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SPECTRAL CRITERIA OF ABSTRACT FUNCTIONS; INTEGRAL AND DIFFERENCE PROBLEMS

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Abstract. Let X be a complex Banach space and let M be a closed subspace of $L^\infty(J,X)$, where $J \in \{\mathbf{R},\mathbf{R}^+\}$. We answer the following question: Under what conditions $\phi_s - \phi \in M\ \forall s \in J$ implies that $\phi \in M$. Some conditions will be imposed on M to obtain the main result concerning the indefinite integral. These conditions guarantee the following implication : $F \in E(J,X) \Longrightarrow F \in M$, where F is the integral $\int_0^t f(s)\, ds$ of $f \in M \cap C_{ub}(J,X)$. Also, we generalize Loomis' Theorem for almost periodic functions [19, Theorem 5], to a more general class of functions $M \subseteq L^\infty(\mathbf{R},X)$ containing $AP(\mathbf{R},X)$. The main result of Part IV is: If ϕ is uniformly continuous, bounded, such that the M-spectrum $\sigma_M(\phi)$ of ϕ is at most countable and, for every $\lambda \in \sigma_M(\phi)$, the function $e^{-i\lambda t}\phi(t)$ is ergodic, then $\phi \in M$.

1. Introduction

A continuous scalar function f on $\mathbf{R}$ is called almost periodic (a.p) if the set of all translates $\{f_w : w \in \mathbf{R}\}$ is relatively compact (r.c) in $C_b(\mathbf{R})$ ($C_b(\mathbf{R})$ is the space of all scalar continuous bounded functions). Bohl and Bohr [8] proved that if f is a scalar almost periodic on $\mathbf{R}$, then $F(t) = \int\limits_0^t f(s)\, ds$ is a.p iff F is bounded (see also [22]). The almost periodicity of a function with values in a Banach space is defined similarly. M. I. Kadets [18] generalized this theorem and proved that: if f is an a.p from $\mathbf{R}$ to X which does not contain c_0, then F is a.p iff F is bounded. Here, c_0 is the space of all numerical sequences tending to 0. Thereafter, he proved this theorem for arbitrary Banach spaces X when the range of F is weakly relatively compact (w.r.c) in X (see [19]). Instead of the above mentioned integral problem B. Basit [2] considered the difference problem and proved the following result: Suppose that $f \in C_{ub}(G,X)$ such that $f_s - f$ is a.p $\forall s \in G$. If either

(i) X does not contain a subspace isomorphic to c_0,

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or

(ii) $f(G)$ is w.r.c in X,

then f is a.p.

The case $X = \mathbf{R}$ is proved by R. Doss [12]. See also F. Galvin, G. Muraz and P. Szeptycki [15] for a general group (nonabelian) and C. Datry and G. Muraz [11] for G-modules. See also E. Emmam [14] for almost automorphic functions. Here G is a group and $f_s(t) = f(ts)$. Mary L. Boas and R. P. Boas [15] proved that if f is bounded and $f_s - f$ is continuous for every $s \in \mathbf{R}$, then f is continuous. This result is generalized by F. Galvin, G. Muraz and P. Szeptycki [15] and C. Datry and G. Muraz [11], for the uniformly continuous functions defined on a group with values in a Banach space. Levitan [20] proved the almost periodicity of the integral F, provided that F is bounded and $\lim\limits_{T\to\infty}(1/2T)\int\limits_{-T}^{T} F(t+s)\,ds$ exists uniformly on $\mathbf{R}$. Basit [4] extended Levitan' s result to recurrent functions. C. Datry and G. Muraz [11] extended the result of Levitan to Banach G-modules.

Throughout this paper, X is a complex Banach space with the norm $\| \ \|$ and $J \in \{\mathbf{R}, \mathbf{R}^+\}$. We denote by $L^\infty(J,X)$ the Banach space of all essentially bounded measurable functions with the norm $\|f\|_\infty = ess \sup\limits_{t\in J} \|f(t)\|$. A function f is called measurable if there exists a sequence of simple functions $\{f_n\}$ such that $f_n \to f$ a.e with respect to the Lebesgue measure m. By a simple function it is meant a function of the form $\sum\limits_{i=1}^{n} x_i \chi_{A_i}$, $x_i \in X$ and χ_{A_i} is the characteristic function of the Lebesgue measurable set A_i with finite measure. Finally, M denotes a closed subspace of $L^\infty(J,X)$.

In the sequel, we impose on M at least one of the following two conditions:

(P1) M is invariant under translations, i.e. $\forall f \in M \ \forall s \in J \ (f_s \in M)$, where $f_s(t) = f(t+s)$.

(P2) M contains the constant functions.

In Section 2, we study examples of closed subspaces of $L^\infty(\mathbf{R}, X)$ which satisfy one or both of the conditions (P1-P2).

