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# P-ADIC HYPERBOLICITY OF THE COMPLEMENT OF HYPERPLANES IN $P^n(C_p)$

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Abstract. An algorithm in terms of linear algebra to determine the Brody hyperbolicity of $P^n(C_p) - |\mathcal{H}|$ is given, where $\mathcal{H}$ is a set of hyperplanes in $P^n(C_p)$ which are not necessary in general position.

## 1. Introduction

A complex space $X$ is called *Brody hyperbolic* if every holomorphic map $f : C \to X$ is constant. Recent studies suggest that the hyperbolicity of a complex space $X$ is related to the finiteness of the number of rational or integral points of $X$ (see S.Lang's Conjecture in [8]).

For the complex case, the study of hyperbolicity of the complement of hyperplanes has a long history back to Bloch and H. Cartan. Bloch, Dufresnoy, Green, Fujimoto showed that the complement of $2n + 1$ hyperplanes in general position in $P^n(C)$ is hyperbolic. However P. Kiernan [5] proved that the complement of $2n$ hyperplanes in general position in $P^n(C)$ is not hyperbolic.

For the p-adic case, Mai Van Tu [7] found an analogue of Kiernan's result by showing that the complement of $n$ hyperplanes in $P^n(C_p)$ is not hyperbolic. Recall that a variety $X$ is said to be *p-adic Brody hyperbolic* if the only p-adic holomorphic maps $f : C_p \to X$ are the constant maps, where $C_p$ is the completion of algebraic closure of the field $Q_p$ of p-adic numbers (see [3], [4]). In this paper we consider the following general question: Given a set $\mathcal{H}$ of hyperplanes in $P^n(C_p)$ which are not necessary in general position. What is a necessary and sufficient condition for $\mathcal{H}$ such that $P^n(C_p) - |\mathcal{H}|$ is hyperbolic and how do we verify it ? Here $|\mathcal{H}|$ denotes the union of hyperplanes in $\mathcal{H}$. We answer this question by providing an algorithm of linear algebra to determine the p-adic Brody hyperbolicity of $P^n(C_p) - |\mathcal{H}|$. As a consequense, we obtain a p-adic anologue of Bloch,

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Green, Fujimoto's Theorem in [6] and Mai Van Tu's result in [7].

## 2. P-adic Brody hyperbolicity of the complement of hyperplanes in the projective space $P^n(C_p)$

**Definition 2.1.** $m$ hyperplanes of $P^n(C_p)$ are said to be in *general position* if $m \geq n$ and any $n+1$ of these hyperplanes are linearly independent.

Let $f$ be a p-adic holomorphic curve in the projective space $P^n(C_p)$, i.e. a holomorphic map from $C_p$ to $P^n(C_p)$. We identify $f$ with its representation by a collection of holomorphic functions on $C_p$:

$$f = (f_0, f_1, ..., f_n),$$

where the functions $f_i$ have no common zero.

**Definition 2.2.** The holomorphic curve $f$ is said to be *degenerate* if the image of $f$ is contained in some proper subspace of $P^n(C_p)$.

**Theorem 2.3** (p-adic Picard's Theorem). *Every non–constant holomorphic function on $C_p$ is a surjective map onto $C_p$.*

The proof of Theorem 2.3 is easy by using the geometric interpretation of height (see [1], [2], [7]).

**Theorem 2.4** (p-adic analogue of Bloch, Green and Fujimoto's Theorem [6]). *The complement in $P^n(C_p)$ of $(n+1)$ hyperplanes in general position is hyperbolic.*

*Proof.* Let

$$f : C_p \to P^n(C_p)$$

be a holomorphic map with the image lying in the complement of $n+1$ hyperplanes in general position. Then there is a linear change of coordinates such that these hyperplanes are defined by the equations

$$x_0 = 0, \cdots, x_n = 0.$$

Now we can write $f$ in homogeneous coordinates

$$f = (f_0, ..., f_n).$$

By the hypotheses in the theorem, $f_i(z) \neq 0$ for all $i$ and all $z$. By Picard's Theorem, $f_i$ is constant $(i = 0, ..., n)$. So $f = (f_0, ..., f_n)$ is constant.

