Fourier transform has a compact support. If $S$ is a sequence of distinct real numbers satisfying the condition* (2), *then*

$$\overline{\text{span}}\Big\{f(\mathbf{x}+\mathbf{s}) : \mathbf{s} \in S^n\Big\} = \overline{\text{span}}\ \mathcal{D}(R^n) = H(\mathbb{R}^n),$$

*where we take the closure span in* $C^\infty(\mathbb{R}^n)$*.*

*Proof.* First we shall prove that

$$\overline{\text{span}}\ \Big\{f(\mathbf{x}+\mathbf{s}) : \mathbf{s} \in S^n\Big\} = \overline{\text{span}}\ \mathcal{D}(\mathbb{R}^n). \tag{15}$$

It is sufficient to check that each continuous linear functional on $C^\infty(\mathbb{R}^n)$ which annihilates the collection $\Big\{f(\mathbf{x}+\mathbf{s}) : \mathbf{s} \in S^n\Big\}$ also annihilates on $\mathcal{D}(\mathbb{R}^n)$.

According to standard results on distribution (see Theorem 22, 29 [3] p.64–p.68), each continuous linear functional $T$ on $C^\infty(\mathbb{R}^n)$ is a distribution with compact support having the representation

$$T = \sum_{\boldsymbol{\ell} \leq \mathbf{m}} g_\ell^{(\boldsymbol{\ell})}, \tag{16}$$

where $g_\ell$ are functions of $C_c(\mathbb{R}^n)$. Furthermore, assuming that $T$ annihilates the set $\Big\{f(\mathbf{x}+\mathbf{s}) : \mathbf{s} \in S^n\Big\}$. That is,

$$T(f(\mathbf{x}+\mathbf{s})) = 0, \quad \mathbf{s} \in S^n. \tag{17}$$

Substituting (16) into (17) we obtain

$$\sum_{\boldsymbol{\ell} \leq \mathbf{m}} \int_{\mathbb{R}^n} (-1)^{\boldsymbol{\ell}} f^{(\boldsymbol{\ell})}(\mathbf{x}+\mathbf{s}) g_\ell(\mathbf{x}) d\mathbf{x} = 0, \quad \mathbf{s} \in S^n. \tag{18}$$

Since $f \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\widehat{f^{(\boldsymbol{\ell})}}(\mathbf{x}+\mathbf{s}) = \widehat{f^{(\boldsymbol{\ell})}}(\mathbf{x}) e^{i\langle \mathbf{x},\mathbf{s}\rangle} = (i\mathbf{x})^{\boldsymbol{\ell}} \hat{f}(\mathbf{x}) e^{i\langle \mathbf{x},\mathbf{s}\rangle}.$$

Hence, by using the Parseval formula and (18) we get

$$\sum_{\boldsymbol{\ell} \leq \mathbf{m}} \int_{\mathbb{R}^n} (-i\mathbf{u})^{\boldsymbol{\ell}} \hat{f}(\mathbf{u}) \hat{g}_\ell(\mathbf{u}) e^{i\langle \mathbf{u},\mathbf{s}\rangle} d\mathbf{u} = 0, \quad \forall \mathbf{s} \in S^n,$$

i.e.,

$$\int\limits_{\mathbb{R}^n} e^{i\langle \mathbf{u},\mathbf{s}\rangle}\hat{h}(\mathbf{u})\Big(\sum_{\boldsymbol{\ell}\le\mathbf{m}}(-i\mathbf{u})^{\boldsymbol{\ell}}\hat{g}_\ell(\mathbf{u})\Big)d\mathbf{u}=0 \quad \forall \mathbf{s}\in S^n, \tag{19}$$

We define

$$\psi(\mathbf{z}) := \sum_{\boldsymbol{\ell}\le\mathbf{m}}(-i\mathbf{z})^{\boldsymbol{\ell}}\hat{g}_\ell(\mathbf{z}), \quad \forall \mathbf{z}\in\mathbb{C}^n.$$

Then (19) is of the form

$$\int\limits_{\mathbb{R}^n} e^{i\langle \mathbf{u},\mathbf{s}\rangle}\hat{f}(\mathbf{u})\psi(\mathbf{u})d\mathbf{u}=0, \quad \forall \mathbf{s}\in S^n. \tag{20}$$

