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ON GRÄTZER'S PROBLEM

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1. Introduction

This paper is concerned with Grätzer's problem: find conditions under which $\mathrm{Sub}(L)$ determines L up to isomorphism (see [5], Problem 1.4).

In [1], [2] we proposed the concept of contractible sublattice and gave a condition on a lattice L without contractible sublattices, such that $\mathrm{Sub}(L)$ determines L up to an isomorphism or a dual isomorphism. In [3] we described a class $\mathbf{K}$ of lattices satisfying this condition.

The main aim of this paper is to study the lattices which have contractible sublattices. By contractible sublattice method we construct such lattices L which are determined by $\mathrm{Sub}(L)$ up to an isomorphism or a dual isomorphism (see Theorem 2.5). It is worth to mention that these lattices do not belong to $\mathbf{K}$.

2. Results

First, we recall some concepts and results from [1], [2], [3].

Definition I. A proper sublattice A of the lattice L with $|A| > 1$ is called a contractible sublattice if A satisfies the following conditions:

(a) A is convex

(b) $c \in A \Leftrightarrow d \in A$, for any square $\langle a, b; c, d\rangle$ in L.

Remark. Suppose that A is contractible and $\langle a, b; c, d\rangle$ is a square in L. According to (a) and (b), if an element of $\{c, d\}$ belongs to A then the sublattice $\{a, b, c, d\}$ is contained in A. Therefore, instead of (b), we can shortly say that *"sublattice A absorbs the squares"*.

In what follows, it will be denoted by $a\,S\,b$ (resp. $a\|b$), when a is comparable (resp. incomparable) with b.

Lemma II. *Let A be a contractible sublattice of L and $k \in L \setminus A$, $a \in A$, Then:*

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(P$_1$) *If* $k < a$ *then* $k < x$, $\forall x \in A$.
(P$_2$) *If* $k > a$ *then* $k > x$, $\forall x \in A$.
(P$_3$) *If* $k \| a$ *then* $k \| x$, $\forall x \in A$.

We recall that a one-to-one and onto map $\varphi : L \to L'$ for two arbitrary lattices L. L' is called *a square preserving bijection* if: $\langle a, b; c, d\rangle$ is a square in $L \Leftrightarrow \langle \varphi(a), \varphi(b); \varphi(c), \varphi(d)\rangle$ is a square in L'.

Proposition III. *Let L be a lattice having no contractible sublattices and $\varphi : L \to L'$ a square preserving bijection. Then φ is either an isomorphism or a dual isomorphism.*

Now, we prove some lemmas concerning the contractible sublattices and the lattices having no linear decomposition.

We say that a lattice L has *a linear decomposition* if there exist a chain I with $|I| > 1$ and sublattices L_i, $i \in I$ of L such that $L = \bigcup\limits_{i \in I} L_i$ and for $i, j \in I$, $i < j$ then $a < b$ for every $a \in L_i$, $b \in L_j$.

Lemma 2.1. *If A, B are contractible sublattices of L such that $A \nsubseteq B$, $B \nsubseteq A$ and $A \cap B \neq \emptyset$, then $A \cup B$ is a linearly decomposable sublattice.*

Proof. Let $C = A \cap B$ and $X = A \setminus C$, $Y = B \setminus C$. Clearly, C is a sublattice. Take $x, y \in X$ such that $x \| y$. If at least one of two elements $x \wedge y$, $x \vee y$ belongs to C, then $x, y \in B$ because of the contractibility of B. This is a contradiction, since $X \cap B = \emptyset$. Therefore $x \wedge y$, $x \vee y \in X$, i.e. X is a sublattice. Analogously, Y is also a sublattice.