**Theorem 2.5** (p-adic analogue of Ru's Theorem [8]). *Let*

$$f : C_p \to P^n(C_p)$$

*be a holomorphic map. If the image of $f$ omits at least two distinct hyperplanes in $P^n(C_p)$, then $f$ must be degenerate.*

*Proof.* Let $H_1, ..., H_q, q \geq 2$ be the distinct hyperplanes that the image of $f$ omits. Let $L_1(x), ..., L_q(x)$ denote the linear forms defining the hyperplanes. Then $L_i(f) = L_i o f : C_p \to C_p$ is a holomorphic function on $C_p, i = 1, ..., q$, By the hypotheses of the theorem, $L_i(f)(z) \neq 0$ for all $i = 1, ..., q$ and all $z$. Then, Picard's Theorem implies that $L_i(f)$ is non-zero constant, $i = 1, ..., q$.

First, let $\mathcal{H} = \{H_1, ..., H_q\}$ be a set of hyperplanes in $P^n(C_p)$ which are linearly dependent. By the linearly dependence assumption, there exist non-zero constants $c_i$ such that

$$\sum_{i=1}^{q} c_i L_i(x) \equiv 0.$$

Without loss of generality, we may assume that $q$ is the smallest integer such that we have such a relation (i.e. $c_i \neq 0$ for all $i$). Because the hyperplanes are distinct, we have $q \geq 3$. Since $c_i L_i(f), i = 1, ..., q-1$, are constant, there exist constants $d_i$, not all zero such that

$$\sum_{i=1}^{q-1} d_i c_i L_i(f) = 0 \quad (q \geq 3).$$

Therefore, the image of $f$ is contained in the subspace (hyperplane) defined by the equation

$$\sum_{i=1}^{q-1} (d_i c_i) L_i(x) = 0 \quad (q \geq 3),$$

and this subspace is a proper subspace, since $q$ was chosen to be minimal.

Now, let $\mathcal{H} = \{H_1, ..., H_q\}$ be a set of hyperplanes in $P^n(C_p)$ which are linearly independent. Since $L_1(f), ..., L_q(f)$ are constants, there exist constants $a_i$, not all zero such that

$$\sum_{i=1}^{q} a_i L_i(f) = 0.$$

By the independence assumption, $\sum_{i=1}^{q} a_i L_i(x) \not\equiv 0$. Therefore, the image of $f$ is contained in the proper subspace defined by the equation

$$\sum_{i=1}^{q} a_i L_i(x) = 0.$$

**Definition 2.6.** Let $\mathcal{H}$ be a set of hyperplanes in the projective space $P^n(C_p)$ and let $V$ be a subspace of $P^n(C_p)$. The subspace $V$ is called $\mathcal{H}$–*admissible* if $V$ is not contained in any of the hyperplanes in $\mathcal{H}$. The set $\mathcal{H}$ is called nondegenerate if for every $\mathcal{H}$–admissible subspace $V$ of positive dimension, $\mathcal{H} \cap V$ contains at lest two distinct hyperplanes of $V$.

*Remark.* If $V$ is $\mathcal{H}$–admissible, then $H \cap V$ is a hyperplanes of $V$ for any $H \in \mathcal{H}$.

