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A CLASSIFICATION OF CONTRACTIVE MAPPINGS IN PROBABILISTIC METRIC SPACES

DO HONG TAN

ABSTRACT. In this work we define some classes of contractive mappings in probabilistic metric spaces, establish the relation between them and prove a fixed point theorem.

1. INTRODUCTION

In 1922 S. Banach established an important result which is called now the contraction principle. Since then this principle has been generalised by many authors. The main results in this direction can be formulated as follows:

Theorem. *Let* (X, d) *be a complete metric space,* T *a mapping of* X *into itself. Then* T *has a fixed point (i.e.* $Tx^* = x^*$*) if one of the following conditions is satisfied:*

1) *There is a constant* $k \in [0, 1)$ *such that*

$$d(Tx, Ty) \leq kd(x, y)$$

for every $x, y \in X$ (Banach [1]).

2) *There is a non-increasing function* $k : (0, \infty) \to [0, 1)$ *such that*

$$d(Tx, Ty) \leq k(d(x, y))d(x, y) \tag{1}$$

for every $x, y \in X$ (Rakotch [6]).

3) *There is an upper semicontinuous from the right function* $k : (0, \infty) \to [0, 1)$ *such that* (1) *holds for every* $x, y \in X$ (Boyd-Wong [2]).

4) *There is a function* $k : (0, \infty) \to [0, 1)$ *satisfying the condition*

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(2) $$\sup\,\{k(t) : a \le t \le b\} < 1$$

for $0 < a \le b < \infty$ *such that* (1) *holds for every* $x, y \in X$ (Sadovskij [7]).

5) *There is a function* $k : (0,\infty)^2 \to [0,1)$ *such that*

$$d(Tx, Ty) \le k(\alpha, \beta) d(x, y)$$

for every $x, y \in X$ *satisfying* $0 < \alpha \le d(x, y) \le \beta < \infty$ (Krasnoselskij [4]).

6) *For each* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that*

(3) $$\varepsilon \le d(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \varepsilon \quad \text{(Meir-Keeler [5]).}$$

Moreover, the fixed point x^* *is unique and* $T^n x_0 \to x^*$ *as* $n \to \infty$ *for every* $x_0 \in X$.

Remark. In [10] we have remarked that Condition (3) is equivalent to the condition that

$$d(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \varepsilon.$$

The results mentioned above suggest us to introduce the corresponding classes of contractive mappings in probabilistic metric spaces.

2. Classification of contractive mappings

Let us recall the definition of probabilistic metric spaces.

Definition 1. A mapping $F : (-\infty, \infty) \to [0, 1]$ is called a distribution function if it is non-decreasing and left-continuous with $\inf F = 0$, $\sup F = 1$. The set of all distribution functions is denoted by $\mathcal{D}$.

Definition 2. A probabilistic metric space (briefly, a PM-space) is a pair $(X, \mathcal{F})$, where X is a nonempty set and $\mathcal{F}$ is a mapping from $X \times X$ into $\mathcal{D}$. We denote the distribution $\mathcal{F}(x, y)$ by F_{xy} and $F_{xy}(t)$ stands for the value of F_{xy} at t. The function F_{xy} is assumed to verify the following conditions: for all $x, y, z \in X$,

(PM.1) $F_{xy}(t) = 1$ for all $t > 0$ if and only if $x = y$,

(PM.2) $F_{xy}(0) = 0$,

(PM.3) $F_{xy} = F_{yx}$,

(PM.4) If $F_{xy}(t_1) = 1$ and $F_{yz}(t_2) = 1$ then $F_{xz}(t_1 + t_2) = 1$.

Definition 3. A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following conditions for all $a, b, c, d \in [0,1]$:

(T.1) $\Delta(a, 1) = a$,
(T.2) $\Delta(a, b) = \Delta(b, a)$,
(T.3) $\Delta(c, d) \geq \Delta(a, b)$ for $c \geq a, d \geq b$,
(T.4) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

Definition 4. A Menger space is a triplet $(X, \mathcal{F}, \Delta)$, where $(X, \mathcal{F})$ is a PM-space and Δ is a t-norm such that

$$F_{xz}(t_1 + t_2) \geq \Delta(F_{xy}(t_1), F_{yz}(t_2))$$

for all $x, y, z \in X$ and $t_1, t_2 \geq 0$.

More information on PM-spaces can be found in [8].

Remark 1. Some authors consider a special t-norm satisfying the additional condition that for all $t \in [0,1]$,

$$\Delta(t, t) \geq t. \tag{4}$$

In this case Δ has a very simple form: $\Delta(a, b) = \min\{a, b\}$. Indeed, suppose $a \geq b$. Then by (T.3) and (4),

$$\Delta(a, b) \geq \Delta(b, b) \geq b. \tag{5}$$

Further, by (T.1),(T.2),(T.3) and (5) we get

$$b = \Delta(b, 1) = \Delta(1, b) \geq \Delta(a, b) \geq b.$$

So $\Delta(a, b) = b = \min\{a, b\}$, as asserted above.

In what follows we only consider the case $\Delta = \min$.

Definition 5. Let $(X, \mathcal{F}, \min)$ be a Menger space.