Now we consider arbitrary elements $x \in X$, $y \in Y$ and $c \in C$. If $x \| c$ then it is easy to deduce that $x \wedge c$, $x \vee c \notin B$ and so, we have $x \wedge c < c < x \vee c$ with $x \wedge c$, $x \vee c \in A \setminus B$. Since B is contractible and $c \in B$, by Lemma II it implies that $x \wedge c < b < x \vee c$, $\forall b \in B$. Because of the convexity of A we have $B \subseteq A$, which contradicts the assumption of the lemma. Thus, we have xSc and by Lemma II it implies either $x < c < y$ or $x > c > y$. This means that $A \cup B$ is a sublattice which is linearly decomposed into X, C, Y.

The proof is complete.

Lemma 2.2. *If a lattice L has no linear decomposition and A, B are different maximal contractible sublattices of L, then $A \cap B = \emptyset$.*

Proof. Let $C = A \cap B$. If $C \neq \emptyset$, then $A \cup B$ is a linear decomposed sublattice of L as shown in Lemma 2.1. Since L is not linearly decomposable, $A \cup B$ must be a proper sublattice. Evidently, $A \cup B$ absorbs the squares. Now, we show that $A \cup B$ is convex. Take $x \in L$ such that $u < x < v$ with $u, v \in A \cup B$. We have to prove that $x \in A \cup B$. The cases, where $u, v \in A$

or $u, v \in B$, are trivial by virtue of convexity of A and B. Hence we may assume that $u \in A \setminus C$, $v \in B \setminus C$ and $x \notin A$. According to (P_2) we have $x > a$, $\forall a \in A$ and thus $x > c$ for some $c \in C \subseteq B$. From $v > x > c$ with $v, c \in B$ we conclude $x \in B \subseteq A \cup B$.

In conclusion, $A \cup B$ is a contractible sublattice, which contradicts the fact that A is maximal. Thus, we have $A \cap B = \emptyset$ and the lemma is proved.

Lemma 2.3. *Let L be a lattice having no linear decomposition, and $\varphi : L \to L'$ a square preserving bijection for some lattice L'. If A is a maximal contractible sublattice of L, then $\varphi(A)$ is a contractible sublattice of L'.*

Proof. For the sake of convenience we denote $\varphi(x)$ by x' and $\varphi(X)$ by X', where $x \in L$ and $X \subseteq L$. Since φ is a bijection, any element of L' is writte uniquely in the form x', $x \in L$.

Let A be a maximal contractible sublattice of L. Take x', $y' \in A'$ with $x' \| y'$. Then $\langle x, y; x \wedge y, x \vee y \rangle$ is a square in A. It implies that $\langle x', y'; x' \wedge y', x' \vee y' \rangle$ is a square in A'. Thus we have $x' \wedge y'$, $x' \vee y' \in A'$, so A' is a sublattice of L'.

Further, if $\langle a', b'; c', d' \rangle$ is a square in L' with, for example, $c' \in A'$, the $\langle a, b; c, d \rangle$ is a square in L with $c \in A$. According to (b) of Definition I we have $d \in A$, i.e. $d' \in A'$. Thus, A' absorbs the squares.

Now, we verify the convexity of A'. We assume by contrary that there exist $h' \notin A'$ and $u' < h' < v'$ with u', $v' \in A'$. From $u' < h'$ it follows by Lemma II that $h\,S\,u$ and so, $h\,S\,a$, $\forall a \in A$. Therefore we have $h' S\, a'$, $\forall a' \in A'$. Denoting $X' = \{x' \in A' | x' < h'\}$ and $Y' = \{y' \in A' | y' > h'\}$ $(u' \in X', v' \in Y')$, we obtain $A' = X' \cup Y'$ as a linearly decomposed lattice.

Since $h\,S\,u$ and the proof for the case of $u > h$ is similar to the case of $u < h$, we shall prove only the case $h > u$. According to (P_2), we have $h > a$, $\forall a \in A$. We denote:

$Z = \{z \in L | z > a, \forall a \in A \text{ and } x' < z' < y', \forall x' \in X', \forall y' \in Y'\}$,

$K = \{k \in L | \exists z \in Z : z \geq k > a,\ \forall a \in A\}$.

