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ON A FIXED POINT THEOREM FOR NONEXPANSIVE NONLINEAR OPERATOR

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Abstract. A new generalized fixed point Edelstein's theorem and some results generalizing the ones of W. V. Petryshyn and T. E. Williamson [10] for nonexpansive and condensing mapping are proved.

1. Introduction

The aim of this note is to study the existence of solutions of the equation

$$x = T(x, x) \tag{1}$$

and the approximation for nonlinear nonexpansive operator T.

It is clear that the equation $x = T(x, x)$ contains the equation $x = T(x)$ as a special case. Instead of Picard iteration (P. it.) which is widely used in fixed point theory we apply a projection-iteration method (Pr. it. m.) for approximating the solutions of equation (1). This method was studied by N. S. Kurpel in his book [8], in which under some assumptions both P. it. ($x_n = T(x_{n-1}, x_{n-1})$) and Pr. it. m. (for instance, $x_n = T(x_n, x_{n-1})$) converge to a solution of (1). Here we shall extend the fixed point Edelstein's theorem of [4] to the operator $T(.,.)$ and obtain a new generalized fixed point Edelstein's theorem. From this theorem and an example we can see that while the P. it. is not applicable the Pr. it. m. may be applied. Note that the main results presented here were published in [2] without proofs.

Our paper is organized as follows. In Section 2 we shall prove the generalized fixed point Edelstein's theorem. Section 3 is devoted to a more general case when T is a nonexpansive and condensing operator. Finally some examples will be given to illustrate our results.

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2. Generalized fixed point Edelstein's theorem

Theorem 1. *Let D be a closed bounded subset of a normed space X, T a continuous mapping from $D \times D$ into D, which satisfies the following nonexpansive condition:*

$$(2)\quad ||T(x,y)-T(z,t)|| \begin{cases} < \max\{||x-z||, ||y-t||\}, & \text{if } (x,y) \neq (z,t) \\ \qquad\qquad \text{and } x \neq y & \text{or } z \neq t \\ \leq ||x-z|| = ||y-t||, & \text{if } x = y \text{ and } z = t \end{cases}$$

for all $x, y, z, t \in D$. Suppose further that either D is compact or T maps $D \times D$ into a compact subset of D. Then equation (1) *has a solution on D. Moreover, every equation $x_n = T(x_n, x_{n-1})$ has a unique solution x_n, $n \geq 1$, and the sequence $\{x_n\}$ so defined converges to a solution of* (1) *for every $x_0 \in D$.*

Proof. For any fixed $v \in D$ let $T_v(x) = T(x, v)$. We can see by (2) that the operator $T_v : D \longrightarrow D$ is strictly nonexpansive, that is

$$(3)\qquad ||T_v(x) - T_v(y)|| < ||x - y||, \quad \forall x, y \in D,\ x \neq y.$$

Moreover, $T_v(D)$ is a compact set in the case $T(D, D)$ is compact. Hence by Edelstein's theorem [4], there exists only a fixed point $\overline{x}$ of T_v with $\overline{x} = T_v(\overline{x}) = T(\overline{x}, v)$. So $x_n = T(x_n, x_{n-1})$ has a unique solution x_n for any fixed $x_0 \in D$, $n \geq 1$.

Define $a_n = ||x_{n+1} - x_n||$. If for some m, $x_{m+1} = x_m$, then x_m is a solution of (1). Suppose $x_{m+1} \neq x_m$ $\forall m$. Then

$$\begin{aligned} a_n &= ||T(x_{n+1}, x_n) - T(x_n, x_{n-1})|| \\ &< \max\{||x_{n+1} - x_n||, ||x_n - x_{n-1}||\} = \max\{a_n, a_{n-1}\}, \end{aligned}$$

hence $a_n < a_{n-1}$, so $a_n \to a_0 \geq 0$. By the compactness of D (or $T(D, D)$) there exist convergent subsequences of $\{x_n\}$:

$$x_{n_i} \to u, \quad x_{n_i+1} \to u_1, \quad x_{n_i-1} \to u_{-1}.$$

If $a_0 > 0$, then $a_0 = \lim_i a_{n_i} = \lim_i ||x_{n_i+1} - x_{n_i}|| = ||u_1 - u||$, but

$$||u_1 - u|| = ||T(u_1, u) - T(u, u_{-1})|| < \max\{||u_1 - u||, ||u - u_{-1}||\}.$$

Consequently,

$$||u_1 - u|| < ||u - u_{-1}|| = \lim_i ||x_n - x_{n_i-1}|| = \lim_i a_{n_i} = a_0,$$

which is a contradiction. Thus $a_0 = 0$ and $u_1 = u_{-1} = u$ is a solution of $x = T(x,x)$.