(i) A sequence $\{x_n\}$ is said to be convergent to $x_0 \in X$ (we write $x_n \to x$) if for given $\varepsilon > 0$ and $\lambda > 0$ there is a positive integer N such that $F_{xx_n}(\varepsilon) > 1 - \lambda$ whenever $n \geq N$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for given $\varepsilon > 0$ and $\lambda > 0$ there exists a positive integer N such that $F_{x_n x_m}(\varepsilon) > 1 - \lambda$ whenever $n, m \geq N$.

(iii) A Menger space X is said to be complete if each Cauchy sequence in X converges to some point of X.

Now we are going to introduce six classes of contractive mappings some of which have been considered in the fixed point theory in PM-spaces.

Definition 6. A mapping T from a Menger space $(X, \mathcal{F}, \min)$ into itself is said to belong to the class [B] if there is a constant $k \in (0, 1)$ such that

$$F_{TxTy}(t) \geq F_{xy}(t/k)$$

for every $x, y \in X$.

Definition 7. A mapping $T : X \to X$ is said to belong to the class [R] if there is a non-increasing function $k : (0, \infty) \to (0, 1)$ such that

$$F_{TxTy}(t) \geq F_{xy}(t/k(t)) \tag{6}$$

for every $x, y \in X$ and $t > 0$.

Remark 2. Condition (6) implies the inequality

$$F_{TxTy}(t) \geq F_{xy}(t/k(\alpha))$$

for all $x, y \in X$ and $0 < \alpha \leq t$.

Definition 8. A mapping $T : X \to X$ is said to belong to the class [K] if there is a function $k : (0, \infty)^2 \to (0, 1)$ such that

$$F_{TxTy}(t) \geq F_{xy}(t/k(\alpha, \beta)) \tag{7}$$

for every $x, y \in X$ and $0 < \alpha \leq t \leq \beta < \infty$.

Remark 3. The function k in Definition 8 may be assumed to be non-increasing in α and non-decreasing in β. Indeed, one can replace it by the function

$$h(\alpha, \beta) = \inf\{k(\alpha', \beta') : \alpha' \leq \alpha, \beta' \geq \beta\}$$

for all $\alpha, \beta \in (0, \infty)$.

Definition 9. A mapping $T : X \to X$ is said to belong to the class [S] if there is a function $k : (0, \infty) \to (0, 1)$ satisfying the condition

$$\sup \{k(t) : a \leq t \leq b\} < 1$$

for $0 < a \leq b < \infty$ such that

$$F_{TxTy}(t) \geq F_{xy}(t/k(t)) \tag{8}$$

for every $x, y \in X$ and $t > 0$.

Definition 10. A mapping $T : X \to X$ is said to belong to the class [BW] if there exists a function $k : (0, \infty) \to (0, 1)$ upper semicontinuous from the right such that (8) holds for every $x, y \in X$ and $t > 0$.

Definition 11. A mapping $T : X \to X$ is said to belong to the class [MK] if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$F_{TxTy}(\varepsilon) \geq F_{xy}(\varepsilon + \delta) \tag{9}$$

for every $x, y \in X$.

The main result of this note establishes a relation between the above mentioned classes of mappings.

Theorem 1. *The following inclusions hold:*

$$[B] \subset [R] \subset [K] \subset [S] \subset [BW] \subset [MK].$$

Proof.

1) It is clear that $[B] \subset [R]$.

2) To show that $[R] \subset [K]$ using Remark 2 one put $h(\alpha, \beta) = k(\alpha)$ for all $\alpha, \beta \in (0, \infty)$, where k is the function mentioned in the definition of the class $[R]$.

3) We now show that $[K] \subset [S]$. By Remark 3 we may assume that the function k mentioned in definition of the class $[K]$ is non-increasing in α and non-decreasing in β. We put

$$h(t) = \lim_{\substack{\alpha \to t-0 \\ \beta \to t+0}} k(\alpha, \beta)$$

for each $t > 0$. Since $h(t) \leq k(\alpha, \beta)$ for $\alpha < t < \beta$, one has $h(t) < 1$ for each t and moreover, $\sup \{h(t) : a \leq t \leq b\} < 1$ for every $a \leq b$. It remains to show that h satisfies (8) in place of k. Indeed, by the definition of h, for each fixed $t \geq 0$ there are two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_n \leq t \leq \beta_n$ for all n, such that $k(\alpha_n, \beta_n) \to h(t)$ decreasingly.

So $\dfrac{t}{k(\alpha_n, \beta_n)} \to \dfrac{t}{h(t)}$ increasingly. Since k satisfies (7) and F_{xy} is left-continuous, h must satisfy (8).

4) To show that $[S] \subset [BW]$ it suffices to put $h(t) = \limsup_{s \to t} k(s)$ for all $t \geq 0$, where k is the function mentioned in the definition of the class $[S]$. Obviously, h is upper semicontinuous and satisfies (8).