The third section is devoted to extend the previous results of the integral problem or the difference problem to the general space M, i.e. what are the conditions that insure the following implication

$$f \in M \bigcap C_{ub}(J,X) \Longrightarrow F(t) = \int\limits_0^t f(s)\,ds \in M$$

or

$$\phi_s - \phi \in M \quad \forall s \in J \Longrightarrow \phi \in M.$$

When $f \in M = AAP(\mathbf{R}^+, X)$, W. M. Ruess and W. H. Summers [28] proved that if $f \in AAP(\mathbf{R}^+, X)$, then

$$F(t) = \int_0^t f(s)\,ds \in AAP(\mathbf{R}^+, X) \quad \text{iff } F \in W(\mathbf{R}^+, X).$$

In this section, the notion of ergodic function in [13], [11] plays an essential role. A function $\phi \in L^\infty(J, X)$ is called ergodic if there exists $x \in X$ such that

$$\lim_{T\to\infty} \|(1/T)\int_0^T (\phi_s(t) - x)\,ds\|_\infty = 0.$$

We denote by $E(J, X)$ the space of all ergodic functions. We prove that if ϕ (resp. F) of the difference (resp. integral) problem is ergodic, then $\phi \in M$ (resp. $F \in M$).

In Section 4 M denotes a Banach subspace of $L^\infty(\mathbf{R}, X)$ which satisfies one or more of the conditions (P1-P3), where (P1-P2) are stated above and condition (P3) is:

(P3) M is invariant under multiplication by characters, i.e. $\forall f \in M\ \forall \lambda \in \mathbf{R}$ $(\breve{\lambda}\, f \in M)$, where $\breve{\lambda}(t) = e^{i\lambda t}$.

In Subsection 4.1 the M-spectrum of a function $u \in L^\infty(\mathbf{R}, X)$ will be defined by

$$\sigma_M(u) = Z(I_M(u)) = \{\alpha \in \mathbf{R} : \hat{f}(\alpha) = 0\ \forall f \in I_M(u)\},$$

where $\hat{f}(\alpha) = \int_{\mathbf{R}} f(t)e^{-i\alpha t}\,dt$, and $I_M(u)$ is the ideal of all $f \in L^1(\mathbf{R})$ such that $f * u \in M$. In the case $M = \{0\}$, $\sigma_M(u)$ is the well-known classical Beurling spectrum. Some properties of the M-spectrum, which we need in proving our results, will be shown.

When $M = AP(\mathbf{R}, \mathbf{C})$, L. H. Loomis [21] proved that if $u \in C_{ub}(\mathbf{R}, \mathbf{C})$ and $\sigma_{AP(\mathbf{R})}(u)$ (the set of all non-almost periodicity of u) is at most countable, then u is a.p. B. Basit generalized this theorem in [5] to a class of bounded uniformly continuous vector-valued functions defined on $\mathbf{R}$ with certain properties satisfied by many known classes.

In Subsection 4.2, we extend these results to a general closed subspace M of $L^\infty(\mathbf{R}, X)$. In this section, assuming that M satisfies (P1-P3), we prove that if ϕ is uniformly continuous, bounded, such that $\sigma_M(\phi)$ is at most countable, and for every $\lambda \in \sigma_M(\phi)$ the function $(\breve{-\lambda})\phi$ is ergodic, then $\phi \in M$. This theorem plays an essential role in proving the existence of solutions in some classes $M \subseteq L^\infty(\mathbf{R}, X)$ for abstract functional equations defined on $\mathbf{R}$ (see A. Hamza [17]).

Also, we prove the following result : Assume that ϕ is uniformly continuous, bounded, such that $\phi_s - \phi \in M \ \forall s \in \mathbf{R}$. If $0 \notin \sigma_M(\phi)$, then $\phi \in M$.

As a direct consequence, we obtain a result concerning the indefinite integral $F(t) = \int_0^t f(s)\,ds$, where $f \in M \bigcap C_{ub}(\mathbf{R}, X)$: $0 \notin \sigma_M(F)$ implies $F \in M$.

2. Preliminaries and examples

In this section, for the convenience of the reader, we recall some definitions and examples of closed subspaces M satisfying (P1) or (P2) or (P1) and (P2) above. Consider the following closed subspaces of $L^\infty(J, X)$.
(1) $C_b(J, X) = \{f : J \to X : f$ is continuous and bounded$\}$.
(2) $C_{ub}(J, X) = \{f : J \to X : f$ is uniformly continuous and bounded$\}$.
(3) $AP(\mathbf{R}, X)$-the Banach space of all almost periodic (a.p) functions. A function $f \in C_b(\mathbf{R}, X)$ is called a.p if for every $\varepsilon > 0$ the set

$$E_\varepsilon(f) = \{\tau \in \mathbf{R} : \sup_{t \in \mathbf{R}} \|f(t+\tau) - f(t)\| < \varepsilon\}$$