Now again by condition (2) it is clear that

$$||x_n - u|| < ||x_{n-1} - u||.$$

It follows that $\lim_n ||x_n - u||$ exists. Since $a_0 = 0$, we get $\lim_i ||x_{n_i} - u|| = 0$, $\lim_n ||x_n - u|| = 0$. Theorem 1 is thus proved.

Remark 1. It is not difficult to see that if under the assumptions of Theorem 1 we use Picard iteration for $T^*(x) = T(x,x)$ (which has been investigated by M. Edelstein [4], L. P. Belluce and W. A. Kirk [1], F. E. Browder [3], W. V. Petryshyn and T. E. Williamson [9]) we cannot assert either the existence of the solution of equation (1) or its approximate solutions.

Similarly as for Theorem 1 we get

Theorem 1'. *Let D be a closed bounded subset of a metric space (X,d), $g(.,.)$ a function from $D \times D$ into $(0,\infty)$ having the properties*
(i) $g(x,y) = 0$ *if and only if* $x = y$, $\forall x,y \in D$,
(ii) g *is continuous in the pair* (x,y),
(iii) *if* $g(x,y) \to 0$ *then* $d(x,y) \to 0$.
Let T be an operator from $D \times D$ into D which satisfies the nonexpansive condition

$$(1') \quad g(T(x,y),T(z,t)) \begin{cases} < \max\{g(x,z), g(y,t)\}, & \text{if } (x,y) \neq (z,t) \\ \qquad\qquad \text{and } x \neq y & \text{or } z \neq t \\ \leq g(x,z) = g(y,t), & \text{if } x = y \text{ and } z = t \end{cases}$$

for all $x,y,z,t \in D$. Suppose further that either D is compact or T maps $D \times D$ into a compact subset of D. Then the conclusion of Theorem 1 remains valid.

Remark 2. The function g need not to be a metric. Indeed, g may not satisfy the triangle inequality or even the identity $g(x,y) = g(y,x)$. For example, $g(x,y) = (x-y)^2$, or $g(x,y) = |\exp(x-y) - 1|$, $D = [0,1] \subset \mathbb{R}$.

Corollary. *Let $T = U + S$, where U and S are operators from D into D, D is a subset of a linear metric space X, such that $T(x,y) = U(x) + S(y)$ satisfies condition (1') with $g(.,.)$ of Theorem 1'. Suppose futher that either D is compact or D is closed, bounded and U, S are compact. Then T has* *a fixed point on D. Moreover, fixed points of T could be approximated by the Pr. it. m.*

3. Condensing operator

In this section we consider more general cases, when T is a nonexpansive and condensing operator. The concept of a condensing mapping was first introduced by B. N. Sadovsky [12] with the ball-measure of noncompactness $\chi(\Omega)$ of a set Ω ($\chi(\Omega) = 0$ iff Ω is compact) and later by M. Furi and A. Vignoli [6] with the set-measure of noncompactness $\gamma(\Omega)$ (the definitions, differences and common properties of ball-measure and set-measure can be found in [10], [11] and also in the references of those papers).

In our paper the same arguments work for condensing operators defined either in terms of γ or in terms of χ. So we shall use only the notation γ for both measures. One of the results of W. V. Petryshyn and T. E. Williamson [10] is an assertion on the convergence of successive approximations to a fixed point of strictly nonexpansive (or nonexpansive in the case where X is strictly convex) and γ-condensing self-mappings. We extend that result to mappings $T : D \times D \to D$, and instead of successive approximation we use the Pr. it. m.

Theorem 2. *Let D be a closed bounded convex subset of a Banach space X. Suppose that the operator $T : D \times D \to D$ satisfies nonexpansive condition* (1) *and the condensing conditions*

$$\gamma(T(U,U)) < \gamma(U), \text{ for each } U \text{ of } D \text{ with } \gamma(U) > 0, \tag{4}$$

$$\gamma(T(U,x)) < \gamma(U), \tag{5}$$

for each U of D with $\gamma(U) > 0$ and each point x on D. Then every equation $x_n = T(x_n, x_{n-1})$ has a unique solution $x_n, n \geq 1$, and the sequence $\{x_n\}$ so defined converges to a solution of (1) *for any $x_0 \in D$.*

Proof. For any fixed $v \in D$, by (2) and (3) we can see that the operator $T_v : D \to D$, where $T_v(x) = T(x, v)$, is strictly nonexpansive and γ-condensing. Hence the unique solution of the equation $\overline{x} = T_v(\overline{x})$ follows from [12].

