NORMAL STRUCTURE AND FIXED POINT PROPERTY IN LINEAR METRIC SPACES

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A linear metric space \((X, d)\) is said to have the fixed point property if for every non-empty weakly compact convex subset \(K\) of \(X\), every non-expansive map \(T : K \to K\) has a fixed point. In this paper we discuss a class of linear metric spaces which has the fixed point property.

Normal structure is one of the fundamental tools in fixed point theory for non-expansive maps. A central problem in the fixed point theory of non-expansive maps is to determine those spaces which have the fixed point property (f.p.p.). With the appearance of Alspach’s example [1], we know that there is a weakly compact convex set in the Banach space \(L_1[0,1]\) which need not have the f.p.p. for non-expansive self maps. On the other hand, Kirk [4] proved that if a reflexive Banach space has weak normal structure then it has the f.p.p. For general Banach spaces it was proved by Dulst and Sims [2]. Here we prove this result in linear metric spaces by generalising the result of Kirk [4] as well as of Dulst and Sims [2]. We start with a few definitions.

Let \((X, d)\) be a metric space. A continuous mapping \(W : X \times X \times [0,1] \to X\) is said to be a convex structure on \(X\), if for all \(x, y\) in \(X\) and \(\alpha \in [0,1]\) the following condition is satisfied:

\[
d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha)d(u, y)
\]

for all \(u \in X\). A metric space \(X\) with convex structure is called a convex metric space.

This notion of convexity in metric spaces was introduced by W. Takahashi [6] in 1970. Clearly, a Banach space or any convex subset of it is a convex metric space with \(W(x, y, \alpha) = \alpha x + (1 - \alpha)y\). More generally, if \(X\) is a linear space with a translation invariant metric \(d\) satisfying \(d(\alpha x + (1 - \alpha)y, 0) \leq \alpha d(x, 0) + (1 - \alpha)d(y, 0)\), then \(X\) is a convex metric space. There are many convex metric spaces (see Takahashi [6]) which cannot be embedded in any Banach space.

A non-empty set \(K\) of a convex metric space \(X\) is said to be convex if
\( W(x, y, \alpha) \in K \) for all \( x, y \in K \) and \( \alpha \in [0, 1] \). For linear spaces, convexity of \( K \) requires that \( \alpha x + (1 - \alpha)y \in K \) for all \( x, y \in K \) and \( \alpha \in [0, 1] \).

Takahashi [6] proved that in a convex metric space \( X \), open balls \( B(x, r) = \{ y \in X : d(x, y) < r \} \) and closed balls \( B[x, r] = \{ y \in X : d(x, y) \leq r \} \) are convex and if \( \{ K_\alpha : \alpha \in \Lambda \} \) is a family of convex sets in \( X \) then \( \bigcap \{ K_\alpha : \alpha \in \Lambda \} \) is convex.

A convex metric space \((X, d)\) is said to satisfy property (I) if for all \( x, y, p \in X \) and \( \alpha \in [0, 1] \),

\[
    d(W(x, p, \alpha), W(y, p, \alpha)) \leq \alpha d(x, y).
\]

Property (I) is always satisfied in any normed linear space. For details we refer to Guay, Singh and Whitfield [3].

Let \( B \) be a bounded set in a convex metric space \( X \) and let \( \delta(B) \) be its diameter. An element \( x \in B \) is said to be a diametral point of \( B \) if \( \sup_{y \in B} d(x, y) = \delta(B) \). For closed balls in \( X \), diametral points are precisely the boundary points. A point \( x \in B \) is called a non-diametral point of \( B \) if \( \sup_{y \in B} d(x, y) < \delta(B) \).

A convex subset \( S \) of a convex metric space \( X \) is said to have normal structure if every bounded convex subset \( S_1 \) of \( S \) which contains more than one point has a point that is not a diametral point.

Any compact convex subset of a convex metric space has normal structure (Takahashi [6], Proposition 5).

Let \( X \) be a metric space. For subsets \( H, K \) of \( X \), \( H \) bounded, let

\[
    r_x(H) = \sup \{ d(x, y) : y \in H \}, \quad x \in K,
\]

\[
    r(H, K) = \inf \{ r_x(H), x \in K \},
\]

\[
    C(H, K) = \{ x \in K : r_x(H) = r(H, K) \}.
\]

The set \( C(H, K) \) is frequently referred to as the Chebyshev centre of \( H \) with respect to (w.r.t.) \( K \) in \( X \) or the set of best simultaneous approximation in \( K \) to \( H \), and \( r(H, K) \) is called the Chebyshev radius of \( H \) w.r.t. \( K \).

If \( K \) is compact convex and \( H \) is a bounded subset of a convex metric space \( X \), then \( C(H, K) \) is non-empty, closed, convex subset of \( K \) (Naimpally, Singh and Whitfield [5], Lemma 3.1).