Since $g_\ell \in C_c(\mathbb{R}^n)$, we deduce that $\hat{g}_\ell$ is an entire function and hence so is $\psi$. Moreover, because $\hat{f}$ has a compact support, we infer that $e^{\delta|\cdot|}\hat{f}(\cdot)\psi(\cdot)\in L_1(\mathbb{R}^n)$, $\forall\delta>0$. By applying Lemma 2.3 from (20) and the assumption on the set $S$, we obtain for any $f\in\mathbf{A}$

$$\hat{f}(\mathbf{u})\psi(\mathbf{u})=0, \quad \text{a.e. on} \quad \mathbb{R}^n. \tag{21}$$

We shall prove that $\psi\equiv 0$. Indeed, otherwise $\psi$ vanishes on $\mathbb{R}^n$ at a set with zero measure. This and (21) imply that $\hat{f}\equiv 0$ on $\mathbb{R}^n$. Consequently $f\equiv 0$, a contradiction. Thus, we have proved that if a distribution $T$ of the form (16) annihilating the collection $\Big\{f(\mathbf{x}+\mathbf{s}) : \mathbf{s}\in S^n\Big\}$ then

$$\psi(\mathbf{z}) = \sum_{\boldsymbol{\ell}\le\mathbf{m}}(-i\mathbf{z})^{\boldsymbol{\ell}}\hat{g}_\ell(\mathbf{z})=0, \quad \forall \mathbf{z}\in\mathbb{C}^n. \tag{22}$$

Now, it remains to check that $T$ annihilates $\mathcal{D}(\mathbb{R}^n)$. For this let $h\in\mathcal{D}(\mathbb{R}^n)$. Then by virtue of the Parseval formula we get

$$T(h)=\sum_{\boldsymbol{\ell}\le\mathbf{m}}\int\limits_{\mathbb{R}^n} h^{(\boldsymbol{\ell})}(\mathbf{x})g_\ell(\mathbf{x})d\mathbf{x} = \int\limits_{\mathbb{R}^n}\hat{h}(\mathbf{u})\Big\{\sum_{\boldsymbol{\ell}\le\mathbf{m}}(-i\mathbf{u})^{\boldsymbol{\ell}}\hat{g}_\ell(\mathbf{u})\Big\}d\mathbf{u}=0,$$

where the last identity follows from (22). Therefore, $T$ also annihilaties $\mathcal{D}(\mathbb{R}^n)$. Hence (15) is proved.

It remains to prove that

$$\overline{\operatorname{span}}\left\{f(\mathbf{x}+\mathbf{s}) : \mathbf{s} \in S^n\right\} = H(\mathbb{R}^n). \tag{23}$$

Indeed, from the above argument we see that $T$ is a distribution with compact support annihilating the set $\left\{f(\mathbf{x}+\mathbf{s}) : \mathbf{s} \in S^n\right\}$ if and only if $T$ is of the form (16) and the functions $g_\ell$ satisfy the relation

$$\sum_{\boldsymbol{\ell}\leq\mathbf{m}} (-i\mathbf{z})^{\boldsymbol{\ell}} \hat{g}_\ell(\mathbf{z}) = 0, \quad \forall \mathbf{z} \in \mathbb{C}^n,$$

i.e.

$$\sum_{\boldsymbol{\ell}\leq\mathbf{m}} (-i\mathbf{z})^{\boldsymbol{\ell}} \int\limits_{\mathbb{R}^n} g_\ell(\mathbf{u}) e^{-i\langle\mathbf{u},\mathbf{z}\rangle} d\mathbf{u} = 0, \quad \forall \mathbf{z} \in \mathbb{C}^n.$$

Therefore,

$$\sum_{\boldsymbol{\ell}\leq\mathbf{m}} (-i\mathbf{z})^{\boldsymbol{\ell}} \int\limits_{\mathbb{R}^n} g_\ell(\mathbf{u}) \Big( \sum_{k\in\mathbb{Z}^n_+} \frac{(-i\mathbf{u})^{\mathbf{k}}\mathbf{z}^{\mathbf{k}}}{\mathbf{k}!} \Big) d\mathbf{u} = 0 \quad \forall \mathbf{z} \in \mathbb{C}^n.$$

This implies that

$$\sum_{\mathbf{k}\in\mathbb{Z}^n_+} \mathbf{z}^{\mathbf{k}} \sum_{\boldsymbol{\ell}\leq\min(\mathbf{k},\mathbf{m})} \int\limits_{\mathbb{R}^n} \frac{(-1)^{\boldsymbol{\ell}} g_\ell(\mathbf{u}) \mathbf{u}^{\mathbf{k}-\boldsymbol{\ell}}}{(\mathbf{k}-\boldsymbol{\ell})!} d\mathbf{u} = 0, \quad \forall \mathbf{z} \in \mathbb{C}^n.$$