is relatively dense (r.d) in $\mathbf{R}$. A subset $B \subseteq \mathbf{R}$ is said to be r.d if there exists $\ell > 0$ such that $\forall a \in \mathbf{R}$ $(a, a+\ell) \cap B \neq \emptyset$. A function f is a.p iff $H(f) = \{f_\omega : \omega \in \mathbf{R}\}$ is relatively compact (r.c) in $C_b(\mathbf{R}, X)$, (see [1, 9, 20]).
(4) $AP(\mathbf{R}^+, X) = AP(\mathbf{R}, X)|_{\mathbf{R}^+}$, where $AP(\mathbf{R}, X)|_{\mathbf{R}^+}$ is the restriction of the a.p functions on $\mathbf{R}^+$.
(5) $C_0(\mathbf{R}, X) = \{f \in C_b(\mathbf{R}, X) : \lim_{|t| \to \infty} \|f(t)\| = 0\}$.
(6) $C_0(\mathbf{R}^+, X) = \{f \in C_b(\mathbf{R}^+, X) : \lim_{t \to \infty} \|f(t)\| = 0\}$.
(7) $L_0^\infty(J, X) = \{f \in L^\infty(J, X) : \lim_{|t| \to \infty} \|f(t)\| = 0\}$.
(8) $AAP(J, X) = AP(J, X) + C_0(J, X) = \{p + q : p \in AP(J, X), q \in C_0(J, X)\}$-the Banach space of all asymptotically almost periodic (a.a.p) functions from J to X. We notice that the decomposition $p + q$, where $p \in AP(J, X)$ and $q \in C_0(J, X)$, is unique. Indeed, if $p \in AP(J, X)$ then $\|p\|_\infty = \limsup_{t \to \infty} \|p(t)\|$ (see [31]). So, if $p \in AP(J, X) \bigcap C_0(J, X)$, then

$p = 0$. A function $f \in AAP(\mathbf{R}^+, X)$ iff $H(f) = \{f_\omega : \omega \in \mathbf{R}^+\}$ is r.c in $C_b(\mathbf{R}^+, X)$ (see [25, 26]).
(9) $S - AAP(J, X) = AP(J, X) + L_0^\infty(J, X)$-the Banach space of all a.a.p in the sense of Staffans [31]. Also the decomposition $p + q$, where $p \in AP(J, X)$ and $q \in L_0^\infty(J, X)$ is unique.
(10) $AA(\mathbf{R}, X)$-the Banach space of all almost automorphic (a.a) functions from $\mathbf{R}$ to X. A function $f \in C_b(\mathbf{R}, X)$ is called a.a if for each sequence $\{a_n'\} \subset \mathbf{R}$, there exists a subsequence $\{a_n\}$ such that
(i) $\lim\limits_{n\to\infty} f(t + a_n) = g(t)$, $t \in \mathbf{R}$, where g is a continuous function.
(ii) $\lim\limits_{n\to\infty} g(t - a_n) = f(t)$, $t \in \mathbf{R}$.
It is well-known that an a.a function is uniformly continuous and its range is totally bounded. A uniformly continuous function with totally bounded range is a.a iff $\forall \varepsilon > 0$ $\forall r > 0$ the set

$$E_{\varepsilon,r}(f) = \{\tau : \sup_{|t|\le r} \|f(t+\tau) - f(t)\| < \varepsilon\}$$

is r.d in $\mathbf{R}$, (see [3, 10]).
(11) $AA(\mathbf{R}^+, X) = AA(\mathbf{R}, X)|_{\mathbf{R}^+}$.

Lemma 2.1. *If $f \in AA(J, X)$, then $\|f\|_\infty = \limsup\limits_{t\to\infty} \|f(t)\|$.*

Proof. We have

$$\|f\|_\infty \ge \limsup_{t\to\infty} \|f(t)\|. \tag{1}$$

Let $\varepsilon > 0$, there exists $x_\varepsilon \in J$ such that $\|f\|_\infty \le \|f(x_\varepsilon)\| + \varepsilon$. Since $E_{\varepsilon,r}(f)$ is r.d, where $r = |x_\varepsilon| + 1$, there exists a sequence $\{\tau_n\} \subset E_{\varepsilon,r}(f)$ such that $\tau_n \to \infty$. We have $\|f(x_\varepsilon)\| \le \|f(x_\varepsilon + \tau_n)\| + \varepsilon$ $\forall n$, whence $\|f(x_\varepsilon)\| \le \limsup\limits_{t\to\infty} \|f(t)\| + \varepsilon$. Hence

$$\|f\|_\infty \le \limsup_{t\to\infty} \|f(t)\| + 2\varepsilon \quad \forall \varepsilon > 0.$$

Therefore,

$$\|f\|_\infty \le \limsup_{t\to\infty} \|f(t)\|. \tag{2}$$

We get from (1) and (2), that $\|f\|_\infty = \limsup\limits_{t\to\infty} \|f(t)\|$.