To verify the convergence of the sequence $\{x_n\}$ to a solution of (1) it suffices to show its compactness. For we may apply Theorem 1 for the rest of the proof. Supposing $\gamma\big(\{x_n\}_0^\infty\big) > 0$ and setting $C = \{x_n\}_1^\infty$. By (4) and (5) and by the properties of the meassure γ we have

$$\begin{aligned}\gamma(\{x_n\}_0^\infty) &= \gamma(C) = \gamma(\{T(x_n, x_{n-1})\}_1^\infty)\\ &\leq \gamma(T(\{x_n\}_1^\infty, \{x_{n-1}\}_1^\infty)) = \gamma(T(C,C) \cup T(C,x_0))\\ &= \max\{\gamma(T(C,C)), \gamma(T(C,x_0))\} < \gamma(C),\end{aligned}$$

which is a contradiction. Consequently $\gamma(\{x_n\}_0^\infty) = 0$, so $\{x_n\}$ is compact. Theorem 2 is thus proved.

We now deal with nonexpansive condensing operator in uniformly convex Banach space, where the condition on T of Theorem 2 can be relaxed.

Theorem 3. *Let D be a closed bounded convex subset of a uniformly convex Banach space X, T a continuous operator from $D \times D$ into D satisfying the conditions*

(6)
$$||T(x,y) - T(z,t)|| \begin{cases} < \max\{||x-z||, ||y-t||\}, & \text{if } ||x-z|| \neq ||y-t|| \\ \leq ||x-z|| = ||y-t|| \end{cases}$$

for all $x, y, z, t \in D$,

$$\gamma(T(U,V)) < \max\{\gamma(U), \gamma(V)\} \tag{7}$$

for subsets U, V of D such that $\gamma(U \setminus V) > 0$. Then there exist numbers λ_n, $0 < a < \lambda_n < b < 1$, $n \geq 1$, where a, b are constants, such that the sequence $\{x_n\}$ defined by

$$x_n = \lambda_n x_{n-1} + (1-\lambda_n)\overline{x}_n, \tag{8}$$

where $\overline{x}_n = T(\overline{x}_n, x_{n-1})$, converges to a solution of (1) *for any $x_0 \in D$.*

Proof. For any fixed $v \in D$ the operator $T_v : D \longrightarrow D$ is strictly nonexpansive by (6). Note that (5) follows from (7), hence T_v is γ-condensing. So $\overline{x}_n$ are defined uniquely by x_{n-1}.

Now we will show that there exists λ_n such that $\overline{x}_n \neq x_m$, $\forall n, m$. For any $n \geq 1$ we can choose a number $\lambda_n \in (a,b)$ such that $\lambda_n \leq \lambda_{n-1}$ and $x_n \neq \overline{x}_m$, $\forall m = 1, 2, \dots, n-1$, i.e. $\overline{x}_n \neq x_m, \forall m > n$. As in the proof of Theorem 1 we can assume $\overline{x}_n \neq x_{n-1}$, $\forall n \geq 1$. By using the uniform convexity of X and by a theorem by F. Browder [3], T has solutions on D. Let p be one of them. By (6) and by the strict convexity of X for any $n \geq 1$ we have

$$||\overline{x}_n - p|| = ||T(\overline{x}_n, x_{n-1}) - T(p,p)|| \leq ||x_{n-1} - p|| \tag{9}$$

and

$$||x_n - p|| = ||\lambda_n(x_{n-1} - p) + (1 - \lambda_n)(\overline{x}_n - p)|| < ||x_{n-1} - p||.$$

Hence

$$||\overline{x}_n - p|| \leq ||x_{n-1} - p|| < ||x_m - p||, m = 0, 1, \dots, n-2.$$

It follows that

$$\overline{x}_n \neq x_m, m = 0, 1, \dots, n-1.$$

Besides $\overline{x}_n \neq x_n$, we get $\overline{x}_n \neq x_m$, $\forall n, m$.