It is easy to see that $Z \neq \emptyset$ $(h \in Z)$, $Z \subseteq K$ and $K \cap A = \emptyset$. In order to prove the convexity of A' we need the following claims.

Claim 1. Z is a sublattice.

Proof. Considering arbitrary $z_1, z_2 \in Z$ with $z_1 \| z_2$, we have $z_1 \vee z_2 > z_1 > z_1 \wedge z_2 \geq a$, $\forall a \in A$. If $z_1 \wedge z_2 = a$, then $z_1 \wedge z_2 \in A$. From the contractibility of A it follows that $\{z_1, z_2; z_1 \wedge z_2, z_1 \vee z_2\} \subseteq A$, which contradicts the fact that $Z \cap A = \emptyset$. Hence $z_1 \wedge z_2 > a$. On the other hand, for any $x' \in X'$, $y' \in Y'$, we always have $x' < z_1' \wedge z_2'$, $z_1' \vee z_2' < y'$. (If, for example, $x' = z'_1 \wedge z'_2$ then $z_1, z_2 \in A \cap Z$, a contradiction). Consequently, we obtain $z_1 \wedge z_2$, $z_1 \vee z_2 \in Z$, i.e. Z is a sublattice.

Claim 2. K is a sublattice.

Proof. Take $k_1, k_2 \in K$ such that $k_1 \| k_2$. Then $k_1 \vee k_2 > k_1 \wedge k_2 > a$, $\forall a \in A$. Moreover, since $k_1 < z_1$ and $k_2 < z_2$ with $z_1, z_2 \in Z$, we have $k_1 \vee k_2 < z_1 \vee z_2 \in Z$. Therefore $k_1 \wedge k_2$, $k_1 \vee k_2 \in K$.

We observe that K is convex by its definition.

Claim 3. K absorbs the squares.

Proof. Let us consider a square $\langle e, f; c, d\rangle$ in L with $c < d$ we have to show that $c \in K \Leftrightarrow d \in K$.

Necessity. Let $c \in K$. Then $d > e > c > a$, $\forall a \in A$ (see Fig. 1a). Therefore $d' S\, a'$, $\forall a' \in A'$.

We have two alternative cases.

(1) $c \in Z$. In this case $x' < c' < y'$, $\forall x' \in X'$, $\forall y' \in Y'$. Consider the square $\langle c', f'; c', d'\rangle$, where, without loss of generality, we can assume that $c' < d'$ (see Fig. 1b).

Fig. 1

Note that

$$(*) \qquad y' \in Y' \Rightarrow y' > d'.$$

Indeed, since $y \in Y$ and $e, f > c > y$ we have $y' S e', f'$. If $y' < e'$, f', then $y' \leq e' \wedge f' = c'$, which contradicts $c' < y'$. Thus, $y' > e'$, f' and so $y' \geq d'$. Since $d \notin A$, it implies that $y' > d'$.

By the assumption we have $d > a$, $\forall a \in A$ and $x' < d'$, $\forall x' \in X'$. By (*), $d' < y'$, $\forall y' \in Y'$. Hence $d \in Z \subseteq K$.

(2) $c \notin Z$. Since cSa, $\forall a \in A$ and the condition $x' < c' < y'$, $\forall x' \in X'$, $\forall y' \in Y'$ does not hold, we have to examine 4 possibilities:

(2a) $c' < a'$, $\forall a' \in A'$.
(2b) $\exists p', q' \in X' : p' < c' < q'$.
(2c) $c' > a'$, $\forall a' \in A'$.
(2d) $\exists p', q' \in Y' : p' < c' < q'$.

We shall only examine the cases (2a) and (2b). The proof of (2c) and (2d) is similar. We may assume that $c' < d'$.

Case (2a) is shown in Fig. 2a, where z' is an arbitrary element of Z'. Applying (*) to an arbitrary $x' \in X'$, we obtain $d' < x'$ and hence, $d' < z'$.