(12) $AAA(J,X) = AA(J,X) + C_0(J,X)$-the Banach space of all asymptotically almost automorphic functions (a.a.a). We have by the previous lemma that $AA(J,X) \bigcap C_0(J,X) = \{0\}$.
(13) $S - AAA(J,X) = AA(J,X) + L_0^\infty(J,X)$-the Banach space of all a.a.a in the sense of Staffans. Also, $AA(J,X) \bigcap L_0^\infty(J,X) = \{0\}$.
(14) $W(J,X)$ the Banach space of all weakly almost periodic functions in the sense of Eberlien (w.a.p-E). A function $f \in C_b(J,X)$ is called w.a.p-E if $\{f_\omega : \omega \in J\}$ is w.r.c in $C_b(J,X)$. A function f is w.a.p-E iff f satisfies the double limit property, i.e. $\forall \{\omega_n\} \subseteq J \ \ \forall \{t_n\} \subseteq J \ \ \forall \{x_n^*\} \subseteq X^*$ such that $\|x_n^*\| \leq 1$ we have

$$\lim_{n\to\infty} \lim_{m\to\infty} x_n^*(f_{\omega_m}(t_n)) = \lim_{m\to\infty} \lim_{n\to\infty} x_n^*(f_{\omega_m}(t_n))$$

whenever both of the limits exist, see [23].
(15) $E(J,X)$-the Banach space of all ergodic functions. A function $\phi \in L^\infty(J,X)$ is called ergodic if there exists $x \in X$ such that

$$\lim_{T\to\infty} \|1/T \int_0^T (\phi_s(t) - x)\, ds\|_\infty = 0.$$

(16) $TE(J,X) = \{\phi \in L^\infty(J,X) : e^{i\lambda t}\phi(t) \in E(J,X)\ \forall \lambda \in \mathbf{R}\}$.
(17) $E_0(J,X) = \{\phi \in E(J,X) : \lim\limits_{T\to\infty} \|1/T \int\limits_0^T \phi_s\, ds\|_\infty = 0$.

Lemma 2.2. *The spaces in Examples* (1-17) *are closed subspaces of* $L^\infty(J,X)$ *satisfying* (P1) *and the spaces in Examples* (1-4), (8-16) *satisfy* (P2). *The spaces in Examples* (5-7) *and* (17) *don't satisfy* (P2).

Lemma 2.3. $AP(J,X) \subset AAP(J,X) \subset W(J,X) \subset TE(J,X)$.

Proof. We can check that $C_0(J,X) \subset W(J,X)$. Indeed, suppose that $f \in C_0(J,X)$. Let $\{t_n\}$ and $\{\omega_n\}$ be two sequences in J. Let $\{x_n^*\}$ be a sequence in X^* such that $\|x_n^*\| \leq 1$ and both of the following iterated limits

$$\lim_{m\to\infty} \lim_{n\to\infty} x_n^*(f_{\omega_m}(t_n))$$

and

$$\lim_{n\to\infty} \lim_{m\to\infty} x_n^*(f_{\omega_m}(t_n))$$

exist. We have the following cases:
(i) both of $\{t_n\}$ and $\{\omega_n\}$ are unbounded. In this case we can suppose without loss of generality that $t_n \to \infty$ and $\omega_n \to \infty$.

(ii) one of the two sequences, say $\{t_n\}$, is unbounded and the other is bounded. In this case we can suppose that $t_n \to \infty$ and $\omega_n \to a$ for some $a \in J$.
(iii) both of the two sequences are bounded. We can assume in this case that $t_n \to a$ and $\omega_n \to b$ for some a and b in J.
In case (i) both of the iterated limits equal zero. In case (ii), the first iterated limit equals zero. The second iterated limit equals

$$\lim_{n\to\infty} x_n^*(f_a(t_n)) = 0.$$

In case (iii) we use the uniform continuity of f to conclude that the two iterated limits are equal. The fact that $W(J,X) \subset TE(J,X)$ is a result of [13]. To show that both of $AP(J,X)$, $AAP(J,X)$ and $W(J,X)$ are subsets of $C_{ub}(J,X)$, see [20, 23, 30].

3.The difference and the integral problem

As before we assume that M is a closed subspace of $L^\infty(J,X)$. In this section we study the difference problem, viz we answer the following question: Under what conditions, does $\phi_s - \phi \in M \quad \forall s \in J$ imply that $\phi \in M$. As a direct consequence we get a result concerning the indefinite integral problem (see [2, 8, 14, 18, 20, 22, 28]). L. H. Loomis [21] imposed the condition $\phi \in C_{ub}(\mathbf{R},\mathbf{C})$ to get $\phi \in AP(\mathbf{R},\mathbf{C})$. When $M = AP(\mathbf{R},X)$, B. Basit [2] supposed the same condition $\phi \in C_{ub}(\mathbf{R},X)$ to get $\phi \in AP(\mathbf{R},X)$, provided that X does not contain c_0 or the range of ϕ is w.r.c in X. In fact this condition is not necessary because ϕ will be uniformly continuous and bounded according to Theorem 3.0 stated below. In case $M = C_b(\mathbf{R},\mathbf{R})$, Mary L. Boas and R. P. Boas [15] proved that: If $\phi : \mathbf{R} \to \mathbf{R}$ is bounded on a set of positive measure and $\phi_s - \phi$ is continuous for every $s \in \mathbf{R}$, then ϕ is continuous. We can see some generalizations of this result in [11] and [15].
According to a general result of C. Datry and G. Muraz [11], we have the following theorem:

