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# APPROXIMATION ORDERS IN THE CONDITIONAL CENTRAL LIMIT THEOREM FOR WEAKLY DEPENDENT RANDOM VARIABLES

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Abstract. Let $(X_n)_{n\geq 1}$ be a stationary, strong mixing sequence of random variables with $EX_n=0$, $EX_n^2=1$ and let $B\in\sigma(X_1,X_2,...,X_n,...)$ with $P(B)>0$. In this note we establish an estimation for the quantity

$$\Delta_n(B)=\sup_{t\in R}|P(S_n.(ES_n^2)^{-1/2}<t|B)-\Phi(t)|,$$

where $\Phi(t)$ is a standard normal distribution function and $S_n=\sum_{i=1}^{n} X_i$.

## 1. Introduction

Let $(X_n)_{n\geq 1}$ be a sequence of random variables with $EX_n = 0$ and $EX_n^2 = 1$. The sequence $X_n$ is said to be strong mixing (in the sense of Rosenblatt) if

$$\sup |P(E_1 \cap E_2) - P(E_1)P(E_2)| = \varrho(n) \downarrow 0 \quad \text{as} \quad n \to \infty, \tag{1.1}$$

where the supremum in (1.1) is taken over all $E_1 \in \sigma(X_1, ..., X_k)$, $E_2 \in \sigma(X_{k+n}, X_{k+n+1}, ...)$ and over all $k = 1, 2, ....$ The function $\varrho(n)$ of (1.1) is called the mixing coefficient. The sequence $X_n$ is said to be $\phi$-mixing (in the sense of Ibragimov) if

$$|P(E_1 \cap E_2) - P(E_1)P(E_2)| \leq \phi(n)P(E_2) \tag{1.2}$$

for all $E_1 \in \sigma(X_1, ..., X_k)$ and $E_2 \in \sigma(X_{k+n}, X_{k+n+1}, ...)$. For i.i.d. sequences of random variabless, the classical theorem of Berry-Esseen gives an estimation of the rate of convergence to the normal law as follows:

$$\Delta_n(\Omega) = \sup_{t\in R} |P(S_n.(S_n^2)^{-1/2} < t) - \Phi(t)| = 0(n^{-1/2}),$$

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where $\Phi(t)$ is a standard normal distribution function.

A. Renyi [6] firstly showed that $\Delta_n(B) \to 0$ as $n \to \infty$ for arbitrary subset $B$. This theorem (which is called conditional central limmit theorem) plays an important role in the theory of random summation, in problems of random walk, in the sequential estimation ...

Landers, D. and Rogge, L. [5] proved that

$$\Delta_n(B) = O(n^{-1/2})$$

if $E|X_1|^p < \infty$ for some $p > 3$ and

$$\begin{aligned} d(B, \sigma(X_1, X_2, ..., X_n)) &= \inf\{P(B\Delta A) : A \in \sigma(X_1, X_2, ..., X_n)\} \\ &= 0\Big(\frac{1}{n^{1/2}(\log n)^{3/2}}\Big) \cdot \end{aligned}$$

For an unconditional central limit theorem (that means when $B = \Omega$) Stein [7] showed that if $(X_n)$ is stationary, $\phi$-mixing with $E(X_1^8) < \infty$, then

$$\Delta_n(\Omega) = O(n^{-1/2}).$$

In an earlier paper [2] the author extended the result of D. Landers and L. Rogge to the case of stationary, $\phi$-mixing sequences of random variables as follows.

**Theorem 1.1** [2]. *Let $B \in \sigma(X_1, X_2, ...)$ with $P(B) > 0$ and let $(X_n)_{n\geq 1}$ be a stationary, $\phi$-mixing sequence of random variables such that*

(i) $E|X_1|^{p+\varepsilon} < \infty$ *for some* $p > 8, \ \varepsilon > 0$,

(ii) $\phi(n) \leq C.n^{-\theta}, \ C > 0, \ \theta > 0$,

(iii) $d(B, \sigma(X_1, ..., X_n)) = \inf\{P(B\Delta A) : A \in \sigma(X_1, ..., X_n)\}$
$= O(n^{-(\frac{1}{2}+\delta)}(\log n)^{-r}), \ r > 1, \ \delta > 0.$

*Then*

$$\Delta_n(B) = O(n^{-(\frac{1}{2}-\varepsilon(p,\delta))}),$$

*where*

$$\varepsilon(p, \delta) = \frac{1}{p} = \varepsilon + \frac{1}{p + 4\delta p} \cdot$$

However, condition (i) is very strong even when we can obtain such an approximation order as in the unconditional case.