A locally convex linear metric space \((X, d)\) is said to have weak normal structure if every non-trivial weakly compact convex subset of \( X \) has normal structure. The space \( X \) is said to have the fixed point property
(f.p.p.) if for every non-empty weakly compact convex subset $K$ of $X$, every non-expansive map $T : K \to K$ has a fixed point i.e. there exists $x \in K$ such that $Tx = x$.

The following result deals with the continuity of the function $r_x$ defined above:

**Lemma 1.** The function $r_x : K \to \mathbb{R}$ defined above is uniformly continuous.

**Proof.** Let $x, y \in K$. Then $d(x, z) \leq d(x, y) + d(y, z)$ for all $z \in H$. Therefore $\sup_{z \in H} d(x, z) \leq d(x, y) + \sup_{z \in H} d(y, z)$ and so $r_x(H) \leq d(x, y) + r_y(H)$ or $r_x(H) - r_y(H) \leq d(x, y)$. Interchanging $x$ and $y$, we get $r_y(H) - r_x(H) \leq d(y, x)$. So $|r_x(H) - r_y(H)| \leq d(x, y)$ for all $x, y \in K$ and hence the result follows.

The following result deals with the convexity of the function $r_x$:

**Lemma 2.** If $K$ is a convex subset of a convex metric space $(X, d)$ then the function $r_x : K \to \mathbb{R}$ defined above is a convex function i.e. $r_{W(x_1, x_2, t)}(H) \leq tr_{x_1}(H) + (1 - t)r_{x_2}(H)$ for all $x_1, x_2 \in K$ and $0 \leq t \leq 1$.

**Proof.** Let $x_1, x_2 \in K$ and $0 \leq t \leq 1$. Since $K$ is convex, $W(x_1, x_2, t) \in K$. We have

$$
r_{W(x_1, x_2, t)}(H) = \sup_{y \in H} d(W(x_1, x_2, t), y)
\leq \sup_{y \in H} [td(x_1, y) + (1 - t)d(x_2, y)]
\leq t \sup_{y \in H} d(x_1, y) + (1 - t) \sup_{y \in H} d(x_2, y)
= tr_{x_1}(H) + (1 - t)r_{x_2}(H)
$$

and so $r_x$ is convex.

**Remark 1.** The result is true if $K$ is a convex subset of a linear metric space $(X, d)$ satisfying $d(tx + (1 - t)y, 0) \leq td(x, 0) + (1 - t)d(y, 0)$ for all $x, y \in X$ and $t \in [0, 1]$.

It is well known that if $K$ is a weakly compact convex subset of a Banach space $X$ then $C(H, K)$ is non-empty, weakly compact and convex. In linear metric spaces we have:

**Lemma 3.** If $K$ is a weakly compact convex subset of a locally convex linear metric space $(X, d)$ having convex structure and with property (I) then $C(H, K)$ is non-empty, weakly compact and convex.
Proof. Since $K$ is a weakly compact and convex set in the locally convex space $X$, the function $r_x : K \to \mathbb{R}$, being continuous and convex (by Lemmas 1 and 2), is weakly lower semi-continuous. Let $C(H, K) = \{x \in K : r_x(H) = r(H, K)\} = \{x \in K : r_x(H) \leq r(H, K)\}$ be the Chebyshev centre of $H$ w.r.t. $K$. Suppose $C(H, K) = \emptyset$. Then $r_x(H) > r(H, K)$ for all $x \in K$. Let $x$ be any point of $K$. Since $\frac{1}{2}r(H, K) + r_x(H) < r_x(H)$ and $r_x$ is weakly lower semi-continuous, there exists a weak neighbourhood $W_x$ of $x$ in $K$ such that $\frac{1}{2}r(H, K) + r_x(H) \leq r_y(H)$ for all $y \in W_x$. Now the function $r_y(H)$ is weakly lower semi-continuous, there exists a weak neighbourhood $x \in W_x$. Hence $\frac{1}{2}r(H, K) + r_x(H) \leq r_y(H)$, so $r_y(H) > r(H, K)$ for all $x \in K$. Let $y$ be any element of $K$. Then $y \in W_x$ for some $j$, $1 \leq j \leq n$. Hence $r_y(H) > r(H, K)$, and $C(H, K) \neq \emptyset$. A contradiction. Hence $C(H, K) \neq \emptyset$.

We now show that $C(H, K)$ is convex. Let $x_1, x_2 \in C(H, K)$ and $0 \leq t \leq 1$. Then $r_{tx_1 + (1 - t)x_2}(H) \leq tr_{x_1}(H) + (1 - t)r_{x_2}(H) = tr(H, K) + (1 - t)r(H, K) = r(H, K)$. Therefore $tx_1 + (1 - t)x_2 \in C(H, K)$ and so $C(H, K)$ is convex.