Fig. 2

For the case (2b) we denote $P' = \{x' \in X' | x' < c'\}$ and $Q' = \{x' \in X' | x' > c'\}$. This case is shown in Fig. 2b, where z' is an arbitrary element in Z'. Considering $q' \in Q'$ and using statement $(*)$ we have $d' < q' < z'$.

Since e', $f' < d'$, and by (2a) and (2b) we always have $e\,S\,z$ and $f\,S\,z$, $\forall z \in Z$. If $e, f > z$, $\forall z \in Z$, then $c = e \wedge f > z$, $\forall z \in Z$. This contradicts the definition of K. Therefore, there exists $z_0 \in Z$ such that z_0 is greater than one of the elements e, f. Since $e \| f$, $d = e \vee f \leq z_0$ which shows that $d \in K$.

Sufficiency. Let $d \in K$. If $d \in Z$ then $x' < d' < y'$, $\forall x' \in X'$, $\forall y' \in Y'$. As in part (1) of Necessity we have $c \in Z$. Assume that $d \notin Z$. Then $a < d < z_0$ for all $a \in A$ and some $z_0 \in Z$. Consider $\langle e', f'; c', d' \rangle$, where we can assume that $c' < d'$. We show that $a\,S\,e$, f, $\forall a \in A$. Indeed:

a) If $d' < z'_0$ then e', $f' < d' < z'_0 < y'$, $\forall y' \in Y'$. Therefore $e\,S\,y$ and $f\,S\,y$, for some $y \in A$.

b) If $d' > z'_0$, using the condition that e, $f < d < z_0$ we have $e'S\,z'_0$, $f'S\,z'_0$. Therefore $z'_0 < e'$, f'. This means that $e', f' > x'$, $\forall x' \in X'$ and hence $e\,S\,x$ and $f\,S\,x$ for some $x \in A$.

From a) , b) and by Lemma II we obtain that $a\,S\,e$, f, $\forall a \in A$.

If e, $f < a_0$, for $a_0 \in A$ then $d = e \vee f \leq a_0$ which contradicts the fact that $d > a$, $\forall a \in A$. Thus $e, f > a$, $\forall a \in A$ and so $c = e \wedge f > a$, $\forall a \in A$, or in other words, $c \in K$. Claim 3 is proved.

Now we can finish the proof of Lemma 2.3 as follows. Observe that $a < k$, $\forall a \in A$, $\forall k \in K$. This implies that $A \cup K$ is a linearly decomposed sublattice. By the assumption that L is not linearly decomposable, it implies that $A \cup K \neq L$. Furthermore, since A is contractible, K is convex and K absorbs the squares, $A \cup K$ is a contractible sublattice. This contradicts the maximality of A. Hence A' is convex. Summing up, A' is a contractible sublattice.

The proof of Lemma 2.3 is now complete.

Remark. Lemmas 2.2 and 2.3 are not true for the linearly decomposable lattices. Indeed, consider the lattices L and L' in Fig. 3.

Fig. 3

The lattice L is linearly decomposed into A, B, C such that $a < b < c$, $\forall a \in A$, $\forall b \in B$, $\forall c \in C$. If we assume that A, C are not linearly decomposable then $A \cup B$, $B \cup C$ are the maximal contractible sublattices in L, whose intersection is non-empty.

On the other hand, L' consists of the same sublattices A, B, C as in L, which form a linear decomposition of L' satisfying the condition $a < c < b$, $\forall a \in A$, $\forall c < C$, $\forall b \in B$. Using the identities id_A, id_B, id_C as lattice isomorphisms on A, B, C, respectively, we construct a square preserving bijection $\varphi : L \to L'$ such that $\varphi\big|_A = id_A$, $\varphi\big|_B = id_B$, $\varphi\big|_C = id_C$. If C is not linearly decomposable, then $A \cup B$ is a maximal contractible sublattice of L, but sublattice $\varphi(A \cup B)$ is not contractible in L'.