In this note, we investigate approximation order of $\Delta_n(B)$ for the class of stationary, strong mixing process (which is wider than the class of

$\phi$-mixing processes) and under assumption that the stationary sequence $(X_n)$ has only $s$-th order finite moment for $2 < s < 3$.

Our main result is the following theorem

**Theorem 1.2.** *Let $(X_n)_{n\geq 1}$ be a strictly stationary, strong mixing sequence of random variables with mixing coefficient*

$$\varrho(n) < K.n^{-\theta},$$

*where $K > 0$, $\theta > \dfrac{3}{2}$, and $EX_1 = 0$, $EX_1^2 = 1$, $E|X_1|^s < \infty$,*

$$2 < s < \min\left\{\frac{5}{2}, s_0(\theta)\right\}, \quad s_0(\theta) = \frac{\theta - 1}{\theta} + \sqrt{\left(\frac{\theta-1}{\theta}\right)^2 + \frac{4+2\theta}{\theta}}\ .$$

*Assume that*

$$ES_n^2 \geq \mu n EX_1^2, \quad \mu > 0.$$

*Let $B \in \sigma(X_1, X_2, ..., X_n, ...)$ with $P(B) > 0$ such that*

$$\begin{aligned} d(B, \sigma(X_1, X_2, ..., X_n)) &= \inf\{P(A\Delta B) : A \in \sigma(X_1, ..., X_n)\} \\ &= O\Big(\frac{1}{n^{\frac{1}{2}+\delta}(\log n)^r}\Big), \quad \delta > \frac{s-2}{s(4-s)}\ . \end{aligned}$$

*Then*

$$\Delta_n(B) = \sup_{t\in R}\ |P(S_n.(ES_n^2)^{-1/2} < t|B) - \Phi(t)| = O\Big(\frac{\log n}{n^{\frac{s-2}{2}}}\Big)$$

## 2. Proof of Theorem 1.2

We need some auxiliary results.

**Lemma 2.1** (see [3], Lemma 5.4, page 528). *Let $X$ and $Y$ be random variables with $|X| \leq 1$ and $EX = 0$. Then*

$$|E(XY)| \leq 4E|Y|\ \varrho(\sigma(X), \sigma(Y)),$$

*where*

$$\varrho(\sigma(X), \sigma(Y)) = \sup|P(A \cap B) - P(A)P(B)|,$$

*the supremum being taken over all sets $A \in \sigma(X)$, $B \in \sigma(Y)$.*

**Lemma 2.2** (see [8], page 636). *Let $(X_n)_{n\geq 1}$ be a stationary, strong mixing sequence of random variables with mixing coefficient*

$$\varrho(n) < K.n^{-\theta}, \quad \theta > 0,\ K > 0,$$

*and*

$$EX_1 = 0, \quad E|X_1|^s < \infty,$$

$$2 < s < s_0(\theta) = \frac{\theta - 1}{\theta} + \sqrt{\Big(\frac{\theta-1}{\theta}\Big)^2 + \frac{4+2\theta}{\theta}}\ . \tag{2.1}$$

*If*

$$ES_n^2 \geq \mu n EX_1^2, \quad \mu > 0,$$

*there exists a constant $C(s,\theta,K,\mu)$ depending only on $s$, $\theta$, $K$, and $\mu$ such that*

$$\Delta_n(\Omega) = \sup_{t\in R} |P(S_n.n^{-1/2} < t) - \Phi(t)| \leq \quad C(s,\theta,K,\mu)\frac{\beta_s}{n^{s-2/2}}\ , \tag{2.2}$$