Next we show that $C(H, K)$ is weakly compact. Let $x$ belong to the weak closure of $K$ in $H$ and $\varepsilon > 0$ be given. Then there exists a weak neighbourhood $W$ of $x$ in $K$ such that $r_x(H) - \varepsilon \leq r_y(H)$ for all $y \in W$. As $x$ belongs to the weak closure of $C(H, K)$ in $K$, there exists a $y \in C(H, K) \cap W$. For this $y$, $r_H(y) = r(H, K)$. So $r_x(H) - \varepsilon \leq r(H, K) = r_x(H)$. Since $\varepsilon > 0$ is arbitrary, $r_x(H) \leq r(H, K) + \varepsilon$. In $C(H, K)$ is weakly closed subset of the weakly compact set $K$ and hence $C(H, K)$ is weakly compact.

The following result gives a relation between the diameters of $C(K, K)$ and $K$:

**Lemma 4.** Let $K$ be a weakly compact convex subset of a locally convex linear metric space $(X, d)$ having convex structure and with property (I) and $C(K, K)$ be the Chebyshev centre of $K$ w.r.t. itself. If $K$ has normal structure then $\text{diam } C(K, K) < \text{diam } K$.

**Proof.** Since $K$ has normal structure, there exists $x \in K$ such that $\sup \{d(x, y) : y \in K\} < \text{diam } K$ or $r_x(K) < \text{diam } K$. Let $x_1, x_2$ be
any two points of $C(K,K)$. Then $d(x_1,x_2) \leq r_{x_1}(K) = r(K,K)$. So $\text{diam} C(K,K) \leq r(K,K) \leq r_x(K) < \text{diam} K$.

Alspach [1] proved that the Banach space $L_1[0,1]$ does not have the f.p.p.. Kirk [4] proved that if a reflexive Banach space has weak normal structure then it has the f.p.p.. The following theorem shows that there are certain linear metric spaces which have the f.p.p.:

**Theorem.** Let $(X,d)$ be a locally convex linear metric space having convex structure and with property (I). Then $X$ has the f.p.p. if $X$ has weak normal structure.

**Proof.** Let $K$ be a non-empty weakly compact convex subset of $X$ and $T : K \to K$ a non-expansive map. Let $\mathcal{F}$ be the family of all non-empty weakly compact convex subsets of $K$ which are invariant under $T$. $\mathcal{F}$ is non-empty as $K \in \mathcal{F}$. For $K_1, K_2 \in \mathcal{F}$ let $K_1 \leq K_2$ if $K_1 \supseteq K_2$. This is a partial ordering in $\mathcal{F}$. Let $\{K_\alpha\}$ be any chain in $\mathcal{F}$. Obviously, $\{K_\alpha\}$ has the finite intersection property. Since $K$ is weakly compact, $\{K_\alpha\}$ must have a non-empty intersection, say $C_1$. Then $C_1$ is a non-empty weakly compact convex subset of $K$ which is invariant under $T$. Obviously, $C_1$ is an upper bound of $\{K_\alpha\}$ in $\mathcal{F}$. By Zorn’s lemma, $\mathcal{F}$ must have a maximal element, say $M$. If $M$ is a singleton, then obviously $T$ has a fixed point in $K$. So assume that $M$ has more than one point. Since $X$ has weak normal structure, $M$ has normal structure. By Lemma 4, $\text{diam} C(M,M) < \text{diam} M$ and so $C(M,M) \subseteq M$ but $C(M,M) \neq M$. Also by Lemma 3, $C(M,M)$ is a non-empty weakly compact and convex set. We now show that $C(M,M)$ is invariant under $T$. Let $x \in C(M,M)$. We have $d(Tx,Ty) \leq d(x,y)$ for all $y \in M$. So $d(Tx,Ty) \leq r_x(M) = r(M,M)$ for all $y \in M$. Let $B$ be the closed ball in $X$ with centre $Tx$ and radius $r(M,M)$. Then $Ty \in B$ for all $y \in M$, i.e. $T(M) \subseteq B$. Consequently, $T(B \cap M) \subseteq T(M) \subseteq M \cap B$. Since $M \cap B$ is a non-empty weakly compact convex subset of $K$, $B \cap M \in \mathcal{F}$. By the maximality of $M$, $M \cap B = M$ or $M \subseteq B$. Hence for all $y \in M$, $d(Tx,y) \leq r(M,M)$ or $r_{Tx}(M) \leq r(M,M)$ and so $Tx \in C(M,M)$. Therefore $C(M,M) \in \mathcal{F}$ and so by the maximality of $M$ we must have $C(M,M) = M$, a contradiction. Hence $M$ must be a singleton proving thereby that $T$ has a fixed point in $K$.

**Remark 2.** Since Banach spaces are locally convex linear metric spaces having convex structure and with property (I), the above theorem generalizes the corresponding result of Kirk [4] as well as of Dulst and Sims [2].
References


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