Consequently,

$$\sum_{\boldsymbol{\ell}\leq\min(\mathbf{k},\mathbf{m})} \int\limits_{\mathbb{R}^n} | \frac{(-1)^{\boldsymbol{\ell}} g_\ell(\mathbf{u}) \mathbf{u}^{\mathbf{k}-\boldsymbol{\ell}}}{(\mathbf{k}-\boldsymbol{\ell})!} d\mathbf{u} = 0, \quad \forall \mathbf{k} \in \mathbb{Z}^n_+.$$

On the other hand, it is easy to verify that

$$T(\mathbf{u}^{\mathbf{k}}) = \sum_{\boldsymbol{\ell}\leq\min(\mathbf{k},\mathbf{m})} \int\limits_{\mathbb{R}^n} \frac{(-1)^{\boldsymbol{\ell}} g_\ell(\mathbf{u}) \mathbf{u}^{\mathbf{k}-\boldsymbol{\ell}} \mathbf{k}!}{(\mathbf{k}-\boldsymbol{\ell})!} d\mathbf{u}, \quad \forall \mathbf{k} \in \mathbb{Z}^n_+.$$

Hence,

$$\overline{\operatorname{span}}\left\{f(\mathbf{x}+\mathbf{s}) : \mathbf{s} \in S^n\right\} = \overline{\operatorname{span}}\left\{\mathbf{u}^{\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^n_+\right\}. \tag{24}$$

It follows from the Taylor expansion theorem that

$$\overline{\operatorname{span}}\left\{\mathbf{u}^{\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^n_+\right\} = H(\mathbb{R}^n). \tag{25}$$

Thus, by combining (24), (25) we complete the proof of the theorem. □

## Acknowledgements

The author wishes to thank Professor Dinh Dung (Din'Zung) for his comments and suggestions during the preparation of this paper. My thanks are due to Professor Allan Pinkus for many valuable disscusions.

## References

1. M. L. Agranovski and E. T. Quinto, *Injectivity sets for the Radon transforms over circles and complete systems of radial functions*, preprint.
2. Dinh Dung, *The sampling theorem $L^t_q$-Approximation and $\varepsilon$-Dimension*, J. Approximation. Theory. **70** (1992), 1-15.
3. A. Friedman, *Generalized Functions and Partial Differential Equations*, Prentice Hall, 1963.
4. Khang Zhao, *Density of dialates of a shift-invariant subspace*, J. Math. and Appl. **184** (1994), 517-532.
5. S. N. Nikol'skii, *Approximation of Functions of Several Variables and Embedding Theorems*, Math. Nauka, 1997 (in Russian)
6. A. Pinkus and B. Wajnryb, *Multivariate polynomials: a spanning question*, to appear in Constructive Approximation.
7. A. Pinkus and B. Wajnryb, *A problem of approximation using multivariate polynomials*, Russian Mathematical Surveys **50** (1995), 319-341.
8. A. Pinkus, *TDI subspaces of $C(\mathbb{R}^d)$ and some density Problems from Neural networks*, Journal of Approximation Theory **85** (1996), 269-287.
9. A. Pinkus, *Some density problems in multivariate approximation*, Proc. Dortmund Conference 1996.
10. V. V. Volchkov, *Approximation of functions on bounded domain in $\mathbb{R}^n$ by linear combination of shifts*, Dolk. USSR. **334**, No 5 (1994), 560-561, (in Russian).
11. N. Wiener, *The Fourier integral and certain of its applications*, Cambridge University Press, 1933.
12. R. A. Zalik, *On approximation by shifts and a theorem of Wiener*, Trans. Amer. Math. Soc. **243** (1987), 299-308.
13. R. A. Zalik, *Approximation by nonfundamental sequences of translates*, Proc. Amer. Math. Soc. **78** (1980), 261-266.
14. R. A. Zalik. *On fundamental sequences of translates*, Proc. Amer. Math. Soc. **79** (1980), 255-259.
15. R. A. Zalik, *The Munt-Szasz Theorem and the closure of translates*, J. Math. Anal. Appl. **82** (1981), 361-369.

Faculty of Mathematics-Mechanics-Informatics
College of Natural Sciences
National University of Hanoi
90 Nguyen Trai, Hanoi, Vietnam