*where* $\quad \beta_s = \dfrac{E|X_1|^s}{(EX_1^2)^{s/2}}\ .$

*Proof of Theorem 1.2.* By [1, page 170-172] we have $ES_n^2 \sim \sigma.n$, where

$$\sigma^2 = EX_1^2 + 2\sum_{k=2}^{\infty} EX_1X_k < \infty.$$

So, without lost of generality we may assume that $ES_n^2 = n$. We put

$$S_{i,j} = \sum_{k=i+1}^{j} X_k \quad \text{for} \quad i < j \quad \text{and} \quad N_1 = \{2^i :\ i \geq 1\}.$$

Consider the following sets:

$$A_n^k = (S_n < t\sqrt{n}) = \Big(\frac{S_{2k,n}}{\sqrt{n-2k}} < \frac{t\sqrt{n}}{\sqrt{n-2k}} - \frac{S_{2k}}{\sqrt{n-2k}}\Big)\,,$$

$$B_n^k = \Big(\frac{S_{2k,n}}{\sqrt{n-2k}} < \frac{t\sqrt{n}}{\sqrt{n-2k}} + \frac{C(k)}{\sqrt{n-2k}}\Big)\,,$$

$$C_n^k = \Big(\frac{S_{2k,n}}{\sqrt{n-2k}} < \frac{t\sqrt{n}}{\sqrt{n-2k}} - \frac{C(k)}{\sqrt{n-2k}}\Big)\,,$$

$$D_k = (|S_2k| \geq C(k))\,,$$

$$D_k^c = (|S_{2k}| < C(k))\,,$$

where $C(k)$ is a constant depending on $k$. On one hand, we have

$$A_n^k = (A_n^k D_k \cup A_n^k D_k^c) \subseteq D_k \cup B_n^k. \tag{2.3}$$

On the other hand, we have

$$C_n^k = C_n^k D_k \cup C_n^k D_k^c \subseteq A_n^k D_k \cup A_n^k D_k^c \subseteq D_k \cup A_n^k. \tag{2.4}$$

These relations imply

$$1_{A_n^k} \leq 1_{D_k} + 1_{B_n^k}, \tag{2.5}$$

$$1_{A_n^k} \geq 1_{C_n^k} - 1_{D_k}, \tag{2.6}$$

where $1_.$ denotes the indicater function of the given event. Combining (2.5) and (2.6) we finaly obtain

$$1_{C_n^k} - 1_{D_k} - \Phi(t) \leq 1_{A_n^k} - \Phi(t) \leq 1_{B_n^k} + 1_{D_k} - \Phi(t). \tag{2.7}$$

Now we choose $B_k \in \sigma(X_1, ..., X_k)$ such that

$$P(B \Delta B_k) \leq \frac{C}{k^{\frac{1}{2}+\delta} (\log n)^r},$$

where $C$ is a constant. Then

$$\begin{aligned} |P(B).\Delta_n(B)| &= |P(A_n^k B) - \Phi(t) P(B)| \\ &= |E[1_{A_n^k} - \Phi(t)] 1_B| \\ &\leq |E[1_{A_n^k} - \Phi(t)][1_B - 1_{B_{n_0}}]| \\ &\quad + \sum_{\substack{n_1 \leq k \leq n_0 \\ k \in N_1}} |E[1_{A_n^k} - \Phi(t)][1_{B_k} - 1_{B_{\frac{k}{2}}}]| \\ &\quad + |E[1_{A_n^k} - \Phi(t)] 1_{B_{n_1}}| = I_1 + I_2 + I_3, \end{aligned} \tag{2.8}$$

where $n_0$, $n_1 \in N_1$, $n_1 < n_0$, will be choosen later.

The term $I_2$ will be estimative as follows. By (2.7) we get

$$\begin{aligned} E[1_{C_n^k} - \Phi(t)] 1_{B_k} - E(1_{D_k} 1_{B_k}) &\leq E[1_{A_n^k} - \Phi(t)] 1_{B_k} \\ &\leq E[1_{B_n^k} - \Phi(t)] 1_{B_k} + E(1_{D_k} 1_{B_k}), \end{aligned} \tag{2.9}$$

$$
\begin{aligned}
&E[1_{C_n^k} - \Phi(t)]1_{B_{k/2}} - E(1_{D_k}1_{B_{k/2}}) \le E[1_{A_n^k} - \Phi(t)]1_{B_{k/2}} \\
&\qquad\qquad \le E[1_{B_n^k} - \Phi(t)]1_{B_{k/2}} + E(1_{D_k}1_{B_{k/2}}).
\end{aligned} \tag{2.10}
$$