Theorem 3.0. *A bounded function $\phi : \mathbf{R} \to X$ is uniformly continuous iff $\phi_s - \phi$ is uniformly continuous for every $s \in \mathbf{R}$.*

Lemma 3.1. *Let $\phi \in C_{ub}(J,X)$. If $\phi_s - \phi \in M \quad \forall s \in J$, then $\int_J(\phi_s - \phi)\, d\mu(s) \in M$ for every bounded Borel measure on J.*

Proof. The function $g : J \to M$ defined by $g(s) = \phi_s - \phi$ is bounded and continuous, since $\phi \in C_{ub}(J,X)$. Suppose that μ is a bounded Borel measure. Hence, g is measurable with respect to μ (see Pettis' Theorem [32,

p. 131]). We apply Bochner's Theorem [32, p. 133] to get $\int_J g(s)\,d\mu_(s) \in M$ i.e. $\int_J(\phi_s - \phi)\,d\mu(s) \in M$.

Lemma 3.2. *Let $\phi \in L^\infty(J,X)$ be such that $\phi_s - \phi \in M$ $\forall s \in J$. If there exists a bounded Borel measure μ such that $\int_J d\mu(s) \neq 0$ and $\int_J \phi_s\,d\mu(s) = 0$, then $\phi \in M$.*

Theorem 3.3. *Let M satisfy* (P2). *Suppose that $\phi \in C_{ub}(J,X) \bigcap E(J,X)$. If $\forall s \in J$ $\phi_s - \phi \in M$, then $\phi \in M$.*

Proof. There exists $x \in X$ such that $\phi - x \in E_0(J,X)$. By condition (P2) M contains the constant function $x(t) = x$, $t \in J$. We apply Lemma 3.1 to get $\int_J(\phi_s - \phi)\,d\mu_T(s) \in M$, where $d\mu_T(s) = (1/T)\chi_{[0,T]}(s)\,ds$, $T > 0$. Hence, $(1/T)\int\limits_0^T(\phi_s - \phi)\,ds \in M$ $\forall T > 0$. Taking the limit as $T \to \infty$, we get $\phi \in M$ (M is closed).

Theorem 3.4. *Let M satisfy* (P1-P2). *Suppose that $f \in M \bigcap C_{ub}(J,X)$. If the function F defined by $F(t) = \int_0^t f(s)\,ds$ belongs to $E(J,X)$, then $F \in M$.*

Proof. We have $(F_s - F)(t) = \int\limits_0^s f_u(t)\,du$, $s \in J$, $t \in J$. Fix s and let I be the interval with end points 0 and s. The function $g : I \to M$ defined by $g(u) = f_u$ is continuous. Since $\int_J \|g(u)\|_\infty\,ds \leq |s|\|f\|_\infty < \infty$, then $\int\limits_0^s f_u\,du \in M$. Hence $F_s - F \in M$ $\forall s \in J$. By Theorem 3.3, we get $F \in M$.

Corollary 3.5. *Let $f \in C_b(J,X)$ and $F(t) = \int\limits_0^t f(s)\,ds$. Then the following statements are true.*

(1) *If $f \in AP(J,X)$ and $F \in E(J,X)$, then $F \in AP(J,X)$.*
(2) *If $f \in AAP(J,X)$ and $F \in E(J,X)$, then $F \in AAP(J,X)$.*
(3) *If $f \in AAP(J,X)$ and $F \in W(J,X)$, then $F \in AAP(J,X)$.*
(4) *If $f \in AA(J,X)$ and $F \in E(J,X)$, then $F \in AA(J,X)$.*
(5) *If $f \in AAA(J,X)$ and $F \in E(J,X)$, then $F \in AAA(J,X)$.*

Proof. The statements (1), (2), (4) and (5) are true, since all of the following spaces $AP(J,X)$, $AAP(J,X)$, $AA(J,X)$ and $AAA(J,X)$ satisfy (P1-P2). The statement (3) is true, since $W(J,X) \subset TE(J,X)$.

4. Spectral criteria of abstract functions

In this section M denotes a closed subspace of $L^\infty(\mathbf{R}, X)$ which satisfies one or more conditions on M from the following list:

(P1) M is invariant under translations, i.e. $\forall f \in M \ \forall s \in \mathbf{R} \ (f_s \in M)$, where $f_s(t) = f(t+s)$.

(P2) M contains the constant functions.

(P3) M is invariant under multiplication by characters, i.e. $\forall f \in M \ \forall \lambda \in \mathbf{R}(\breve{\lambda} f \in M)$, where $\breve{\lambda}(t) = e^{i\lambda t}$.

