ACTA MATHEMATICA VIETNAMICA
Volume 23, Number 2, 1998, pp. 269–293

HOPF'S FORMULA FOR LIPSCHITZ SOLUTIONS OF HAMILTON-JACOBI EQUATIONS WITH CONCAVE-CONVEX HAMILTONIAN

HA TIEN NGOAN

Abstract. We extend Hopf's formula for Lipschitz solutions of Hamilton-Jacobi equation to the case where the Hamiltonian $H(p) = H(p', p'')$ is a concave-convex function.

1. Introduction

We consider the Cauchy problem for Hamilton-Jacobi equations of the form

$$u_t + H(u_x) = 0, \quad (t, x) \in \Omega, \tag{1.1}$$

with initial condition

$$u(0, x) = \sigma(x), \quad x \in R^n, \tag{1.2}$$

where $\Omega = (0, T) \times R^n$. We denote by $Lip\,(\Omega)$ the set of all locally Lipschitz continuous functions on Ω.

Definition 1.1. A function $u(t, x) \in Lip\big([0, T) \times R^n\big) = Lip\,(\Omega) \cap C\big([0, T) \times R^n\big)$ is said to be a global Lipschitz solution of the problem (1.1) - (1.2) if $u(t, x)$ satisfies (1.1) a.e. in Ω and $u(0, x) = \sigma(x)$ for $x \in R^n$.

In Theorem 5a of the paper [1] E. Hopf proved the followings:

1) If $H(p)$ is a strictly concave function in R^n satisfying the growth condition:

Received December 23, 1996; in revised form December 30, 1997.
1991 Mathematics Subject Classification. 35F25.
Key words and phrases. Lipschitz solution, concave-convex function, Legendre transformation.
Supported in part by the National Basic Research Program in Natural Science, Vietnam.

$$\lim_{|p|\to\infty} \frac{H(p)}{|p|} = -\infty, \tag{1.3}$$

and $\sigma(x)$ is globally Lipschitz continuous in R^n, then the function

$$u(t,x) = \sup_{\xi\in R^n} \Big(\sigma(\xi) + tH^*\big(\frac{x-\xi}{t}\big)\Big) \tag{1.4}$$

is a global Lipschitz solution of the problem (1.1), (1.2).

2) If $H(p)$ is a strictly convex function in R^n satisfying the growth condition:

$$\lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty, \tag{1.5}$$

and $\sigma(x)$ is globally Lipschitz continuous in R^n, then the function

$$u(t,x) = \inf_{\xi\in R^n} \Big(\sigma(\xi) + tH^*\big(\frac{x-\xi}{t}\big)\Big) \tag{1.6}$$

is a global Lipschitz solution of the problem (1.1), (1.2). Here, in the formulas (1.4), (1.6), $H^*(z)$ is the Legendre transform of the function $H(p)$.

There are many papers devoted to extensions of Hopf's formulas (1.4), (1.6). See [2], [3] and the references therein. In this paper we extend Hopf's results just mentioned by permitting the Hamiltonian $H(p)$ to be a concave-convex function. We assume that the variable p is separated into two groups: $p = (p', p'') \in R^n = R^{n_1} \times R^{n_2}$.

Definition 1.2 ([4]). The function $H(p', p'')$ is said to be concave-convex if for each fixed $p'' \in R^{n_2}$ the function $H(p', p'')$ is concave with respect to p' and for each fixed $p' \in R^{n_1}$ the function $H(p', p'')$ is convex with respect to p''.

In Section 2 we introduce the Legendre transformation for some class of concave-convex functions. In Section 3 we give a simple version of Hopf's Lemma ([1]) for the minimax and maximin cases. In Section 4 we obtain various intermediate cases of Hopf's formulas (1.4), (1.6).

2. The Legendre transformation for concave-convex functions

In [4] a notion of conjugate for concave-convex functions was introduced. In this paragraph we introduce a notion of Legendre transformation for some class of concave-convex functions. First we recall some well-known definitions of Legendre transformation for convex or concave functions.

For a convex function $H(p)$, defined on R^n, we introduce the following conditions:

(C_1): *$H(p)$ is twice continuously differentiable.*

(C_2): *The matrix $\dfrac{\partial^2 H(p)}{\partial p^2}$ is positively definite on R^n. Besides that, $H(p)$ is co-finite; i.e.,*

$$\lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty.$$

The Legendre transform $H^*(z)$ of the convex function $H(p)$ satisfying the conditions (C_1), (C_2) is defined by the formula:

$$H^*(z) = -H(p) + \langle z, p\rangle \tag{2.1}$$

where $p = p(z)$ is the solution of the following system of equations:

$$\frac{\partial H(p)}{\partial p} = z. \tag{2.2}$$

Here and in what follows $\dfrac{\partial H(p)}{\partial p}$, z,... are understood as row-vectors, i.e. $1 \times n$ matrices. Since the matrix $\dfrac{\partial^2 H(p)}{\partial p^2}$ is continuous and is positively definite, the solution $p = p(z)$ of (2.2) is unique and is continuously differentiable in z. The function $H^*(z)$ is also defined on R^n and is convex. It coincides with the Fenchel conjugate of $H(p)$; i.e.,

$$H^*(z) = \sup_{p\in R^n} \big(\langle z, p\rangle - H(p)\big), \tag{2.3}$$

where $\langle ., .\rangle$ is the scalar product. Moreover, there is the following formula:

$$\frac{\partial H^*(z)}{\partial z} = p(z). \tag{2.4}$$

Indeed, from (2.1) and (2.2) we have

$$\frac{\partial H^*(z)}{\partial z} = -\frac{\partial H(p(z))}{\partial p}\frac{\partial p(z)}{\partial z} + p(z) + z\frac{\partial p(z)}{\partial z} = p(z).$$

From (2.4) it follows that $H^*(z)$ is twice continuously differentiable. It is also co-finite. The matrix $\dfrac{\partial^2 H^*(z)}{\partial z^2}$ is positively definite and is equal to $\big(\dfrac{\partial^2 H(p(z))}{\partial p^2}\big)^{-1}$.

For a concave function $H(p)$, defined on R^n, we introduce the following condition:

(C_3): *The matrix* $\dfrac{\partial^2 H(p)}{\partial p^2}$ *is negatively definite on* R^n. *The function* $H(p)$ *is co-finite; i.e.,*

$$\lim_{|p|\to\infty} \frac{H(p)}{|p|} = -\infty.$$

The Legendre transform $H^*(z)$ of the concave function $H(p)$ satisfying the conditions (C_1), (C_3) is defined by the same formulas (2.1)-(2.2). In this case the function $H^*(z)$ is also defined on R^n and is concave. It coincides with the Fenchel conjugate of $H(p)$; i.e.,

$$H^*(z) = \inf_{p\in R^n} \big(\langle z, p\rangle - H(p)\big). \tag{2.5}$$

Since the matrix $\dfrac{\partial^2 H(p)}{\partial p^2}$ is continuous and negatively definite, the solution $p = p(z)$ of (2.2) is unique and is continuously differentiable; and there holds the formula (2.4). The function $H^*(z)$ is twice continuously differentiable and is co-finite. The matrix $\dfrac{\partial^2 H^*(z)}{\partial z^2}$ is negatively definite and is equal to $\big(\dfrac{\partial^2 H(p(z))}{\partial p^2}\big)^{-1}$.

Now for a concave-convex function $H(p', p'')$, defined on $R^n = R^{n_1} \times R^{n_2}$, we introduce the following conditions:

(C_4): $H(p', p'')$ *is twice continuously differentiable function on* $R^n = R^{n_1} \times R^{n_2}$.