Now, let L be a lattice which has contractible sublattices. If we denote by $\mathbf{C}$ the family of all contractible sublattices of L, then $\mathbf{C}$ is partially ordered with the inclusion relation $\subseteq$. Suppose that $\{C_i \big| i \in I\}$ is a chain in $\mathbf{C}$, it is easy to check that $C = \bigcup\limits_{i \in I} C_i$ is a sublattice of L satisfying (a), (b) of Definition I. But in general C is not contractible, since it is not always proper. Consider the lattice L in Fig. 4. We observe that $A_n = [a_n, b_n]$, $n \in \mathbf{N}$ (natural numbers) are contractible sublattices of L and $\bigcup_{n\in\mathbf{N}} A_n = L$.

Fig. 4

We say that *condition (M) holds for a lattice* L if every contractible sublattice of L is included in a maximal one. By Zorn's Lemma, this is equivalent to the fact that if $\{C_i \big| i \in I\}$ is a chain in $\mathbf{C}$, then $\bigcup_{i\in I} C_i \in \mathbf{C}$.

In what follows, we consider only the lattices which are not linearly decomposable and satisfy condition (M).

Let $\{A_i \big| i \in I\}$ be the family of all maximal contractible sublattices of L. According to Lemma 2.2, $A_i \cap A_j = \emptyset$, $\forall i, j \in I$, $i \neq j$. This allows us to define an equivalence ρ on L, whose equivalence classes are the sets A_i, $i \in I$, and the one-elements sets $\{x\}$, $x \in L \setminus \bigcup_{i\in I} A_i$. The equivalence relation ρ is said to be *induced by the family* $\{A_i \big| i \in I\}$.

Lemma 2.4. *The equivalence ρ is a congruence.*

Proof. Let (a, a'), $(b, b') \in \rho$. We have to prove that $(a \wedge b, a' \wedge b') \in \rho$ and $(a \vee b, a' \vee b') \in \rho$. When $a = a'$, $b = b'$ or $a, b, a', b' \in A_i$ for some $i \in I$, it is trivial. For the remaining cases, it is sufficient to examine only the case where $a \neq a'$, $b \neq b'$ and $a, a' \in A_i$, $b, b' \in A_j$, for some $i, j \in I$, $i \neq j$. Put $c = a \wedge b$ and $c' = a' \wedge b'$.

If $c \in A_i$ then also $b \in A_i$ since A_i absorbs the squares. This is impossible, because $A_i \cap A_j = \emptyset$. Hence $c \notin A_i$. Since $c < a$, by (P_1) we get $c < a'$.

Analogously, we have $c < b'$. Thus $c \leq a' \wedge b' = c'$. By the symmetrical role of c and c' we also have $c' \leq c$ and hence, $c = c'$, i.e. $(c, c') \in \rho$.

By duality we can show that $(a \vee b, a' \vee b') \in \rho$ and the proof is complete.

Remark. Lemma 2.4 is valid for an arbitrary family of contractible sublattices $\{A_i \big| i \in I\}$ such that $A_i \cap A_j = \emptyset$, $\forall i, j \in I$, $i \neq j$. Here the maximality of A_i, $i \in I$ is necessary for obtaining the quotient lattice L/ρ having no contractible sublattices.

Before proving the main theorem we recall a theorem of N. D. Filippov [4], which states that:

(F) Let L, L' be arbitrary lattices. Then $\mathrm{Sub}(L) \cong \mathrm{Sub}(L')$ if and only if there exists a square preserving bijection $\varphi : L \to L'$.

For brevity we say that *condition (G) holds for a lattice* L if L is determined by $\mathrm{Sub}(L)$ up to an isomorphism, that is, if $\mathrm{Sub}(L) \cong \mathrm{Sub}(L')$ for some lattice L' then $L \cong L'$.