These relations imply

$$
\begin{aligned}
&E[1_{C_n^k} - \Phi(t)]1_{B_k} - 2P(D_k) - E[1_{B_n^k} - \Phi(t)]1_{B_{k/2}} \\
&\qquad \le E[1_{A_n^k} - \Phi(t)][1_{B_k} - 1_{B_{k/2}}] \\
&\qquad \le E[1_{B_n^k} - \Phi(t)]1_{B_k} + 2P(D_k) - E[1_{C_n^k} - \Phi(t)]1_{B_{k/2}}.
\end{aligned} \tag{2.11}
$$

It follows from (2.11) that

$$
\begin{aligned}
&E[1_{C_n^k} - \Phi(t)][1_{B_k} - 1_{B_{k/2}}] - 2P(D_k) - E[1_{B_n^k} - 1_{C_n^k}]1_{B_{k/2}} \\
&\qquad \le E[1_{A_n^k} - \Phi(t)][1_{B_k} - 1_{B_{k/2}}] \\
&\qquad \le E[1_{B_n^k} - \Phi(t)][1_{B_k} - 1_{B_{k/2}}] \\
&\qquad\quad + 2P(D_k) + E[1_{B_n^k} - 1_{C_n^k}]1_{B_{k/2}}.
\end{aligned} \tag{2.12}
$$

Finally we get

$$
\begin{aligned}
&|E[1_{A_n^k} - \Phi(t)][1_{B_k} - 1_{B_{k/2}}]| \le |E[1_{B_n^k} - \Phi(t)][1_{B_k} - 1_{B_{k/2}}]| + \\
&\qquad + |E[1_{C_n^k} - \Phi(t)][1_{B_k} - 1_{B_{k/2}}]| \\
&\qquad + E[1_{B_n^k} - 1_{C_n^k}]1_{B_{k/2}} + 2P(D_k).
\end{aligned} \tag{2.13}
$$

From (2.8) and (2.13) we have

$$
\begin{aligned}
I_2 \le & \sum_{\substack{k\in N_1 \\ n_1 \le k \le n_0}} |E[1_{B_n^k} - \Phi(t)][1_{B_k} - 1_{B_{\frac{k}{2}}}]| \quad + \\
& + \sum_{\substack{k\in N_1 \\ n_1 \le k \le n_0}} |E[1_{C_n^k} - \Phi(t)][1_{B_k} - 1_{B_{\frac{k}{2}}}]| + \\
& + \sum_{\substack{k\in N_1 \\ n_1 \le k \le n_0}} E[1_{B_n^k} - 1_{C_n^k}]1_{B_{\frac{k}{2}}} + \sum_{\substack{k\in N_1 \\ n_1 \le k \le n_0}} 2P(D_k) \\
= & \, T_1 + T_2 + T_3 + T_4.
\end{aligned} \tag{2.14}
$$

We shall estimate each term of the right-hand side of (2.14) as follows.

For the first term $T_1$ we have

$$T_1 \le \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} |E[1_{B_n^k} - \Phi(t)][1_{B_k} - 1_{B_{k/2}}]|$$

$$\le \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} |E[(1_{B_n^k} - \Phi(t)) - (P(B_k) - \Phi(t))][1_{B_k} - 1_{B_{k/2}}]$$

$$+ \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} |[P(B_n^k) - \Phi(t)]E[1_{B_k} - 1_{B_{k/2}}]|.$$