(C_5): *For each fixed* $p'' \in R^{n_2}$, *the matrix* $\dfrac{\partial^2 H(p', p'')}{\partial {p'}^2}$ *is negatively definite on* R^{n_1}, *and* $H(p', p'')$ *is co-finite in* p':

$$\lim_{|p'|\to\infty} \frac{H(p', p'')}{|p'|} = -\infty. \tag{2.6}$$

(C_6): *For each fixed* $p' \in R^{n_1}$, *the matrix* $\dfrac{\partial^2 H(p',p'')}{\partial p''^2}$ *is positively definite on* R^{n_2}, *and* $H(p',p'')$ *is co-finite in* p'':

$$\lim_{|p''|\to\infty} \frac{H(p',p'')}{|p''|} = +\infty. \tag{2.7}$$

Proposition 2.1. *Suppose that the function* $H(p',p'')$, *defined on* $R^n = R^{n_1} \times R^{n_2}$, *satisfies the conditions* $(C_4), (C_5), (C_6)$. *Then the matrix*

$$\frac{\partial^2 H(p',p'')}{\partial p^2} = \begin{bmatrix} \dfrac{\partial^2 H(p',p'')}{\partial p'^2} & \dfrac{\partial^2 H(p',p'')}{\partial p''\partial p'} \\ \dfrac{\partial^2 H(p',p'')}{\partial p'\partial p''} & \dfrac{\partial^2 H(p',p'')}{\partial p''^2} \end{bmatrix} = \begin{bmatrix} -A & C \\ C^t & B \end{bmatrix}$$

is nondegenerate on $R^n = R^{n_1} \times R^{n_2}$, *where* $C = \dfrac{\partial^2 H(p',p'')}{\partial p''\partial p'}$ *is an* $n_1 \times n_2$ *matrix and* C^t *is the transpose of the matrix* C.

Proof. Let $\xi = (\xi', \xi'') \in R^n = R^{n_1} \times R^{n_2}$. If $\xi \dfrac{\partial^2 H(p',p'')}{\partial p^2} = 0$, then

$$-\xi' A + \xi'' C = 0', \xi' C^t + \xi'' B = 0''.$$

Here $0'$ stands for the zero-vector in R^{n_1}, $0''$ for the zero-vector in R^{n_2}, respectively. It follows that $\xi' = \xi'' C A^{-1}$ and $\xi''(CA^{-1}C^t + B) = 0''$. The matrix $A = \dfrac{\partial^2 H(p',p'')}{\partial p'^2}$ is positively definite, hence the matrix $CA^{-1}C^t$ is nonnegatively definite. Since the matrix $B = \dfrac{\partial^2 H(p',p'')}{\partial p''^2}$ is positively definite, so is the matrix $CA^{-1}C^t + B$. Then $\xi'' = 0''$, and $\xi' = 0'$. This proves the nondegenerateness of the matrix $\dfrac{\partial^2 H(p',p'')}{\partial p^2}$. □

Definition 2.1. Suppose that the function $H(p',p'')$ defined on $R^n = R^{n_1} \times R^{n_2}$ satisfies the conditions (C_4), (C_5), (C_6). The partial Legendre transform $H^{*_2}(p',z'')$ of the concave-convex function $H(p',p'')$ with respect to variable $p'' \in R^{n_2}$ is defined, for each fixed $p' \in R^{n_1}$, by the formula:

$$H^{*_2}(p',z'') = -H(p',p'') + \langle z'', p'' \rangle, \tag{2.8}$$

where $p'' = p''(p', z'')$ is the solution of the following system of equations:

$$\frac{\partial H(p', p'')}{\partial p''} = z''. \tag{2.9}$$

Since the matrix $\dfrac{\partial^2 H(p', p'')}{\partial p''^2}$ is continuous in (p', p'') and is positively definite, the solution $p'' = p''(p', z'')$ of (2.9) is unique and is continuously differentiable. We have the following formula:

$$\frac{\partial H^{*_2}(p', z'')}{\partial z''} = p''(p', z''). \tag{2.10}$$

For each fixed $p' \in R^{n_1}$, the function $H^{*_2}(p', z'')$ is convex and is twice continuously differentiable in z'', and

$$H^{*_2}(p', z'') = \sup_{p'' \in R^{n_2}} \big(\langle z'', p''\rangle - H(p', p'')\big). \tag{2.11}$$

Moreover, the matrix $\dfrac{\partial^2 H^{*_2}(p', z'')}{\partial z''^2}$, which is equal to $\Big(\dfrac{\partial^2 H(p', p''(p', z''))}{\partial p''^2}\Big)^{-1}$, is positively definite; and

$$\lim_{|z''| \to \infty} \frac{H^{*_2}(p', z'')}{|z''|} = +\infty. \tag{2.12}$$

Definition 2.2. Suppose that the function $H(p', p'')$ defined on $R^n = R^{n_1} \times R^{n_2}$ satisfies the conditions (C_4), (C_5), (C_6). The partial Legendre transform $H^{*_1}(z', p'')$ of the concave-convex function $H(p', p'')$ with respect to variable $p' \in R^{n_1}$ is defined, for each fixed $p'' \in R^{n_2}$, by the formula:

$$H^{*_1}(z', p'') = -H(p', p'') + \langle z', p'\rangle, \tag{2.13}$$

where $p' = p'(z', p'')$ is the solution of the following system of equations:

$$\frac{\partial H(p', p'')}{\partial p'} = z'. \tag{2.14}$$

Since the matrix $\dfrac{\partial^2 H(p', p'')}{\partial p'^2}$ is continuous in (p', p'') and is negatively definite, the solution $p' = p'(z', p'')$ of (2.14) is unique and is continuously differentiable with

$$\frac{\partial H^{*_1}(z', p'')}{\partial z'} = p'(z', p''). \tag{2.15}$$

For each fixed $p'' \in R^{n_2}$, the function $H^{*_1}(z', p'')$ is concave, twice continuously differentiable in z', and

$$H^{*_1}(z', p'') = \inf_{p' \in R^{n_1}} \big(\langle z', p' \rangle - H(p', p'') \big). \tag{2.16}$$

Moreover, the matrix $\dfrac{\partial^2 H^{*_1}(z', p'')}{\partial z'^2}$ is negatively definite, and

$$\lim_{|z'| \to \infty} \frac{H^{*_1}(z', p'')}{|z'|} = -\infty. \tag{2.17}$$

Proposition 2.2. *Suppose that the function $H(p', p'')$ satisfies the conditions (C_4), (C_5), (C_6). Then*

1) *For each fixed $z'' \in R^{n_2}$ the function $H^{*_2}(p', z'')$ is convex and satisfies the conditions (C_1) and (C_2) with respect to variable p'.*

2) *Moreover, the function $H^{*_2}(p', z'')$ is twice continuously differentiable on $R^n = R^{n_1} \times R^{n_2}$, and is convex in (p', z''). The matrix:*

$$G_2 = \begin{bmatrix} \dfrac{\partial^2 H^{*_2}(p', z'')}{\partial p'^2} & \dfrac{\partial^2 H^{*_2}(p', z'')}{\partial z'' \partial p'} \\ \dfrac{\partial^2 H^{*_2}(p', z'')}{\partial p' \partial z''} & \dfrac{\partial^2 H^{*_2}(p', z'')}{\partial z''^2} \end{bmatrix}$$

is positively definite on $R^n = R^{n_1} \times R^{n_2}$.

Proof. 1) Since the function $H(p', p'')$ is concave in p', for z'' and p'' fixed the function $\langle z'', p'' \rangle - H(p', p'')$ is convex with respect to p'. So from (2.11) it follows that for each fixed $z'' \in R^{n_2}$ the function $H^{*_2}(p', z'')$ is convex with respect to variable p'. The solution $p'' = p''(p', z'')$ of (2.9) is unique and is continuously differentiable in (p', z''). We now prove that

$$\frac{\partial H^{*_2}(p', z'')}{\partial p'} = -\frac{\partial H(p', p''(p', z''))}{\partial p'} \ . \tag{2.18}$$

Differentiating both sides of (2.8) with respect to p', from (2.9) we obtain

$$\begin{aligned} \frac{\partial H^{*_2}(p', z'')}{\partial p'} &= -\frac{\partial H(p', p''(p', z''))}{\partial p'} - \frac{\partial H(p', p''(p', z''))}{\partial p''} \frac{\partial p''(p', z'')}{\partial p'} \\ &\quad + z'' \frac{\partial p''(p', z'')}{\partial p'} = -\frac{\partial H(p', p''(p', z''))}{\partial p'}; \end{aligned}$$

i.e., (2.18) is satisfied. Here $\dfrac{\partial p''(p',z'')}{\partial p'}$ is an $n_2 \times n_1$ matrix.