5) Finally, we show that $[BW] \subset [MK]$. Let $T \in [BW]$ and k satisfy (8). Since k is upper semicontinuous from the right, so is the function $h(t) = tk(t)$ defined for $t > 0$. For a given $\varepsilon > 0$ we have $h(\varepsilon) = \varepsilon k(\varepsilon) < \varepsilon$. By upper semicontinuity from the right of h, there is $\delta > 0$ such that $tk(t) < \varepsilon$ for $\varepsilon \leq t < \varepsilon + \delta$. Since F_{TxTy} is non-decreasing,

$$F_{TxTy}(\varepsilon) \geq F_{TxTy}(tk(t))$$

for all $\varepsilon \leq t < \varepsilon + \delta$. Then by (8) we get

$$F_{TxTy}(\varepsilon) \geq F_{xy}(t)$$

for all $\varepsilon \leq t < \varepsilon + \delta$. Letting $t \to \varepsilon + \delta$ and using left-continuity of F_{xy} we get

$$F_{TxTy}(\varepsilon) \geq F_{xy}(\varepsilon + \delta),$$

so $T \in [MK]$. The proof of the theorem is complete.

3. Remark on a fixed point theorem

In [3] Chang, Lee, Cho, Chen, Kang and Jung have proved the following theorem:

Let $(X, \mathcal{F}, \Delta)$ be a complete Menger space with a t-norm Δ satisfying $\Delta(t,t) \geq t$ for all $t \in [0,1]$. Let $T : X \to X$ be a mapping satisfying the condition:

$$F_{TxTy}(t) \geq F_{xy}(t/k(\alpha, \beta))$$

for all $x, y \in X$, $t > 0$ and $\alpha, \beta \in (0, \infty)$ with $F_{xy}(\alpha) > 0$ and $F_{xy}(\beta) < 1$, where $k(\alpha, \beta) : (0, \infty)^2 \to (0, 1)$ is a function. Then T has exactly one fixed point in X.

We do not know the relation between the mappings considered in the above theorem and the ones mentioned in Section 2, but we observe that they are quite similar to that of the class [K], for which the conclusion of the theorem is also valid. In fact, we can prove a more general result:

Theorem 2. *Let $(X, \mathcal{F}, \min)$ be a complete Menger space and $T : X \to X$ belong to the class $[MK]$. Then T has a unique fixed point x^* and we have $T^n x_0 \to x^*$ as $n \to \infty$ for each $x_0 \in X$.*

This is a corollary of Theorem 6 in [9]. For the convenience of the reader we reproduce here the sketch of its proof.

First we note that if we define for all $x, y \in X$, $\lambda \in (0, 1)$,

$$d_\lambda(x, y) = \sup\{t : F_{xy}(t) \leq 1 - \lambda\}, \tag{10}$$

then, with the assumption $\Delta = \min$, for each λ, d_λ is a pseudo-metric, that is,

$$d_\lambda(x, y) \geq 0, d_\lambda(x, x) = 0, d_\lambda(x, y) = d_\lambda(y, x),$$
$$d_\lambda(x, y) \leq d_\lambda(x, z) + d_\lambda(z, y).$$

Moreover, condition (PM.1) implies that $d_\lambda(x, y) = 0$ for all $\lambda \in (0, 1)$ if and only if $x = y$.

By the left-continuity of F_{xy} we get from (10) the inequality

$$F_{xy}(d_\lambda(x, y)) \leq 1 - \lambda \tag{11}$$

for all $x, y \in X$ and $\lambda \in (0, 1)$.

In the space X with the family of pseudo-metrics $\{d_\lambda : \lambda \in (0, 1)\}$ we introduce the following

Definition 12.

(i) A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if for each $\lambda \in (0, 1)$, $d_\lambda(x_n, x) \to 0$ as $n \to \infty$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\lambda \in (0, 1)$, $d_\lambda(x_n, x_m) \to 0$ as $n, m \to \infty$.

(iii) The space $(X, \{d_\lambda\})$ is said to be complete if each Cauchy sequence in X converges to some point of X.

In this way we may consider the complete space $(X, \{d_\lambda\})$ instead of the complete Menger space $(X, \mathcal{F}, \min)$.

We now show that if $T \in [MK]$ then T satisfies the condition that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$d_\lambda(x, y) < \varepsilon + \delta \text{ implies } d_\lambda(Tx, Ty) < \varepsilon \tag{12}$$

for each $\lambda \in (0, 1)$. Indeed, given $\varepsilon > 0$ we choose $\delta > 0$ such that (9) holds. Let $\lambda \in (0, 1)$ and $d_\lambda(x, y) < \varepsilon + \delta$. Then by (10) we have

$F_{xy}(\varepsilon+\delta) > 1-\lambda$. From (9) we get $F_{TxTy}(\varepsilon) > 1-\lambda$ which together with (11) give $d_\lambda(Tx, Ty) < \varepsilon$.

Since T satisfies (12) for each $\lambda \in (0,1)$, by slightly extending a theorem of Meir and Keeler in [5] (see the theorem and the remark in Introduction) to complete spaces with a family of pseudo-metrics one can prove that T has a unique fixed point x^* and $T^n x_0 \to x^*$ for every $x_0 \in X$.

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Institute of Mathematics
P.O. BOX 631, Boho, Hanoi, Vietnam