We consider the closed subspaces of $L^\infty(\mathbf{R}, X)$ which are given in Section 2. We can check that all of them satisfy (P3) except the spaces in Examples (15) and (17). We can prove that $W(\mathbf{R}, X)$ satisfies (P3), by showing that for every $f \in W(\mathbf{R}, X)$ and for every $\lambda \in \mathbf{R}$ the function $\breve{\lambda} f$ satisfies the double limit property, where $\breve{\lambda}(t) = e^{i\lambda t}$.

4.1. The M-spectrum of functions in $L^\infty(\mathbf{R}, X)$

Definition 4.1.1. For a function $u \in L^\infty(\mathbf{R}, X)$ and $f \in L^1(\mathbf{R})$ denote by

$$(f * u)(t) = \int_{\mathbf{R}} f(t-s)u(s)\, ds, \qquad t \in \mathbf{R}.$$

Lemma 4.1.2 (see also [5]). *If M is a closed subspace of $L^\infty(\mathbf{R}, X)$ satisfying* (P1)*, then*

$$\forall f \in L^1(\mathbf{R}) \ \forall u \in M \bigcap C_{ub}(\mathbf{R}, X) \, (f * u \in M) \, .$$

Proof. Let $f \in L^1(\mathbf{R})$ and $u \in M \bigcap C_{ub}(\mathbf{R}, X)$. Define the function $g : \mathbf{R} \to M$ by

$$g(s) = u_{-s}.$$

The function g is continuous and bounded, since u is uniformly continuous. Applying Bochner's Theorem [32, p. 133], we get $\int_{\mathbf{R}} f(s) u_{-s}\, ds \in M$, whence $f * u \in M$

Lemma 4.1.3. *If $u \in L^\infty(\mathbf{R}, X)$, then the following conditions are equivalent*

(i) $u \in C_{ub}(\mathbf{R}, X)$,

(ii) $\lim_{t \to 0} \|u_t - u\|_\infty = 0$,

(iii) $\lim_{T \to 0} \|\rho_T * u - u\|_\infty = 0$, *where* $\rho_T = \frac{1}{T}\chi_{[-T,0]}$, $T > 0$. *Here* $\chi_{[-T,0]}$ *is the characteristic function of the interval* $[-T, 0]$.

This is a classical result in the theory of $L^1(G)$-modules. We can replace $\{\rho_T\}$ by any bounded approximate of identity (see [11]).

Definition 4.1.4 (see [6, 14, 5]). Suppose that M is a closed subspace of $L^\infty(\mathbf{R}, X)$ such that (P1) holds. Let $u \in L^\infty(\mathbf{R}, X)$. We denote by

$$I_M(u) = \{f \in L^1(\mathbf{R}) : f * u \in M\}.$$

The set $I_M(u)$ is a closed ideal of $L^1(\mathbf{R})$.

We denote the M-spectrum $\sigma_M(u)$ of $u \in L^\infty(\mathbf{R}, X)$ by

$$\sigma_M(u) = Z(I_M(u)) = \{\alpha \in \mathbf{R} : \hat{f}(\alpha) = 0 \quad \forall f \in I_M(u)\},$$

where $\hat{f}(\alpha) = \int_{\mathbf{R}} f(t)e^{-i\alpha t}\, dt$. The spectrum $\sigma(u)$ is denoted by $\sigma(u) =: \sigma_{\{0\}}(u)$. It is clear that $\sigma_M(u) \subseteq \sigma(u)$.

Lemma 4.1.5 (see also [14, 5]). *Let $u \in L^\infty(\mathbf{R}, X)$. If M is a closed subspace of $L^\infty(\mathbf{R}, X)$ satisfying* (P1)*, then the following hold:*

1. $\sigma_M(u) = \emptyset$ *iff* $\forall f \in L^1(\mathbf{R})$ $f * u \in M$.
2. *If* $u \in C_{ub}(\mathbf{R}, X)$ *then* $\sigma_M(u) = \emptyset$ *iff* $u \in M$.
3. *If* $\sigma_M(u) = \{0\}$, *then* $f * (u_s - u) \in M$ $\forall f \in L^1(\mathbf{R}) \quad \forall s \in \mathbf{R}$.
4. *If* $u \in C_{ub}(\mathbf{R}, X)$, *then* $\sigma_M(u) = \{0\} \Longrightarrow u_s - u \in M$ $\forall s \in \mathbf{R}$.
5. $\sigma_M(f * u) \subseteq \operatorname{supp} \hat{f} \bigcap \sigma_M(u)$ $\forall f \in L^1(\mathbf{R})$.
6. *If in addition M satisfies* (P3), *then*

$$\sigma_M(\breve{\gamma} u) = \sigma_M(u) + \gamma \ \forall \gamma \in \mathbf{R},$$

where $\breve{\gamma}(t) = e^{i\gamma t}$.