Next we verify the compactness of $\{\overline{x}_n\}$ and $\{x_n\}$. Note that the sequence $\{\lambda_n\}$ converges to some $\lambda \in [a, b]$ since $\lambda_n \leq \lambda_{n-1}$. Hence

$$\begin{aligned}\gamma(\{\lambda_n x_{n-1}\}) &\leq \gamma(\{(\lambda_n - \lambda)x_{n-1}\}) + \gamma(\{\lambda x_{n-1}\}) \\ &= \gamma(\{\lambda x_{n-1}\}) = \lambda\gamma(\{x_{n-1}\}).\end{aligned}$$

Similarly we get

$$\gamma(\{(1 - \lambda_n)\overline{x}_n\}) \leq (1 - \lambda)\gamma(\{\overline{x}_n\}).$$

Therefore

$$\begin{aligned}\gamma(\{x_n\}_0^\infty) = \gamma(\{x_n\}_1^\infty) &= \gamma(\{\lambda_n x_{n-1} + (1 - \lambda)\overline{x}_n\}_1^\infty) \\ &\leq \gamma(\{\lambda_n x_{n-1}\}) + \gamma(\{(1 - \lambda_n)\overline{x}_n\}) \\ &\leq \lambda\gamma(\{x_n\}_0^\infty + (1 - \lambda)\gamma(\{\overline{x}_n\}_1^\infty).\end{aligned}$$

Thus

$$\gamma(\{x_n\}) \leq \gamma(\{\overline{x}_n\}).$$

On the other hand, if $\{\overline{x}_n\}$ is not compact, i.e. $\gamma(\{\overline{x}_n\}) > 0$, then by (7)

$$\begin{aligned}\gamma(\{\overline{x}_n\}) = \gamma(\{T(\overline{x}_n, x_{n-1})\}) &\leq \gamma(T(\{\overline{x}_n\}, \{x_n\})) \\ &< \max\{\gamma(\{\overline{x}_n\}, \gamma(\{x_n\})\} \leq \gamma(\{\overline{x}_n\}),\end{aligned}$$

which is a contradiction. Consequently $\gamma(\{\overline{x}_n\}) = 0$ and hence $\gamma(\{x_n\}) = 0$. So both $\{\overline{x}_n\}$ and $\{x_n\}$ are compact.

It is easy to see that there exists an index subsequence n_i such that $x_{n_i} \to u,\ x_{n_i-1} \to u_{-1},\ \overline{x}_{n_i} \to \overline{u},\ \lambda_{n_i} \to \lambda$, as $i \to \infty$. By (8) we have

$$u = \lambda u_{-1} + (1 - \lambda)\overline{u} \quad \text{and } \overline{u} = T(\overline{u}, u_{-1}).$$

It will be proved that

$$u = u_{-1} = \overline{u}.$$

Assume the contrary $\overline{u} \neq u_{-1}$. Then by (9) we have

$$\begin{aligned} ||\overline{u} - p|| \leq ||u_{-1} - p|| = \lim_i ||x_{n_i - 1} - p|| = \lim_n ||x_n - p|| \\ = \lim_i ||x_{n_i} - p|| = ||u - p||. \end{aligned}$$

By the strict convexity of X,

$$\begin{aligned} ||u - p|| &= ||\lambda(u_{-1} - p) + (1 - \lambda)(\overline{u} - p)|| \\ &< ||u_{-1} - p|| = ||u - p||, \end{aligned}$$

which is a contradiction. So $\overline{u} = u_{-1} = u$ is a solution of (1). Finally, taking u instead of p we get

$$\lim_n ||x_n - u|| = \lim_i ||x_{n_i} - u|| = 0.$$

The proof of Theorem 3 is complete.

4. Examples

Example 1. Let us consider the set $D = [0, 1]$ and

$$T(x, y) = sin \frac{x + y}{2}$$

It is not difficult to show that the operator T satisfies all conditions of Theorem 1. Note that T is not contractive.

Example 2. Let

$$D = l_p,\ 1 < p < \infty, \quad T(x, y) = \frac{\tau y}{\exp(||x - y||)}$$

where $\tau : x = (x_1, x_2, \dots) \to \tau x = (0, x_1, x_2, \dots)$, $x \in l_p$. It is not hard to check that for arbitrary $x_0 = (x_{01}, x_{02}, \dots)$, $x_{01} \neq 0$, the sequence defined by

$$x_n = T(x_n, x_{n-1})$$

converges to 0 as $n \to \infty$, but the sequence constructed by

$$x_n = T(x_{n-1}, x_{n-1})$$

does not.

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