By differentiating both sides of (2.18) with respect to p' we have

$$\frac{\partial^2 H^{*_2}(p',z'')}{\partial p'^2} = -\frac{\partial^2 H(p',p''(p',z''))}{\partial p'^2} - \frac{\partial^2 H(p',p''(p',z''))}{\partial p''\partial p'}\frac{\partial p''(p',z'')}{\partial p'}. \tag{2.19}$$

On the other hand, by differentiating both sides of (2.9) with respect to p' we have

$$\frac{\partial^2 H(p',p''(p',z''))}{\partial p'\partial p''} + \frac{\partial^2 H(p',p''(p',z''))}{\partial p''^2}\frac{\partial p''(p',z'')}{\partial p'} = 0.$$

The last equality implies

$$\begin{aligned}\frac{\partial^2 H(p',p''(p',z''))}{\partial p''\partial p'} &= \Big(\frac{\partial^2 H(p',p''(p',z''))}{\partial p'\partial p''}\Big)^t \\ &= -\Big(\frac{\partial p''(p',z'')}{\partial p'}\Big)^t\frac{\partial^2 H(p',p''(p',z''))}{\partial p''^2}.\end{aligned} \tag{2.20}$$

By (2.19) and (2.20) we obtain

$$\begin{aligned}\frac{\partial^2 H^{*_2}(p',z'')}{\partial p'^2} &= -\frac{\partial^2 H(p',p''(p',z''))}{\partial p'^2} \\ &+ \Big(\frac{\partial p''(p',z'')}{\partial p'}\Big)^t\frac{\partial^2 H(p',p''(p',z''))}{\partial p''^2}\frac{\partial p''(p',z'')}{\partial p'}.\end{aligned} \tag{2.21}$$

The matrix $\dfrac{\partial^2 H(p',p''(p',z''))}{\partial p''^2}$ is positively definite; hence the matrix

$$\Big(\frac{\partial p''(p',z'')}{\partial p'}\Big)^t\frac{\partial^2 H(p',p''(p',z''))}{\partial p''^2}\frac{\partial p''(p',z'')}{\partial p'}$$

is nonnegatively definite. Since the matrix $-\dfrac{\partial^2 H(p',p''(p',z''))}{\partial p'^2}$ is positively definite, so is the matrix $\dfrac{\partial^2 H^{*_2}(p',z'')}{\partial p'^2}$.

From (2.11) it follows that $H^{*_2}(p',z'') \geq -H(p',0'')$. Therefore, (2.6) shows that for fixed z'' the function $H^{*_2}(p',z'')$ is co-finite.

2) From (2.18) and (2.10) it follows that $H^{*_2}(p', z'')$ is twice continuously differentiable on $R^n = R^{n_1} \times R^{n_2}$. Moreover, (2.10) gives

$$\frac{\partial p''(p', z'')}{\partial p'} = \frac{\partial^2 H^{*_2}(p', z'')}{\partial p' \partial z''} = \Big(\frac{\partial^2 H^{*_2}(p', z'')}{\partial z'' \partial p'}\Big)^t .$$

Further, together with (2.18) it implies that

$$\begin{aligned}\frac{\partial^2 H^{*_2}(p', z'')}{\partial z'' \partial p'} &= -\frac{\partial^2 H(p', p''(p', z''))}{\partial p'' \partial p'} \frac{\partial p''(p', z'')}{\partial z''} \\ &= -\frac{\partial^2 H(p', p''(p', z''))}{\partial p'' \partial p'} \frac{\partial^2 H^{*_2}(p', z'')}{\partial {z''}^2}.\end{aligned}$$

We denote

$$A = -\frac{\partial^2 H(p', p''(p', z''))}{\partial {p'}^2}, \quad B = \frac{\partial^2 H(p', p''(p', z''))}{\partial {p''}^2},$$

$$C = \frac{\partial^2 H(p', p''(p', z''))}{\partial p'' \partial p'} .$$

Since $\dfrac{\partial^2 H^{*_2}(p', z'')}{\partial {z''}^2} = B^{-1}$, we have

$$\begin{aligned}\frac{\partial^2 H^{*_2}(p', z'')}{\partial' \partial z''} &= -CB^{-1}, \\ \frac{\partial p''(p', z'')}{\partial p'} &= \Big(\frac{\partial^2 H^{*_2}(p', z'')}{\partial z'' \partial p'}\Big)^t = (-CB^{-1})^t = -B^{-1}C^t.\end{aligned}$$

From (2.21) it follows that

$$\frac{\partial^2 H^{*_2}(p', z'')}{\partial {p'}^2} = A + CB^{-1}BB^{-1}C^t = A + CB^{-1}C^t.$$

So

$$G_2 = \begin{bmatrix} A + CB^{-1}C^t & -CB^{-1} \\ -B^{-1}C^t & B^{-1} \end{bmatrix}.$$

Let $\xi = (\xi', \xi'') \in R^n = R^{n_1} \times R^{n_2}$. Then

$$\xi G_2 = (\xi'(A + CB^{-1}C^t) - \xi'' B^{-1}C^t, -\xi' CB^{-1} + \xi'' B^{-1}),$$

$$\begin{aligned}\langle \xi G_2, \xi\rangle &= \langle \xi'(A + CB^{-1}C^t) - \xi'' B^{-1}C^t, \xi'\rangle + \langle -\xi' CB^{-1} + \xi'' B^{-1}, \xi''\rangle \\ &= \langle \xi' A, \xi'\rangle + \langle (\xi'' - \xi' C)B^{-1}, \xi'' - \xi' C\rangle.\end{aligned}$$

Since the matrices A, B^{-1} are positively definite, the matrix G_2 is non-negatively definite. Suppose that $\langle \xi G_2, \xi\rangle = 0$. Then $\langle \xi' A, \xi'\rangle = 0, \langle (\xi'' - \xi C)B^{-1}, \xi'' - \xi' C\rangle = 0$. Consequently $\xi' = 0'$, and $\xi'' - \xi' C = 0''$. So $\xi'' = 0''$ and $\xi = 0$. That means the matrix G_2 is in fact positively definite. □

Proposition 2.3. *Suppose that the function $H(p', p'')$ satisfies the conditions (C_4), $(C_5), (C_6)$. Then*

1) *For each fixed $z' \in R^{n_2}$ the function $H^{*_1}(z', p'')$ is concave and satisfies the conditions (C_1) and (C_3) with respect to variable p''.*

2) *Moreover, the function $H^{*_1}(z', p'')$ is twice continuously differentiable on $R^n = R^{n_1} \times R^{n_2}$, and is concave in (z', p''). The matrix*

$$G_1 = \begin{bmatrix} \dfrac{\partial^2(H^{*_1}(z', p''))}{\partial {z'}^2} & \dfrac{\partial^2(H^{*_1}(z', p''))}{\partial p''\partial z'} \\ \dfrac{\partial^2(H^{*_1}(z', p''))}{\partial z'\partial p''} & \dfrac{\partial^2(H^{*_1}(z', p''))}{\partial {p''}^2} \end{bmatrix}$$

is negatively definite on $R^n = R^{n_1} \times R^{n_2}$.

The proof is analogous to that of Proposition 2.2.