Proof. (1) We have $Z(I_M(u)) = \emptyset$ iff $I_M(u) = L^1(\mathbf{R})$. Hence $\sigma_M(u) = \emptyset$ iff $\forall f \in L^1(\mathbf{R})$ $f * u \in M$.
(2) Let $u \in C_{ub}(\mathbf{R}, X)$. Suppose that $\sigma_M(u) = \emptyset$. Hence, by (1) we have $f * u \in M$ $\forall f \in L^1(\mathbf{R})$. By Lemma 4.1.3 $\lim_{T \to 0} \|\rho_T * u - u\|_\infty = 0$ whence $u \in M$. Conversely, suppose that $u \in M$. By Lemma 4.1.2, we get that $f * u \in M$ $\forall f \in L^1(\mathbf{R})$, which in return implies that $\sigma_M(u) = \emptyset$.
(3) Suppose that $a_M(u) = \{0\}$, i.e. $Z(I_M(u)) = \{0\}$, where $\{0\}$ is a set of spectral synthesis. We have $I_M(u) = \{f \in L^1(\mathbf{R}) : \hat{f}(0) = 0\}$. Hence, $(f_s - f) * u \in M$ $\forall s \in \mathbf{R}$ $\forall f \in L^1(\mathbf{R})$, whence $f * (u_s - u) \in M$ $\forall s \in \mathbf{R}$ $\forall f \in L^1(\mathbf{R})$.
(4) is a direct consequence of (3) and (2).
(5) Let $f \in L^1(\mathbf{R})$. Let $\alpha \in \sigma_M(f * u)$. To show that $\alpha \in \sigma_M(u)$, let

$h \in L^1(\mathbf{R})$ be such that $h * u \in M$. Then $h * (f * u) = f * (h * u) \in M$. Therefore $\hat{h}(\alpha) = 0$ and we get $\alpha \in \sigma_M(u)$.
Now we show that $\alpha \in \text{supp } \hat{f}$. Suppose on the contrary that $\alpha \notin \text{supp } \hat{f}$. Then there exists $g \in L^1(\mathbf{R})$ such that $\hat{g}(\alpha) \neq 0$ and $\hat{g}(\text{supp } \hat{f}) = \{0\}$. We have $g * f = 0$ whence $g * f * u = 0 \in M$. Hence $\hat{g}(\alpha) = 0$ which is a contradiction.
(6) We denote by $g = \breve{\gamma_0} u$, $\gamma_0 \in \mathbf{R}$. Let $\gamma \in \sigma_M(g)$. Let $f \in L^1(\mathbf{R})$ be such that $f * u \in M$. A simple calculation shows that

$$(\breve{\gamma_0} f) * g = \breve{\gamma_0}(f * u).$$

Hence $(\breve{\gamma_0} f) * g \in M$, whence $(\breve{\gamma_0} f)\hat{}(\gamma) = 0$, i.e. $\hat{f}(\gamma - \gamma_0) = 0$ and we get $\gamma - \gamma_0 \in \sigma_M(u)$. Conversely, let $\gamma \in \sigma_M(u)$ and $f \in L^1(\mathbf{R})$ be such that $f * g \in M$. We have

$$f * g = (\breve{\gamma_0})[((-\gamma_0)\breve{}\ f) * u].$$

Hence, $((-\gamma_0)\breve{}\ f) * u \in M$, whence $((-\gamma_0)\breve{}\ f)\hat{}(\gamma) = 0$, i.e. $\hat{f}(\gamma + \gamma_0) = 0$ and we get $\gamma + \gamma_0 \in \sigma_M(g)$.

4.2. **Spectral characterization of the classes** M

A theorem of Loomis [21] states that: If $\phi \in C_{ub}(\mathbf{R})$ and $\sigma_{AP(\mathbf{R})}(\phi)$ is at most countable, then $\phi \in AP(\mathbf{R})$. In this section, we generalize this theorem to more general classes of functions $M \subseteq L^\infty(\mathbf{R}, X)$ containing $AP(\mathbf{R}, X)$. We prove the following result: If ϕ is uniformly continuous, bounded, such that the M-spectrum $\sigma_M(\phi)$ of ϕ is at most countable and, for every $\lambda \in \sigma_M(\phi)$, the function $e^{-i\lambda t}\phi(t)$ is ergodic, then $\phi \in M$.

Lemma 4.2.1. *If $\lambda_0 \in \mathbf{R}$ is such that $(-\lambda_0)\breve{}\phi \in E(\mathbf{R}, X) \bigcap C_{ub}(\mathbf{R}, X)$, then λ_0 cannot be an isolated point of $\sigma_M(\phi)$.*