**Theorem 2.7.** *If $\mathcal{H}$ is a set of hyperplanes in $P^n(C_p)$ and $|\mathcal{H}| = \bigcup_{H \in \mathcal{H}} H$, then $P^n(C_p) - |\mathcal{H}|$ is p-adic Brody hyperbolic if and only if $\mathcal{H}$ is nondegenerate.*

*Proof.* Let $\mathcal{H}$ be a set of hyperplanes of $P^n(C_p)$. We first prove that if $\mathcal{H}$ is nondegenerate, then every holomorphic map $f : C_p \to P^n(C_p) - |\mathcal{H}|$ is constant. Since $\mathcal{H}$ is nondegenerate, $\mathcal{H}$ contains at least two distinct hyperplanes of $P^n(C_p)$ (choose $V = P^n(C_p)$). By Theorem 2.5 the image of $f$ is contained in some proper subspace $W$ of $P^n(C_p)$. Since the image of $f$ omits the hyperplanes in $\mathcal{H}$, $W$ is $\mathcal{H}$–admissible. We consider $\mathcal{H} \cap W$. Since $\mathcal{H}$ is nondegenerate, $\mathcal{H} \cap W$ still contains at least two distinct hyperplanes of $W$. So we can apply Theorem 2.5 again. By induction, eventually we can conclude that the image of $f$ is contained in a 0–dimensional subspace of $P^n(C_p)$. This means $f$ is constant.

Now let $\mathcal{H}$ be a set of distinct hyperplanes in $P^n(C_p)$ which is not degenerate. Because $\mathcal{H}$ is not degenerate there exists a positive dimensional subspace $V$ of $P^n(C_p)$ which is $\mathcal{H}$–admissible, but such that $\mathcal{H} \cap V$ does not contain at least two distinct hyperplanes of $V$. Without loss of generality, we may assume that $V = P^n(C_p)$. Now, $\mathcal{H}$ contain only one hyperplane $H_0$ of $P^n(C_p)$. We may assume that $H_0$ is the hyperplane $x_0 = 0$. Let

$$f : C_p \to P^n(C_p) - |\mathcal{H}|$$

be the holomorphic map: $z \mapsto (1, z, ..., z)$. Then $f$ is not constant, so $P^n(C_p) - |\mathcal{H}|$ is not Brody hyperbolic.

For a set $\mathcal{L}$ of linear forms, we denote by $(\mathcal{L})$ the vector space generated by the linear forms in $\mathcal{L}$ over $C_p$.

**Theorem 2.8** (p-adic analogue of Ru's Theorem [8]). *Let $\mathcal{H}$ be a set of hyperplanes in $P^n(C_p)$. Let $\mathcal{L}$ denote the corresponding set of linear forms defining the hyperplanes in $\mathcal{H}$. The set $\mathcal{H}$ is nondegenerate if and only if*

$$\dim(\mathcal{L}) = n + 1.$$

*Proof.* Since $\dim(\mathcal{L}) = n + 1$, $\mathcal{H}$ contains a subset $\mathcal{H}'$ of $n + 1$ hyperplanes which are linearly independent. Let $\mathcal{H}' = \{H_0, ..., H_n\}$. There is a linear change of coordinates such that the hyperplanes are defined by the equations:

$$x_0 = 0, ..., x_n = 0.$$

We have to show that $\mathcal{H}$ is nondegenerate. Let $V$ be an $\mathcal{H}$–admissible subspace of positive dimension. If $\mathcal{H} \cap V$ contains only one hyperplane of $V$, then

$$H_0 \cap V = \cdots = H_n \cap V.$$

Let $x = (x_0, ..., x_n) \in H_0 \cap V$ be an arbitrary element. Then $x \in H_i \cap V$, which implies $x_i = 0$ for all $i = 0, ..., n$. Hence $x = (0, ..., 0)$. This is a contradiction. So $\mathcal{H}$ is nondegenerate.