Definition 2.3. Under the hypotheses of Proposition 2.2 for each fixed $z'' \in R^{n_2}$ we define the partial Legendre transform $\big(-H^{*_2}(., z'')\big)^{*_1}(z')$ of the function $-H^{*_2}(p', z'')$ with respect to variable $p' \in R^{n_1}$ by the formula:

$$\big(-H^{*_2}(., z'')\big)^{*_1}(z') = H^{*_2}(p', z'') + \langle z', p'\rangle, \tag{2.22}$$

where $p' = p'(z', z'')$ is the solution of the following system of equations:

$$\frac{\partial(-H^{*_2}(p', z''))}{\partial p'} = z'. \tag{2.23}$$

Since the matrix $\dfrac{\partial^2(-H^{*_2}(p', z''))}{\partial {p'}^2}$ is continuous in (p', z'') and is negatively definite, the solution $p' = p'(z', z'')$ of (2.23) is unique and is continuously differentiable in (z', z''). We have the following formulas:

$$\frac{\partial\big(-H^{*_2}(., z'')\big)^{*_1}(z')}{\partial z'} = p'(z', z''), \tag{2.24}$$

$$\big(-H^{*_2}(.,z'')\big)^{*_1}(z') = \inf_{p'\in R^{n_1}} \big(\langle z',p'\rangle + H^{*_2}(p',z'')\big). \tag{2.25}$$

By a proof analogous to that of (2.18) one can obtain

$$\frac{\partial\big(-H^{*_2}(.,z'')\big)^{*_1}(z')}{\partial z''} = \frac{\partial H^{*_2}(p'(z',z''),z'')}{\partial z''} \; . \tag{2.26}$$

Let $m < 0$ be arbitrary. From (2.25) we have

$$\big(-H^{*_2}(.,z'')\big)^{*_1}(z') \le \big\langle z', m\frac{z'}{|z'|}\big\rangle + H^{*_2}\Big(m\frac{z'}{|z'|}, z''\Big),$$

which implies

$$\frac{\big(-H^{*_2}(.,z'')\big)^{*_1}(z')}{|z'|} \le m + \frac{1}{|z'|} \max_{|\zeta'|=-m} H^{*_2}\big(\zeta',z''\big).$$

Consequently,

$$\lim_{|z'|\to\infty} \frac{\big(-H^{*_2}(.,z'')\big)^{*_1}(z')}{|z'|} = -\infty, \tag{2.27}$$

where the convergence is locally uniform with respect to $z'' \in R^{n_2}$. For each fixed $z'' \in R^{n_2}$ the matrix $\dfrac{\partial^2\big(-H^{*_2}(.,z'')\big)^{*_1}(z')}{\partial {z'}^2}$ is negatively definite and

$$\frac{\partial^2\big(-H^{*_2}(.,z'')\big)^{*_1}(z')}{\partial {z'}^2} = -\Big(\frac{\partial^2 H^{*_2}(p'(z',z''),z'')}{\partial {p'}^2}\Big)^{-1} .$$

Moreover, from (2.24), (2.26) and from Proposition 2.2 it follows that $\big(-H^{*_2}(.,z'')\big)^{*_1}(z')$ is twice continuously differentiable in $(z',z'') \in R^n = R^{n_1} \times R^{n_2}$.

Definition 2.4. Under the hypotheses of Proposition 2.3 for each fixed $z' \in R^{n_1}$ we define the partial Legendre transform $\big(-H^{*_1}(z',.)\big)^{*_2}(z'')$ of the function $-H^{*_1}(z',p'')$ with respect to variable $p'' \in R^{n_2}$ by the formula:

$$\big(-H^{*_1}(z',.)\big)^{*_2}(z'') = H^{*_1}(z',p'') + \langle z'',p''\rangle, \tag{2.28}$$

where $p'' = p''(z', z'')$ is the solution of the following system of equations:

$$\frac{\partial(-H^{*_1}(z', p''))}{\partial p''} = z''. \tag{2.29}$$

Since the matrix $\dfrac{\partial^2(-H^{*_1}(z', p''))}{\partial p''^2}$ is continuous in (z', p'') and is positively definite, the solution $p'' = p''(z', z'')$ of (2.29) is unique and is continuously differentiable in (z', z''). We have the following formulas:

$$\frac{\partial\big(-H^{*_1}(z', .)\big)^{*_2}(z'')}{\partial z''} = p''(z', z''), \tag{2.30}$$

$$\big(-H^{*_1}(z', .)\big)^{*_2}(z'') = \sup_{p'' \in R^{n_2}} \big(\langle z'', p''\rangle + H^{*_1}(z', p'')\big). \tag{2.31}$$

$$\frac{\partial\big(-H^{*_1}(z', .)\big)^{*_2}(z'')}{\partial z'} = \frac{\partial H^{*_1}(z', p''(z', z''))}{\partial z'}, \tag{2.32}$$

$$\lim_{|z''| \to \infty} \frac{\big(-H^{*_1}(z', .)\big)^{*_2}(z'')}{|z''|} = +\infty, \tag{2.33}$$

where the convergence is locally uniform with respect to $z' \in R^{n_1}$. For each fixed $z' \in R^{n_1}$ the matrix $\dfrac{\partial^2\big(-H^{*_1}(z', .)\big)^{*_2}(z'')}{\partial z''^2}$ is positively definite. Moreover, from (2.30), (2.32) and from Proposition 2.3 it follows that $\big(-H^{*_1}(z', .)\big)^{*_2}(z'')$ is twice continuously differentiable in $(z', z'') \in R^n = R^{n_1} \times R^{n_2}$.

Definition 2.5. Suppose that the concave-convex function $H(p', p'')$ satisfies the conditions (C_4), (C_5), (C_6). The function

$$\overline{H}^*(z', z'') \equiv \big(-H^{*_2}(., z'')\big)^{*_1}(z') \tag{2.34}$$

is called the upper Legendre transform of the function $H(p', p'')$. The function

$$\underline{H}^*(z', z'') \equiv \big(-H^{*_1}(z', .)\big)^{*_2}(z'') \tag{2.35}$$

is called the lower Legendre transform of the function $H(p', p'')$.

Proposition 2.4. *Suppose that the function $H(p', p'')$ satisfies the conditions (C_4), (C_5), (C_6). Then*

1) *The function* $\overline{H}^*(z',z'')$ *satisfies the conditions* (C_4), (C_5).

2) *The function* $\overline{H}^*(z',z'')$ *coincides with the upper conjugate* ([4]) *of the concave-convex function* $H(p',p'')$; *i.e.,*

$$\overline{H}^*(z',z'') = \inf_{p'\in R^{n_1}} \sup_{p''\in R^{n_2}} \big(\langle z',p'\rangle + \langle z'',p''\rangle - H(p',p'')\big). \tag{2.36}$$

Moreover,

$$\overline{H}^*(z',z'') + H(p',p'') = \langle z',p'\rangle + \langle z'',p''\rangle, \tag{2.37}$$

where

$$\frac{\partial H(p',p'')}{\partial p'} = z', \quad \frac{\partial H(p',p'')}{\partial p''} = z'', \tag{2.38}$$

or

$$\frac{\partial \overline{H}^*(z',z'')}{\partial z'} = p', \quad \frac{\partial \overline{H}^*(z',z'')}{\partial z''} = p''. \tag{2.39}$$

Proof. The assertion 1) follows from the above-mentioned properties of the function $\big(-H^{*_2}(.,z'')\big)^{*_1}(z')$. We prove the assertion 2). The identity (2.36) follows from (2.34), (2.25) and (2.11). The identity (2.37) follows from (2.34), (2.22) and (2.8). The relations (2.38) follow from (2.9), (2.18) and (2.23). The relations (2.39) follow from (2.34), (2.24) (2.26) and (2.10). □