Proof. Let $\lambda_0 \in \mathbf{R}$ be such that $(-\lambda_0)\breve{}\phi \in E(\mathbf{R}, X) \bigcap C_{ub}(\mathbf{R}, X)$. Suppose on the contrary that λ_0 is an isolated point of $\sigma_M(\phi)$. There exists a compact neighbourhood V of λ_0 such that $V \bigcap (\sigma_M(\phi) \setminus \{\lambda_0\}) = \emptyset$. Choose $f \in L^1(\mathbf{R})$ such that $\hat{f}(\lambda_0) \neq 0$ and $\hat{f}(\boldsymbol{\mathcal{C}} V) = \{0\}$. Here, $\boldsymbol{\mathcal{C}} V$ is the complement of V. Hence, $\sigma_M(f * \phi) \subseteq \sigma_M(\phi) \bigcap \text{ supp } \hat{f} \subseteq \{\lambda_0\}$, whence $\sigma_M(f * \phi) = \{\lambda_0\}$. By Lemma 4.1.5., we get

$$[(-\lambda_0)\breve{}(f * \phi)]_s - [(-\lambda_0)\breve{}(f * \phi)] \in M \quad \forall s \in \mathbf{R}.$$

Since $(-\lambda_0)\check{}(f * \phi) = (-\lambda_0)\check{}f * (-\lambda_0)\check{}\phi \in E(\mathbf{R}, X) \bigcap C_{ub}(\mathbf{R}, X)$, then, by Theorem 3.3, we get $(-\lambda_0)\check{}(f * \phi) \in M$, whence $f * \phi \in M$. Hence, $\hat{f}(\lambda_0) = 0$, which is a contradiction.

Theorem 4.2.2. *Let $\phi \in C_{ub}(\mathbf{R}, X)$. If $\sigma_M(\phi)$ is at most countable such that the function $(-\lambda)\check{}\phi \in E(\mathbf{R}, X)$ for every $\lambda \in \sigma_M(\phi)$, then $\phi \in M$.*

Proof. Suppose that $\phi \in C_{ub}(\mathbf{R}, X)$ satisfies the hypothesis of the theorem. We show that $\sigma_M(\phi) = \emptyset$. Suppose on the contrary that $\sigma_M(\phi) \neq \emptyset$. Then $\sigma_M(\phi)$ (at most countable) has an isolated point λ_0 [7]. Since $(-\lambda_0)\check{}\phi \in E(\mathbf{R}, X) \bigcap C_{ub}(\mathbf{R}, X)$, then by Lemma 4.2.1, we get that λ_0 is not an isolated point of $\sigma_M(\phi)$ which is a contradiction.

Corollary 4.2.3. *Let M be as in Theorem 4.2.2 and let $\phi \in C_{ub}(\mathbf{R}, X) \bigcap TE(\mathbf{R}, X)$. If $\sigma_M(\phi)$ is at most countable, then $\phi \in M$.*

Proof. It is an immediate consequence of the previous theorem.

In the following theorem we impose a condition on $\sigma_M(\phi)$ that insures the following implication:

$$\phi_s - \phi \in M \ \forall s \in \mathbf{R} \Longrightarrow \phi \in M.$$

Theorem 4.2.4. Assume that M is a closed subspace of $L^\infty(\mathbf{R}, X)$. Let $\phi \in L^\infty(\mathbf{R}, X)$ be such that $0 \notin \sigma_M(\phi)$. If $\phi_s - \phi \in M \bigcap C_{ub}(\mathbf{R}, X)$ $\forall s \in \mathbf{R}$, then $\phi \in M$.

Proof. By Theorem 3.0, $\phi \in C_{ub}(\mathbf{R}, X)$. Since $0 \notin \sigma_M(\phi)$, there exists $f \in L^1(\mathbf{R})$ such that $f * \phi \in M$ and $\hat{f}(0) = 1$. By Lemma 3.1, we get $f * \phi - \phi \in M$, whence $\phi \in M$.

Corollary 4.2.5. *Assume that M is a closed subspace of $L^\infty(\mathbf{R}, X)$ satisfying* (P1). *Let $f \in M \bigcap C_{ub}(\mathbf{R}, X)$. Define F by $F(t) = \int_0^t f(s)\, ds$. If $0 \notin \sigma_M(F)$, then $F \in M$.*

Proof. We have $(F_s - F)(t) = \int_0^s f_u(t)\, du$, $s \in \mathbf{R}$, $t \in \mathbf{R}$. By the same argument as in Theorem 3.4, it follows that $F_s - F \in M$ $\forall s \in \mathbf{R}$. By Theorem 4.2.4, we get $F \in M$.

In the applications of all these results given by A. Hamza in [17] for the solutions of functional equations or differential equations, the condition "ϕ is continuous" is not a restriction, because in general case, it is more than continuous.

To stay complete, we must report the Banach space $BAUC(J,X)$ of all bounded asymptotically uniformly continuous functions from J to X (see [31]); sometimes such function is called *slowly oscillating* [24]. $\phi \in BAUC(J,X)$ if $\phi \in L^\infty(J,X)$ and $\lim_{(t,x)\to(\infty,0)} |\phi_x(t) - \phi(t)| = 0$.

In fact, we have $BAUC(J,X) = C_{ub}(J,X) + L^\infty_o(J,X)$. This space verifies the properties (P1), (P2), (P3) and have some importance for the application in [17].

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