Using Lemma 2.1 and noting that $X = (1_{B_n^k} - \Phi(t)) - (P(B_k) - \Phi(t))$ and $Y = 1_{B_k} - 1_{B_{k/2}}$ we get

$$T_1 \le \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} 4\varrho(k)E|1_{B_k} - 1_{B_{k/2}}|$$

$$+ \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} |[P(B_n^k) - \Phi(t)]E[1_{B_k} - 1_{B_{k/2}}]|$$

$$= \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} 4\varrho(k)E|1_{B_k} - 1_{B_{k/2}}|$$

$$+ \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} |[P\Big(\frac{S_{2k,n}}{\sqrt{n-2k}} < \frac{t\sqrt{n}}{\sqrt{n-2k}} - \frac{C(k)}{\sqrt{n-2k}}\Big)$$

$$- \Phi\Big(\frac{t\sqrt{n}}{\sqrt{n-2k}} - \frac{C(k)}{\sqrt{n-2k}}\Big) + \Phi\Big(\frac{t\sqrt{n}}{\sqrt{n-2k}} - \frac{C(k)}{\sqrt{n-2k}}\Big)$$

$$- \Phi(t)]E[1_{B_k} - 1_{B_{k/2}}]|. \tag{2.15}$$

In view of Lemma 2.2, (2.15) and the following inequalities

$$|\Phi(x) - \Phi(y)| \le \frac{|x-y|}{\sqrt{2\pi}}\,, \tag{2.16}$$

$$|\Phi(\frac{t\sqrt{n}}{\sqrt{n-2k}}) - \Phi(t)| \le \frac{1}{\sqrt{8\pi e}} \cdot \frac{2k}{n-2k}\,, \tag{2.17}$$

we have

$$T_1 \le \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} \Big[4\varrho(k) + \frac{C_1}{(n-2k)^{\frac{s-2}{2}}} + \frac{1}{\sqrt{2\pi}}\frac{C(k)}{\sqrt{n-2k}}$$

$$+ \frac{1}{\sqrt{8e\pi}}\frac{2k}{n-2k}\Big]E|1_{B_k} - 1_{B_{k/2}}|. \tag{2.18}$$

Applying the same procedures as in estimating $T_1$ we obtain that

$$
\begin{aligned}
T_2 &= \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} |E[1_{C_n^k}-\Phi(t)][1_{B_k}-1_{B_{\frac{k}{2}}}]| \\
&\le \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} [4\varrho(k)+\frac{C_1}{(n-2k)^{\frac{s-2}{2}}}+\frac{1}{\sqrt{2\pi}}\frac{C(k)}{\sqrt{n-2k}}+ \\
&\qquad +\frac{1}{\sqrt{8e\pi}}\frac{2k}{n-2k}]E|1_{B_k}-1_{B_{k/2}}|. \qquad (2.19)
\end{aligned}
$$

Since $1_{B_n^k}\ge 1_{C_n^k}$ and using Lemma 2.2 we obtain

$$
\begin{aligned}
T_3 &= \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} E[1_{B_n^k}-1_{C_n^k}]1_{B_{\frac{k}{2}}} \le \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} E[1_{B_n^k}-1_{C_n^k}] \\
&= \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} [P(B_n^k)-\Phi\Big(\frac{t\sqrt{n}}{\sqrt{n-2k}}+\frac{C(k)}{\sqrt{n-2k}}\Big) \\
&\quad +\Phi\Big(\frac{t\sqrt{n}}{\sqrt{n-2k}}+\frac{C(k)}{\sqrt{n-2k}}\Big)-\Phi\Big(\frac{t\sqrt{n}}{\sqrt{n-2k}}-\frac{C(k)}{\sqrt{n-2k}}\Big) \\
&\quad +\Phi\Big(\frac{t\sqrt{n}}{\sqrt{n-2k}}-\frac{C(k)}{\sqrt{n-2k}}\Big)-P(C_n^k)] \\
&\le \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} \frac{2C_1}{(n-2k)^{\frac{s-2}{2}}}+\sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} \frac{2}{\sqrt{2\pi}}\frac{C(k)}{\sqrt{n-2k}}\ . \qquad (2.20)
\end{aligned}
$$

Finally, from (2.14) and (2.20) we get

$$
\begin{aligned}
I_2 &\le \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} 4\varrho(k)E|1_{B_k}-1_{B_{k/2}}| \\
&\quad +\sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} \frac{C_1}{(n-2k)^{\frac{s-2}{2}}}E|1_{B_k}-1_{B_{k/2}}| \\
&\quad +\frac{1}{\sqrt{2\pi}}\sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} \frac{C(k)}{\sqrt{n-2k}}E|1_{B_k}-1_{B_{k/2}}|
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{\sqrt{8e\pi}} \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} \frac{2k}{n-2k} E|1_{B_k} - 1_{B_{k/2}}| \\
&+ \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} \frac{2C_1}{(n-2k)^{\frac{s-2}{2}}} + \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} \frac{2}{\sqrt{2\pi}} \frac{C(k)}{\sqrt{n-2k}} \\
&+ \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} P(|S_{2k}| \ge C(k)).
\end{aligned} \tag{2.21}
$$