From the Proposition 2.4 it is easy to get the following

Corollary 2.1. *Suppose that the function* $H(p',p'')$ *satisfies the conditions* (C_4), (C_5), (C_6). *Then for any* $z=(z',z'')\in R^{n_1}\times R^{n_2}$ *we have*

$$\begin{aligned} &\overline{H}^*(z',z'') + H\Big(\frac{\partial \overline{H}^*(z',z'')}{\partial z'}, \frac{\partial \overline{H}^*(z',z'')}{\partial z''}\Big) \\ &= \Big\langle z', \frac{\partial \overline{H}^*(z',z'')}{\partial z'}\Big\rangle + \Big\langle z'', \frac{\partial \overline{H}^*(z',z'')}{\partial z''}\Big\rangle. \end{aligned} \tag{2.40}$$

For the function $\underline{H}^*(z',z'')$ we also have:

Proposition 2.5. *Suppose that the function $H(p', p'')$ satisfies the conditions (C_4), (C_5), (C_6). Then*

1) *The function $\underline{H}^*(z', z'')$ satisfies the conditions (C_4), (C_6).*

2) *The function $\underline{H}^*(z', z'')$ coincides with the lower conjugate ([4]) of the concave-convex function $H(p', p'')$; i.e.,*

$$\underline{H}^*(z', z'') = \sup_{p'' \in R^{n_2}} \inf_{p' \in R^{n_1}} \big(\langle z', p'\rangle + \langle z'', p''\rangle - H(p', p'')\big); \tag{2.41}$$

Moreover,

$$\underline{H}^*(z', z'') + H(p', p'') = \langle z', p'\rangle + \langle z'', p''\rangle, \tag{2.42}$$

where

$$\frac{\partial H(p', p'')}{\partial p'} = z', \quad \frac{\partial H(p', p'')}{\partial p''} = z'', \tag{2.43}$$

or

$$\frac{\partial \underline{H}^*(z', z'')}{\partial z'} = p', \quad \frac{\partial \underline{H}^*(z', z'')}{\partial z''} = p''. \tag{2.44}$$

Since the functions $\overline{H}^*(z', z'')$ and $\underline{H}^*(z', z'')$ are both defined in whole $R^n = R^{n_1} \times R^{n_2}$, from (2.37), (2.38), (2.42), (2.43) and from the nondegenerateness of the matrix $\dfrac{\partial^2 H(p', p'')}{\partial p^2}$ it follows that for all $(z', z'') \in R^n = R^{n_1} \times R^{n_2}$:

$$\overline{H}^*(z', z'') = \underline{H}^*(z', z''). \tag{2.45}$$

So we can make the following.

Definition 2.6. Suppose that the concave-convex function $H(p', p'')$ satisfies the conditions (C_4), (C_5), (C_6). Then the function

$$H^*(z', z'') \equiv \overline{H}^*(z', z'') = \underline{H}^*(z', z'')$$

is called the Legendre transform of the concave-convex function $H(p', p'')$.

From Definition 2.6, (2.45), Corollary 2.1 and Propositions 2.4, 2.5 it is easy to obtain the following.

Proposition 2.6. *Suppose that the function $H(p', p'')$ satisfies the conditions (C_4), (C_5), (C_6). Then*

1) *The function $H^*(z', z'')$ satisfies the conditions (C_4), (C_5), (C_6) too.*

2) *For any $\xi = (\xi', \xi'') \in R^{n_1} \times R^{n_2}$, the function*

$$v(t, x', x'', \xi', \xi'') = tH^*\Big(\frac{x' - \xi'}{t}, \frac{x'' - \xi''}{t}\Big)$$

is a classical solution of the equation (1.1).

3. Hopf's Lemma in the minimax and maximin cases

In [1] E. Hopf proved that the infimum or maximum of a family of Lipschitz solutions of the Hamilton-Jacobi equation is in general also a Lipschitz solution. In this paragraph we formulate a simple version of his lemma, applied to the minimax and maximin cases.

Let $\{v(., \xi', \xi'')\}$ be a family of functions defined in D, where D is a open domain in R^n, and $(\xi', \xi'') \in R^{n_1} \times R^{n_2}$.

Lemma 3.1. *Suppose that the family $v(x, \xi', \xi'')$ satisfies the following conditions:*

1) *The function $v(x, \xi', \xi'')$ is continuously differentiable in $D \times R^{n_1} \times R^{n_2}$, and for any fixed $(\xi', \xi'') \in R^{n_1} \times R^{n_2}$ satisfies every where in D the equation:*

$$F\big(x, v(x, \xi', \xi''), v_x(x, \xi', \xi'')\big) = 0; \tag{3.1}$$

2) *For each fixed $\xi'' \in R^{n_2}$ (respectively, $\xi' \in R^{n_1}$), one has*

$$v(x, \xi', \xi'') \to -\infty \text{ as } |\xi'| \to \infty \tag{3.2}$$

$$\big(\text{respectively}, v(x, \xi', \xi'') \to +\infty \text{ as } |\xi''| \to \infty\big) \tag{3.3}$$

locally uniformly with respect to $x \in D$;

3) *For any fixed $(x, \xi'') \in D \times R^{n_2}$ (respectively, $(x, \xi') \in D \times R^{n_1}$), there exists a unique stationary point $\xi' = \xi'(x, \xi'')$ (respectively, $\xi'' = \xi''(x, \xi')$) for the function $v(x, \xi', \xi'')$. Moreover the function $\xi' = \xi'(x, \xi'')$ (respectively, $\xi'' = \xi''(x, \xi')$) is differentiable in x;*

4) *If we set*

$$w(x, \xi'') = \sup_{\xi' \in R^{n_1}} v(x, \xi', \xi'') = v(x, \xi'(x, \xi''), \xi'') \tag{3.4}$$

$\big($*respctively,*

$$w(x,\xi') = \inf_{\xi''\in R^{n_2}} v(x,\xi',\xi'') = v(x,\xi',\xi''(x,\xi'))\ \big), \tag{3.5}$$

then

$$w(x,\xi'') \to +\infty \quad as\ |\xi''| \to +\infty \ locally uniformly\ with\ respect\ to\ x \in D\big) \tag{3.6}$$

$\big($*respectively,*

$$w(x,\xi') \to -\infty \ as\ |\xi'| \to \infty\ locally\ uniformly\ with\ respect\ to\ x \in D\big). \tag{3.7}$$

Then the function:

$$u(x) = \inf_{\xi''\in R^{n_2}} w(x,\xi'') = \inf_{\xi''\in R^{n_2}} \sup_{\xi'\in R^{n_1}} v(x,\xi',\xi'')$$

$$\big(respectively,\ u(x) = \sup_{\xi'\in R^{n_1}} w(x,\xi'') = \sup_{\xi'\in R^{n_1}} \inf_{\xi''\in R^{n_2}} v(x,\xi',\xi'')\ \big)$$

is a Lipschitz solution of the equation:

$$F(x,u(x),u_x(x)) = 0$$

in the domain D.