Conversely, if $\mathcal{H} = \{H_1, ..., H_q\}$ is nondegenerate, we are going to prove that $\dim(\mathcal{L}) = n + 1$. Assume $\dim(\mathcal{L}) < n + 1$. Denote by $\mathcal{L}^*$ a maximal subset of $\mathcal{L}$ such that all of the linear forms in $\mathcal{L}^*$ are linearly independent. Let $\mathcal{L}^* = \{L_1, ..., L_r\}$. If $r = \dim(\mathcal{L}) = 1$, then $\mathcal{H}$ contains only one hyperplane. So $\mathcal{H}$ is not degenerate and $r > 1$. Since there are only finitely many elements in $\mathcal{L}$ and all of the elements in $\mathcal{L}^*$ are linearly independent, there exist non-zero constants $c_2, ..., c_r$ such that

$$(L_2 - c_2L_1, ..., L_r - c_rL_1) \cap \mathcal{L} = \emptyset.$$

Put $W = (L_2 - c_2L1, ..., L_r - c_rL_1)$ and

$$V = \{x \in P^n(C_p) : L(x) = 0, \text{ for all } L \in W\}.$$

Then $\dim V = n - \dim(W) = n - (\dim(\mathcal{L}^*) - 1) = n + 1 - \dim(\mathcal{L}) > 0$.

Assume that $V$ is not $\mathcal{H}$–admissible. Then $V$ is contained in some hyperplane $H_t \in \mathcal{H}$. Hence $L_t(x) = 0$ for all $x \in V$, where $L_t \in \mathcal{L}$ is the corresponding form to $H_t$. We consider two systems of linear equation

$$\text{(I)} \qquad \begin{cases} L_i - c_iL_1 = 0, \\ i = 2, ..., r, \end{cases}$$

and

$$\text{(II)} \qquad \begin{cases} L_i - c_i L_1 = 0, \\ L_t = 0, \\ i = 2, ..., r, \end{cases}$$

Since two systems (I) and (II) are equivalent, the system (II) is linearly dependent. So we can find $a_i \in C_p, i = 2, ..., r$, such that

$$L_t = \sum_{i=2}^{r} a_i (L_i - c_i L_1).$$

From this it follows that $L_t \in W$, i.e. $W \cap \mathcal{L} \neq \emptyset$. This is a contradiction. Thus $V$ is $\mathcal{H}$–admissible.

Now we are going to prove that $\mathcal{H} \cap V$ contains only one hyperplane of $V$. Indeed, for all $x \in V$ and $i = 2, ..., r$ we have $(L_i - c_i L_1)(x) = 0$. Hence $L_i(x) = c_i L_1(x)$.

Let $H \in \mathcal{H}$ and $L \in \mathcal{L}$ be the coresponding linear form. Since $\mathcal{L}^*$ is maximal, the system $\{L_1, ..., L_r, L\}$ is linearly dependent. Therefore one can find $t_i \in C_p, i = 1, ..., r$ such that $L = \sum_{i=1}^{r} t_i L_i$. For all $x \in V$ we then have:

$$L(x) = \sum_{i=1}^{r} t_i L_i(x) = t_1 L_1(x) + \sum_{i=2}^{r} t_i L_i(x) =$$
$$= t_1 L_1(x) + \sum_{i=2}^{r} (t_i c_i) L_1(x) = t L_1(x),$$

where $t \in C_p$. This means $H \cap V = H_1 \cap V$. Thus the only hyperplane in $\mathcal{H} \cap V$ is the hyperplane $H_1 \cap V$ defined by the linear form $L_1$. Theorem 2.8 is proved.

As immediate consequences of Theorem 2.7 and 2.8 we get again Theorem 2.4 and the following extension of [7]:

**Corollary 2.9.** *The complement of at most $n$ hyperplanes in $P^n(C_p)$ is not hyperbolic.*

**Example.** Let

$$\mathcal{L} = \{x_0, x_1, x_2, x_0 + x_1, x_0 + x_1 + x_2\}$$

be the set of linear forms associated to five distinct hyperplanes in $\mathcal{H}$ in $P^2(C_p)$. One can check that $\mathcal{L}$ satisfies the conditions of Theorem 2.8. Hence $\mathcal{H}$ is nondegenerate, although the hyperplanes in $\mathcal{H}$ are not in general position. Theorem 2.7 then implies that $P^2(C_p)-|\mathcal{H}|$ is hyperbolic.

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