Each term of the right-hand side of (2.21) will be estimated as follows. First, we have

$$
\begin{aligned}
\sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} 4\varrho(k) E|1_{B_k} - 1_{B_{k/2}}| &\le \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} 4C \frac{1}{k^\theta} \cdot \frac{1}{k^{\frac{1}{2}+\delta}} \cdot \frac{1}{(\log k)^r} \\
&\le \frac{C_2}{n_1^{\frac{1+2\theta+2\delta}{2}}} \le \frac{C_2}{n^{\frac{s-2}{2}}} \, ,
\end{aligned} \tag{2.22}
$$

if we choose

$$
n_1 \le n^{\frac{s-2}{1+2\theta+2\delta}} \; , \tag{2.22a}
$$

where $\quad C_2 = 4C \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} \frac{1}{(\log k)^r}$ . Since $k \le n_0 \le \frac{n}{4}$ we obtain

$$
\begin{aligned}
&\sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} \frac{C_1}{(n-2k)^{\frac{s-2}{2}}} E|1_{B_k} - 1_{B_{k/2}}| \\
&= \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} \frac{C_1}{(n-2k)^{\frac{s-2}{2}}} \cdot \frac{1}{k^{\frac{1}{2}+\delta}} \cdot \frac{1}{(\log k)^r} \\
&\le \frac{1}{n^{\frac{s-2}{2}}} \cdot \sum_{\substack{k\in N_1\\ n_1\le k\le n_0}} \frac{2^{\frac{s-2}{2}} C_1}{(\log k)^r} = \frac{C_3}{n^{\frac{s-2}{2}}} \; \cdot
\end{aligned} \tag{2.23}
$$

Choosing $C(k) = (2k)^{\frac{1}{2}+\delta}$ we have

$$\frac{1}{\sqrt{2\pi}} \sum_{\substack{k\in N_1 \\ n_1\le k\le n_0}} \frac{C(k)}{\sqrt{n-2k}} E|1_{B_k} - 1_{B_{k/2}}|$$

$$= \frac{2^{\frac{1}{2}+\delta}}{\sqrt{2\pi}} \sum_{\substack{k\in N_1 \\ n_1\le k\le n_0}} \frac{1}{\sqrt{n-2k}} \frac{1}{(\log k)^r} \le \frac{C_4}{n^{\frac{1}{2}}}\ , \tag{2.24}$$

$$\frac{1}{\sqrt{8e\pi}} \sum_{\substack{k\in N_1 \\ n_1\le k\le n_0}} \frac{2k}{n-2k} E|1_{B_k} - 1_{B_{k/2}}|$$

$$\le \frac{1}{\sqrt{8e\pi}} \sum_{\substack{k\in N_1 \\ n_1\le k\le n_0}} \frac{2k^{\frac{1}{2}-\delta}}{n-2k} \cdot \frac{1}{(\log k)^r} \le \frac{C_5}{n^{\frac{1}{2}}}\ . \tag{2.25}$$

For the three last term of the right-hand side of (2.21) we have

$$\sum_{\substack{k\in N_1 \\ n_1\le k\le n_0}} \frac{2C_1}{(n-2k)^{\frac{s-2}{2}}} \le \frac{C_6.\log n}{n^{\frac{s-2}{2}}}\ , \tag{2.26}$$

$$\sum_{\substack{k\in N_1 \\ n_1\le k\le n_0}} \frac{2}{\sqrt{2\pi}} \frac{C(k)}{\sqrt{n-2k}} = \sum_{\substack{k\in N_1 \\ n_1\le k\le n_0}} \frac{2}{\sqrt{2\pi}} \frac{(2k)^{\frac{1}{2}+\delta}}{\sqrt{n-2k}}$$

$$\le \quad \frac{2^{\frac{1}{2}+\delta}.n_0^{\frac{1}{2}+\delta}.\log n}{\frac{n}{2}^{\frac{1}{2}}} \le \frac{C_7.\log n}{n^{\frac{s-2}{2}}}\ , \tag{2.27}$$

if we choose $n_0$ such that $n_0 \le n^{\frac{3-s}{1+2\delta}}$.