Proof. From the conditions of the Lemma and Hopf's Lemma 2.2 in [1] it follows that for any fixed ξ'' (resp. ξ') the function $w(x,\xi'')$ (resp. $w(x,\xi')$) defined by (3.4) (resp. (3.5)) is differentiable in x and satisfies everywhere in D the equation:

$$F(x,w(x,\xi''),w_x(x,\xi'')) = 0. \tag{3.8}$$

From (3.8), (3.6) (resp. (3.7)) and Hopf's Lemma it follows that $u(x)$ is a Lipschitz solution of the equation:

$$F(x,u(x),u_x(x)) = 0. \qquad \square$$

4. Hopf's formula in the minimax and maximin cases

In this paragraph we shall prove the following main result:

Theorem 4.1. *Assume that the Hamiltonian $H(p', p'')$ and the initial function $\sigma(x', x'')$ satisfy the following conditions:*

1) *$H(p', p'')$ defined on $R^n = R^{n_1} \times R^{n_2}$ is a concave-convex function, satisfying the conditions (C_4), (C_5), (C_6);*

2) *$\sigma(x', x'')$ defined on $R^n = R^{n_1} \times R^{n_2}$ is twice differentiable with respect to x' and is globally Lipschitz in x'':*

$$|\sigma(x', \xi'') - \sigma(x', x'')| \leq L(x').|\xi'' - x''|, \tag{4.1}$$

where $L(x') \in L^\infty_{loc}(R^{n_1}, R^+)$;

3) *There exists $T > 0$ such that for any fixed $\xi'' \in R^{n_2}$ one has*

$$\lim_{|\xi'|\to\infty} \Big\{(\sigma(\xi', \xi'') - \sigma(x', \xi'')) + tH^*\Big(\frac{x' - \xi'}{t}, \frac{x'' - \xi''}{t}\Big)\Big\} = -\infty$$

where the convergence is locally uniform with respect to $(t, x', x'') \in (0, T)\times R^{n_1} \times R^{n_2}$;

4) *For any $(t, x', x'', \xi'') \in (0, T] \times R^n \times R^{n_2}$ the following system of equations with respect to ξ':*

$$\xi' - t\frac{\partial H^{*_2}}{\partial p'}\Big(\frac{\partial \sigma(\xi', \xi'')}{\partial \xi'}, \frac{x'' - \xi''}{t}\Big) = x' \tag{4.2}$$

has a unique solution $\xi' = \xi'(t, x', x'', \xi'')$, which is continuously differentiable in (t, x', x'');

5) *There exists a constant M_1, which can be positive, such that for any $\theta' \in R^{n_1}$*

$$\sup_{(\xi', \xi'')\in R^n} \Big\langle \theta' \frac{\partial^2 \sigma(\xi', \xi'')}{\partial {\xi'}^2}, \theta' \Big\rangle \leq M_1 |\theta'|^2; \tag{4.3}$$

6) *There exists a constant M_2 such that for any $t \in [0, T]$, $(p', z'') \in R^{n_1} \times R^{n_2}$*

$$H^{*_2}(p', z'') - tM_1 \Big|\frac{\partial H^{*_2}(p', z'')}{\partial p'}\Big|^2 \geq M_2. \tag{4.4}$$

Then the function

$$u(t,x) = \inf_{\xi''\in R^{n_2}} \sup_{\xi'\in R^{n_1}} \Big\{\sigma(\xi',\xi'') + tH^*\Big(\frac{x'-\xi'}{t}, \frac{x''-\xi''}{t}\Big)\Big\} \tag{4.5}$$

is a global Lipschitz solution of the problem (1.1), (1.2).

Proof. We set

$$v(t,x',x'',\xi',\xi'') = \sigma(\xi',\xi'') + tH^*\Big(\frac{x'-\xi'}{t}, \frac{x''-\xi''}{t}\Big)\ . \tag{4.6}$$

We verify that the function $v(t,x',x'',\xi',\xi'')$ satisfies all the conditions of Lemma 3.1. According to Proposition 2.6 the function $v(t,x',x'',\xi',\xi'')$ is a continuously differentiable function and for any fixed $(\xi',\xi'')\in R^n$ it is a solution of the equation:

$$v_t + H(v_{x'}, v_{x''}) = 0. \tag{4.7}$$

From Condition 3) of the theorem, it follows that the condition (3.2) of Lemma 3.1 for the function $v(t,x',x'',\xi',\xi'')$ is satisfied. Given any fixed $(t,x',x'',\xi'')\in\Omega\times R^{n_2}$, in order to maximize $v(t,x',x'',\xi',\xi'')$ over R^{n_1} we solve the following system of equations

$$\frac{\partial v(t,x',x'',\xi',\xi'')}{\partial\xi'} = 0 \tag{4.8}$$

with respect to ξ'. We have

$$\frac{\partial v(t,x',x'',\xi',\xi'')}{\partial\xi'} = \frac{\partial\sigma(\xi',\xi'')}{\partial\xi'} - \frac{\partial H^*}{\partial z'}\Big(\frac{x'-\xi'}{t}, \frac{x''-\xi''}{t}\Big)\ .$$

So from (4.8) it follows that

$$\frac{\partial H^*}{\partial z'}\Big(\frac{x'-\xi'}{t}, \frac{x''-\xi''}{t}\Big) = \frac{\partial\sigma(\xi',\xi'')}{\partial\xi'}\ . \tag{4.9}$$

Since

$$H^*(z',z'') = \big(-H^{*_2}(.,z'')\big)^{*_1}(z'), \tag{4.10}$$

we may use (2.24) with

$$z' = \frac{x' - \xi'}{t}, \quad z'' = \frac{x'' - \xi''}{t}, \quad p'(z', z'') = \frac{\partial\sigma(\xi', \xi'')}{\partial\xi'}$$

to deduce from (4.9) that

$$-\frac{\partial H^{*_2}}{\partial p'}\Big(\frac{\partial\sigma(\xi', \xi'')}{\partial\xi'}, \frac{x'' - \xi''}{t}\Big) = \frac{x' - \xi'}{t}\,. \tag{4.11}$$

So we have

$$\xi' - t\frac{\partial H^{*_2}}{\partial p'}\Big(\frac{\partial\sigma(\xi', \xi'')}{\partial\xi'}, \frac{x'' - \xi''}{t}\Big) = x'. \tag{4.12}$$

Condition 4) of the theorem guarantees that the solution $\xi' = \xi'(t, x', x'', \xi'')$ of the system (4.12), which is continuously differentiable with respect to (t, x', x''), determines the unique stationary point ξ' of the function $v(t, x', x'', \xi', \xi'')$ for any fixed (t, x', x'', ξ').

Now we set

$$w(t, x', x'', \xi'') = \sup_{\xi'} v(t, x', x'', \xi', \xi''). \tag{4.13}$$

From (4.10), (4.11) and from the definition (2.22) of $\big(-H^{*_2}(., z'')\big)^{*_1}(z')$ we have

$$\begin{aligned} w(t, x', x'', \xi'') &= v(t, x', x'', \xi', \xi'') = \sigma(\xi', \xi'') + tH^*\Big(\frac{x' - \xi'}{t}, \frac{x'' - \xi''}{t}\Big) \\ &= \sigma(\xi', \xi'') + t\Big(- H^{*_2}\big(., \frac{x'' - \xi''}{t}\big)\Big)^{*_1}\Big(\frac{x' - \xi'}{t}\Big) \\ &= \sigma(\xi', \xi'') + t\Big\langle\frac{\partial\sigma(\xi', \xi'')}{\partial\xi'}, \frac{x' - \xi'}{t}\Big\rangle + tH^{*_2}\Big(\frac{\partial\sigma(\xi', \xi'')}{\partial\xi'}, \frac{x'' - \xi''}{t}\Big), \end{aligned} \tag{4.14}$$

where ξ' stands for $\xi'(t, x', x'', \xi'')$. On the other hand, for any $x', \xi' \in R^{n_1}, \xi'' \in R^{n_2}$ there exists $\theta' \in R^{n_1}$ such that

$$\begin{aligned} \sigma(x', \xi'') = \sigma(\xi', \xi'') &+ \Big\langle\frac{\partial\sigma(\xi', \xi'')}{\partial\xi'}, x' - \xi'\Big\rangle \\ &+ \frac{1}{2}\Big\langle(x' - \xi')\frac{\partial^2\sigma(\theta', \xi'')}{\partial\xi'^2}, x' - \xi'\Big\rangle. \end{aligned}$$

So we have

$$\sigma(\xi', \xi'') + \Big\langle \frac{\partial \sigma(\xi', \xi'')}{\partial \xi'}, x' - \xi' \Big\rangle$$

$$= \sigma(x', \xi'') - \frac{1}{2} \Big\langle (x' - \xi') \frac{\partial^2 \sigma(\theta', \xi'')}{\partial {\xi'}^2}, x' - \xi' \Big\rangle. \tag{4.15}$$