Thus, according to (F) whenever the lattice L satisfies (G) then every square preserving bijection $\varphi : L \to L'$ induces an isomorphism $f : L \to L'$.

Now, we are ready to state the main result:

Theorem 2.5. *Let L be a lattice having no linear decomposition and satisfying condition (M). Let $\{A_i \big| i \in I\}$ be the family of all maximal contractible sublattices of L. If A_i satisfies (G) for every $i \in I$, then L is determined by Sub(L) up to an isomorphism or a dual isomorphism.*

Proof. Assume that $\mathrm{Sub}(L) \cong \mathrm{Sub}(L')$ for some lattice L'. We have to prove that $L \cong L'$ or $L \overset{*}{\cong} L'$ (dually isomorphic).

According to (F) there exists a square preserving bijection $\varphi : L \to L'$. Consider A_i for some fixed index $i \in I$. Put $\varphi(A_i) = B_i$. By Lemma 2.3, B_i is a contractible sublattice of L'. Denote by $\varphi_i : A_i \to B_i$ the restriction of φ on A_i. Note that φ_i is also a square preserving bijection. Since A_i satisfies (G), by virtue of (F), φ_i induces an isomorphism $f_i : A_i \to B_i$.

On the other hand, taking the dual mapping $d_i : B_i \to B_i^*$ ($d_i(x) = x$ and $x < y \Leftrightarrow d_i(x) > d_i(y)$, $\forall x, y \in B_i$) we have a square preserving bijection $d_i \circ f_i$: $A_i \to B_i^*$, which determines an isomorphism $h_i : A_i \to B_i^*$ (by virtue of (F)). Let $d_i^{-1} : B_i^* \to B_i$ be the dual isomorphism of d_i. Set $g_i = d_i^{-1} \circ h_i$: $A_i \to B_i$. Clearly, g_i is a dual isomorphism.

Further, applying Lemmas 2.4 to the family $\{A_i | i \in I\}$, we obtain a congruence ρ on L. Since A_i, $i \in I$ are maximal, the quotient lattice L/ρ of L has no contractible sublattice. Since φ is a bijection, we have $B_i \cap B_j = \emptyset$, for all $i, j \in I$, $i \neq j$. Again by Lemma 2.4, $\{B_i | i \in I\}$

defines a congruence ρ' on L'. So we have the quotient lattice L'/ρ' of L'. Obviously φ induces a square preserving bijection $\overline{\varphi} : L/\rho \to L'/\rho'$. Since L/ρ has no contractible sublattice, by Proposition III, $\overline{\varphi}$ is either an isomorphism or a dual isomorphism.

To finish the proof, we consider two cases:

a) If $\overline{\varphi}$ is an isomorphism, then based on $\overline{\varphi}$ and the family of isomorphisms $\{f_i \big| i \in I\}$ we can establish an isomorphism $f : L \to L'$ as follows:

1) $a \in L \setminus \bigcup\limits_{i \in I} A_i,\ \ \overline{\varphi}(\{a\}) = \{b\} \Rightarrow f(a) = b.$

2) $a \in A_i \Rightarrow f(a) = f_i(a),\ \ \forall i \in I.$

b) If $\overline{\varphi}$ is a dual isomorphism, then based on $\overline{\varphi}$ and the family of dual isomorphisms $\{g_i \big| i \in I\}$ we define a dual isomorphism $g : L \to L'$ as follows:

1) $a \in L \setminus \bigcup\limits_{i \in I} A_i,\ \ \overline{\varphi}(\{a\}) = \{b\} \Rightarrow g(a) = b.$

2) $a \in A_i \Rightarrow g(a) = g_i(a), \quad \forall i \in I.$

The theorem is proved.

Examples. We give now two examples of lattices which satisfy Theorem 2.5.

Fig. 5

The maximal contractible sublattices $A_1, \ldots, A_4$ of L satisfy (G) and determine L/ρ, while the maximal contractible sublattices B_1, B_2 of L_1 determine L_1/ρ_1.

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