By Markov inequality we obtain

$$\sum_{\substack{k\in N_1 \\ n_1\le k\le n_0}} P(|S_{2k}| \ge C(k)) \le \log n \cdot \max_{\substack{k\in N_1 \\ n_1\le k\le n_0}} P(|S_{2k}| \ge (2k)^{\frac{1}{2}+\delta})$$

$$\le \log n \cdot \max_{\substack{k\in N_1 \\ n_1\le k\le n_0}} \frac{E|S_{2k}|^s}{[(2k)^{\frac{1}{2}+\delta}]^s}$$

$$= \log n \cdot \max_{\substack{k\in N_1 \\ n_1\le k\le n_0}} \frac{E|S_{2k}|^s}{(2k)^{\frac{s}{2}}.(2k)^{s.\delta}}$$

$$\le C_7 \cdot \log n \cdot \frac{1}{n_1^{s.\delta}} \le \frac{C_7 \cdot \log n}{n^{\frac{s-2}{2}}}\ , \tag{2.28}$$

if we choose $n_1 \ge n^{\frac{s-2}{2s.\delta}}$.

Using (2.9) we estimate the term $I_3$ as follows:

$$\begin{aligned} I_3 &\le 8\varrho(n_1) + \frac{2C_8}{(n-2n_1)^{\frac{s-2}{2}}} + \frac{2}{\sqrt{2\pi}}\frac{C(n_1)}{\sqrt{n-2k}} \\ &\quad + \frac{1}{\sqrt{8\pi e}}\cdot\frac{2n_1}{n-2n_1} + P(|S_{n_1}| \ge \quad C(n_1)) \\ &= S_1 + S_2 + S_3 + S_4 + S_5\,. \end{aligned} \tag{2.29}$$

Note that $\theta \ge \dfrac{3}{2}$, $\delta \le \dfrac{1}{2}$, and $s < 3$. Then we have

$$S_1 \le \frac{8}{n_1^{\theta}} \le \frac{8}{n^{\frac{s-2}{2}}} \tag{2.30}$$

if we take

$$2n^{\frac{s-2}{2s.\delta}} \ge n_1 \ge n^{\frac{s-2}{2s.\delta}}. \tag{2.30a}$$

Since $n_1 \le \dfrac{n}{4}$, $\delta \ge \dfrac{s-2}{s(4-s)}$ and (2.30a), it is easy to see that

$$S_2 \le \frac{2C_7}{n^{\frac{s-2}{2}}}\ , \tag{2.31}$$

$$S_4 \le \frac{C.n^{\frac{s-2}{2s.\delta}}}{\frac{n}{2}} \le \frac{2C}{n^{\frac{s-2}{2}}}\ \cdot \tag{2.32}$$

Now we choose the constant $C(n_1)$ such that

$$n_1^{\frac{1}{2}}.n^{\frac{s-2}{2s}} \le n^{\frac{1}{2}-\frac{s-2}{2}}. \tag{2.33}$$

Then we obtain

$$S_3 \le \frac{2}{\sqrt{2\pi}}\cdot\frac{1}{n^{\frac{s-2}{2}}}, \tag{2.34}$$

$$S_5 \le \frac{c}{n^{\frac{s-2}{2}}}\ \cdot \tag{2.35}$$

To complete the proof of Theorem 1.2, we only need to show that

$$I_1 \le P(B\Delta B_{n_0}) \le \frac{C}{n_0^{\frac{1}{2}+\delta}}\cdot\frac{1}{(\log n)^r} \le \frac{C}{n^{\frac{s-2}{2}}}\ \cdot \tag{2.36}$$

But this is obvious because

$$n_0 \le n^{\frac{3-s}{1+2\delta}} \quad \text{and} \quad 2 < s \le \frac{5}{2}\cdot$$

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