From (4.14), (4.15) it follows that

$$\begin{aligned} w(t, x', x'', \xi'') &= \sigma(x', \xi'') + tH^{*_2}\Big(\frac{\partial \sigma(\xi', \xi'')}{\partial \xi'}, \frac{x'' - \xi''}{t}\Big) \\ &\quad - \frac{1}{2} \Big\langle (x' - \xi') \frac{\partial^2 \sigma(\theta', \xi'')}{\partial {\xi'}^2}, x' - \xi' \Big\rangle \\ &= \sigma(x', \xi'') + \frac{t}{2} H^{*_2}\Big(\frac{\partial \sigma(\xi', \xi'')}{\partial \xi'}, \frac{x'' - \xi''}{t}\Big) \\ &\quad + \frac{t}{2} H^{*_2}\Big(\frac{\partial \sigma(\xi', \xi'')}{\partial \xi'}, \frac{x'' - \xi''}{t}\Big) \\ &\quad - \frac{1}{2} \Big\langle (x' - \xi') \frac{\partial^2 \sigma(\theta', \xi'')}{\partial {\xi'}^2}, x' - \xi' \Big\rangle. \end{aligned} \tag{4.16}$$

It follows from (4.10) that

$$-H^{*_2}(p', z'') = \inf_{z' \in R^{n_1}} \big(\langle z', p' \rangle - H^*(z', z'') \big) \le -H^*(0', z'').$$

So we have

$$H^{*_2}\Big(\frac{\partial \sigma(\xi', \xi'')}{\partial \xi'}, \frac{x'' - \xi''}{t}\Big) \ge H^*\Big(0', \frac{x'' - \xi''}{t}\Big). \tag{4.17}$$

By (4.16), (4.17), (4.11), (4.3), (4.4) and (4.1), for $p' = \dfrac{\partial \sigma(\xi', \xi'')}{\partial \xi'}$ and $\xi' = \xi'(t, x', x'', \xi'')$, we obtain

$$\begin{aligned} w(t, x', x'', \xi'') &\ge \sigma(x', \xi'') + \frac{t}{2} H^*\Big(0', \frac{x'' - \xi''}{t}\Big) + \\ &+ \frac{t}{2} H^{*_2}(p', \frac{x'' - \xi''}{t}) - \frac{1}{2} \Big\langle (x' - \xi') \frac{\partial^2 \sigma(\theta', \xi'')}{\partial {\xi'}^2}, x' - \xi' \Big\rangle \ge \\ &\ge \sigma(x', \xi'') + \frac{t}{2} H^*\Big(0', \frac{x'' - \xi''}{t}\Big) \end{aligned} \tag{4.18}$$

$$
\begin{aligned}
&+\frac{t}{2}\Big(H^{*_2}\big(p',\frac{x''-\xi''}{t}\big)-M_1t\Big|\frac{\partial H^{*_2}}{\partial p'}\big(p',\frac{x''-\xi''}{t}\big)\Big|^2\Big)\geq\\
&\geq\sigma(x',\xi'')+\frac{t}{2}H^*\big(0',\frac{x''-\xi''}{t}\big)+\frac{t}{2}M_2\geq\\
&\geq\sigma(x',x'')+t\Big[-L(x')\frac{|x''-\xi''|}{t}+\frac{1}{2}H^*\big(0',\frac{x''-\xi''}{t}\big)\Big]+\frac{t}{2}M_2.
\end{aligned}
$$

From (4.18) and from the co-finiteness of the function $H^*(0',z'')$ it follows that

$$w(t,x',x'',\xi'')\to+\infty \text{ as } |\xi''|\to\infty$$

locally uniformly in $(t,x',x'')\in\Omega$. So all the conditions of Lemma 3.1 have been verified. This implies that the function

$$u(t,x',x'')=\inf_{\xi''} w(t,x',x'',\xi'') \tag{4.19}$$

is a Lipschitz solution of the equation (1.1). From (4.19), (4.18) it follows that there exists a constant M_3 such that

$$u(t,x',x'')\geq\sigma(x',x'')+tM_3. \tag{4.20}$$

We denote by $\xi_0'(t,x',x'')$ the solution of the system

$$\xi'-t\frac{\partial H^{*_2}}{\partial p'}\Big(\frac{\partial\sigma(\xi',x'')}{\partial\xi'},0''\Big)=x'.$$

It is clear that as $t\to0$

$$\xi_0'(t,x',x'')\to\ x';\quad \frac{x'-\xi_0'(t,x',x'')}{t}\to-\frac{\partial H^{*_2}}{\partial p'}\Big(\frac{\partial\sigma(x',x'')}{\partial x'},0''\Big). \tag{4.21}$$

By (4.19), (4.13), (4.6) we have

$$
\begin{aligned}
&u(t,x',x'')\leq w(t,x',x'',\xi'')|_{\xi''=x''}\\
&=v(t,x',x'',\xi'(t,x',x'',\xi''),\xi'')\big|_{\xi''=x''}=\\
&=\sigma(\xi_0'(t,x',x''),x'')+tH^*\Big(\frac{x'-\xi_0'(t,x',x'')}{t},0''\Big).
\end{aligned} \tag{4.22}
$$

From (4.20), (4.21), (4.22) we conclude that

$$u(t,x',x'')\to\sigma(x',x'')\text{ as } t\to0. \qquad \square$$

Theorem 4.2. *Assume that the Hamiltonian $H(p', p'')$ and the initial function $\sigma(x', x'')$ satisfy the following conditions:*

1) *The function $H(p', p'')$, defined on $R^n = R^{n_1} \times R^{n_2}$, is a concave-convex function satisfying the conditions (C_4), (C_5), (C_6);*

2) *The function $\sigma(x', x'')$, defined on $R^n = R^{n_1} \times R^{n_2}$, is twice differentiable with respect to x'' and is globally Lipschitz in x':*

$$|\sigma(\xi', x'') - \sigma(x', x'')| \leq L(x'').|\xi' - x'| \tag{4.23}$$

where $L(x'') \in L^\infty_{loc}(R^{n_2}, R^+)$;

3) *There exists $T > 0$ such that for any fixed $\xi' \in R^{n_1}$ one has*

$$\lim_{|\xi''|\to\infty} \Big\{(\sigma(\xi', \xi'') - \sigma(\xi', x'')) + tH^*\Big(\frac{x' - \xi'}{t}, \frac{x'' - \xi''}{t}\Big)\Big\} = +\infty,$$

where the convergence is locally uniform with respect to $(t, x', x'') \in (0, T) \times R^{n_1} \times R^{n_2}$;

4) *For any $(t, x', x'', \xi') \in (0, T] \times R^n \times R^{n_1}$ the following system of equations with respect to ξ'':*

$$\xi'' - t\frac{\partial H^{*_1}}{\partial p''}\Big(\frac{x' - \xi'}{t}, \frac{\partial \sigma(\xi', \xi'')}{\partial \xi''}\Big) = x'' \tag{4.24}$$

has a unique solution $\xi'' = \xi''(t, x', x'', \xi')$ which is continuously differentiable in (t, x', x'');

5) *There exists some constant M_1, which can be negative, such that for any $\theta'' \in R^{n_2}$:*

$$\inf_{(\xi', \xi'') \in R^n} \big\langle \theta'' \frac{\partial^2 \sigma(\xi', \xi'')}{\partial \xi''^2}, \theta'' \big\rangle \geq M_1 |\theta''|^2; \tag{4.25}$$

6) *There exists a constant M_2 such that for any $t \in [0, T], (z', p'') \in R^{n_1} \times R^{n_2}$*

$$H^{*_1}(z', p'') - tM_1 \Big|\frac{\partial H^{*_1}(z', p'')}{\partial p''}\Big|^2 \leq M_2. \tag{4.26}$$

Then the function

$$u(t, x) = \sup_{\xi' \in R^{n_1}} \inf_{\xi'' \in R^{n_2}} \Big\{\sigma(\xi', \xi'') + tH^*\big(\frac{x' - \xi'}{t}, \frac{x'' - \xi''}{t}\big)\Big\} \tag{4.27}$$

is a global Lipschitz solution of the problem (1.1), (1.2).

Proof. The proof is analogous to that of Theorem 4.1. □

5. An example

We consider the following Cauchy problem

$$\frac{\partial u}{\partial t}-\frac{1}{2}\Big(\frac{\partial u}{\partial x'}\Big)^2+\frac{1}{2}\Big(\frac{\partial u}{\partial x''}\Big)^2+\lambda\frac{\partial u}{\partial x'}\frac{\partial u}{\partial x''}=0, \tag{5.1}$$

$(t,x',x'')\in(0,T)\times R\times R,$

$$u(t,x',x'')\big|_{t=0}=h_0(x'')+h_1(x'')x'+\frac{1}{2}h_2(x'')(x')^2, \tag{5.2}$$

$(x',x'')\in R\times R,$
where $\lambda=const$, the functions $h_0(x''),h_1(x''),h_2(x'')$ are globally Lipschitz, and

$$h_2(x'')\le M_1. \tag{5.3}$$

In this case,

$$H(p',p'')=-\frac{1}{2}(p')^2+\frac{1}{2}(p'')^2+\lambda p'p'', \tag{5.4}$$

$$\sigma(x',x'')=h_0(x'')+h_1(x'')x'+\frac{1}{2}h_2(x'')(x')^2, \tag{5.5}$$

$$H^{*_2}(p',z'')=\frac{1}{2}(1+\lambda^2)(p')^2+\frac{1}{2}(z'')^2-\lambda p'z'', \tag{5.6}$$

$$H^{*_1}(z',p'')=-\frac{1}{2}(1+\lambda^2)(p'')^2-\frac{1}{2}(z')^2+\lambda p''z', \tag{5.7}$$

$$H^{*}(z',z'')=\frac{1}{2(1+\lambda^2)}\big(-(z')^2+(z'')^2+2\lambda z'z''\big). \tag{5.8}$$

We have

$$\frac{\partial\sigma(x',x'')}{\partial x'}=h_1(x'')+h_2(x'')x', \tag{5.9}$$

$$\frac{\partial^2\sigma(x',x'')}{\partial x'^2}=h_2(x''), \tag{5.10}$$

$$\frac{\partial H^{*_2}(p',z'')}{\partial p'}=(1+\lambda^2)p'-\lambda z'', \tag{5.11}$$

$$\Big|\frac{\partial H^{*_2}(p',z'')}{\partial p'}\Big|^2=(1+\lambda^2)^2(p')^2 -2\lambda(1+\lambda^2)p'z''+\lambda^2(z'')^2. \tag{5.12}$$

One can verify that all the conditions of Theorem 4.1 are satisfied. Indeed,

1) The function $H(p', p'')$ is concave-convex and satisfies the conditions (C_4), (C_5), (C_6).

2) From (5.5) it follows that the condition 2) is satisfied.

3) We choose $T = \dfrac{1}{2M_1(1+\lambda^2)}$ if $M_1 > 0$, and T arbitrary if $M_1 \le 0$. Then for $\xi'' \in R$ fixed

$$\lim_{|\xi'|\to\infty} \big(h_0(\xi'') + h_1(\xi'')\xi' + \frac{1}{2}h_2(\xi'')(\xi')^2 + \\ + \frac{1}{2(1+\lambda^2)t}\big(-(x'-\xi')^2 + (x''-\xi'')^2 + 2\lambda(x'-\xi')(x''-\xi'')\big) = -\infty,$$

where the convergence is locally uniform with respect to $(t, x', x'') \in (0,T) \times R \times R$.

4) The equation (4.2) takes the form

$$\xi' - t\Big((1+\lambda^2)\big(h_1(\xi'') + h_2(\xi'')\xi'\big) - \lambda\frac{x''-\xi''}{t}\Big) = x'. \tag{5.13}$$

This equation has the unique solution:

$$\xi' = \frac{t(1+\lambda^2)h_1(\xi'') + x' - \lambda(x''-\xi'')}{1 - t(1+\lambda^2)h_2(\xi'')},$$

which is continuously differentiable in t, x', x'', where $(t, x', x'') \in [0,T] \times R \times R$.

5) Since $h_2(x'') \le M_1$, from (5.10) it is easy to see that the condition (4.3) is fulfilled with the same M_1 as in (5.3).

6) By (5.6) and (5.12), (4.4) clearly holds if we choose $M_2 = 0$. Indeed,

$$H^{*_2}(p', z'') - tM_1\Big|\frac{\partial H^{*_2}(p', z'')}{\partial p'}\Big|^2 = \frac{1}{2}(1+\lambda^2)(1 - 2tM_1(1+\lambda^2))(p')^2 \\ - \lambda(1 - 2tM_1(1+\lambda^2))p'z'' + \frac{1}{2}(1 - 2tM_1\lambda^2)(z'')^2.$$

Since the discriminant of the quadratic form in the right side is equal to $-1 + 2tM_1(1+\lambda^2)$, the condition (4.4) is satisfied with $M_2 = 0$ for all $t \ge 0$ if $M_1 \le 0$ and for all $t \in [0,T]$, where $T = \dfrac{1}{2M_1(1+\lambda^2)}$, if $M_1 > 0$.

From Theorem 4.1 we conclude that the function

$$u(t,x',x'') = \inf_{\xi''\in R} \sup_{\xi'\in R}\Big\{h_0(\xi'') + h_1(\xi'')\xi' + \frac{1}{2}h_2(\xi'')(\xi')^2 +$$

(5.14)

$$+\frac{1}{2(1+\lambda^2)t}\big(-(x'-\xi')^2 + (x''-\xi'')^2 + 2\lambda(x'-\xi')(x''-\xi'')\big)\Big\}$$

is a solution of the problem (5.1), (5.2) in the domain $\Omega = (0,T)\times R^2$, where T is arbitrary if $M_1 \le 0$, and $T = \frac{1}{2M_1(1+\lambda^2)}$ if $M_1 > 0$.

There are the following two particular cases of the formula (5.14):

1) If $h_0(x'') = h_1(x'') = 0$, then

$$u(t,x',x'') = \inf_{\xi''\in R}\Big\{\frac{1}{2(1-t(1+\lambda^2)h_2(\xi''))}\Big[(h_2(\xi'')(x')^2 - 2\lambda h_2(\xi'')x'(x''-\xi'') + (\frac{1}{t} - h_2(\xi''))(x''-\xi'')^2\Big]\Big\};$$

2) If $h_0(x'') = h_2(x'') = 0, h_1(x'') = ax'', a = const$, then

$$u(t,x',x'') = \frac{a}{2\big(a^2t^2 + (1+\lambda at)^2\big)}\big(-at(x')^2 + 2(1+\lambda at)x'x'' + at(x'')^2\big).$$

References

1. E. Hopf, *Generalized solutions of nonlinear equations of first order*, J. Math. and Mech. **14** (1965), 951-973.
2. Tran Duc Van, Nguyen Hoang and Nguyen Duy Thai Son, *On the explicit representation of global solutions of the Cauchy problem for Hamilton-Jacobi equations*, Acta Math. Vietnam. **19** (1994), 111-120.
3. Tran Duc Van, Mai Duc Thanh and Nguyen Hoang, *On the representation of Lipschitz global solutions of the Cauchy problem for Hamilton-Jacobi equations*, Proc. of Inter. Conference on Appl. Anal. and Mech. of Cont. Media, Ho Chi Minh City, (12/1995), 428-436.
4. R. Tyrrell Rockafella, *Convex Analysis*, Princeton, New Jersey. Princeton University Press, 1970.

HANOI INSTITUTE OF MATHEMATICS,
P.O. BOX 631 BOHO, HANOI, VIETNAM