SOME REMARKS ON
RANDOM SERIES IN BANACH SPACES

MARÍA JESÚS CHASCO, NGUYEN DUY TIEN, AND V. I. TARIELADZE

Abstract. The aim of this note is to give: 1) a characterization of Banach spaces of cotype 2; 2) an example related to Rademacher series in the Schatten classes; 3) a characterization of finite dimensional Banach spaces.

Introduction

The study of convergence of random series in Banach spaces plays the key role in probability theory in Banach spaces and has a lot of important applications in functional analysis. Random series of a special form such as Gaussian, stable and Redemacher ones are the most interesting. Convergence of these series, in general, is related to some geometrical property of a Banach space. There are three main results in this note. Theorem 1 of the note is a comparison theorem for convergence of Gaussian and stable series. It gives a characterization of Banach spaces of cotype 2. Theorem 2 concerns convergence of Rademacher series in well known Schatten classes $S_p$, $1 < p < 2$. Finally, we consider some properties of almost $p$-summing operators. Theorem 3 gives a characterization of finite dimensional Banach spaces.

1. Definitions and notation

Throughout this note we make use of the following notation: $X$ denotes a separable real Banach space, $X^*$ denotes its dual space. $(\varepsilon_n)$ denotes a sequence of Bernoulli independent real variables, i.e., a sequence of i.i.d. real random variables with

$$P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = \frac{1}{2}.$$
\((\gamma_n)\) denotes a standard Gaussian sequence, i.e., a sequence of i.i.d. random real variables with characteristic function (c.f.) \(e^{-t^2/2}\).

\((\gamma_n^{(p)})\) denotes a standard \(p\)-stable sequence, i.e., a sequence of i.i.d. random real variables with c.f. \(e^{-|t|^p}\), \(0 < p \leq 2\).

For our purposes we need the following geometric properties of Banach spaces.

**Definition 1.** A Banach space \(X\) is said to be of type \(p\), \(1 \leq p \leq 2\), if for any sequence \((x_n)\) in \(X\) such that

\[\sum_{n=1}^{\infty} \|x_n\|^p < \infty,\]

the random series

\[\sum_{n=1}^{\infty} \varepsilon_n x_n\]

converges a.s. in \(X\).

A Banach space \(X\) is said to be of cotype \(q\), \(2 \leq q < \infty\), if for any sequence \((x_n)\) in \(X\) such that the random series (1.1) converges a.s. in \(X\), we have

\[\sum_{n=1}^{\infty} \|x_n\|^q < \infty.\]

It is well-known that in this definition one can replace the Bernoulli sequence \((\varepsilon_n)\) by the standard Gaussian sequence \((\gamma_n)\). It is also known that the space \(L_r\), \(1 \leq r < \infty\), is of type \(p = \min(2, r)\) and of cotype \(q = \max(2, r)\). In particular, any Hilbert space is of type 2 and of cotype 2.

We refer the reader to [2, 3, 7, 16] for more information on probability in Banach spaces.

### 2. A characterization of Banach spaces of cotype 2

In [10] there is the following result:

**Proposition 1.** Let \((x_n)\) be a sequence of elements in \(X\) and \(r, p, q\) real numbers with \(0 < r < p < 2\), \(1/r = 1/p + 1/q\). If \((b_n) \in l_q\) and the random series

\[\sum_n \gamma_n^{(p)} x_n\]

...
converges a.s. in $\mathbf{X}$, then so does the random series

$$\sum_n b_n \gamma_n^{(r)} x_n.$$ 

It is natural to raise the problem: does hold true the preceding proposition for $p = 2$? Let us begin with an example which says that, in general, the statement in the above proposition is not valid for $p = 2$. Indeed, consider, for instance, $\mathbf{X} = l_4$, $r = 1$, $p = 2$, $(a_n = 1/\sqrt{n})$, $(x_n = a_n e_n)$, where $(e_n)$ is the natural basis in $l_4$. In this case $q = 2$ and the random series

$$\sum_n \gamma_n a_n e_n$$

converges a.s. in $l_4$ as $(a_n) \in l_4$ (according to Vakhania’s Theorem, see [16, Theorem 5.6, p. 334]). On the other hand, if the random series

$$\sum_n \gamma_n^{(1)} a_n b_n e_n$$

converges a.s. in $l_4$ for any $(b_n) \in l_2$, we have:

$$\sum_n |a_n b_n| < \infty, \forall (b_n) \in l_2,$$

since any Banach space is of 1-stable cotype (see [8]). This implies that $(a_n) \in l_2$ which is impossible.

The full answer to the above problem is:

**Theorem 1.** Let $\mathbf{X}$ be a Banach space and $r$ a number with $1 \leq r < 2$. The following statements are equivalent:

(a) $\mathbf{X}$ is of cotype 2.

(b) The random series

$$\sum_n b_n \gamma_n^{(r)} x_n$$

converges a.s. in $\mathbf{X}$ for all $(b_n) \in l_q$ with $1/r = 1/2 + 1/q$ and any sequence $(x_n)$ of $\mathbf{X}$ such that

$$\sum_n \gamma_n x_n$$

converges a.s. in $\mathbf{X}$. 

To prove this theorem we need the following result of Mushtari [9]. A probability measure \( \mu \) in \( X \) is called Gaussian if every \( x^* \in X^* \) (considered as a random variable from the probability space \((X, \mu)\) into the real line) has Gaussian distribution. It is well-known that a symmetrical probability measure \( \mu \) in \( X \) is Gaussian if and only if there is a sequence \((y_n)\) in \( X \) such that \( \mu \) is the distribution of the random series

\[
\sum_n \gamma_n y_n
\]

which is a.s. convergent in \( X \). In this case we have

\[
\hat{\mu}(x^*) = \int_X \exp(i \langle x^*, x \rangle) d\mu(x) = \exp \left(-\frac{1}{2} \sum_n |\langle x^*, y_n \rangle|^2 \right).
\]

Let \( \tau_\mu \) denote the topology generated by the seminorm

\[
\|x^*\|_\mu = \left( \sum_n |\langle x^*, y_n \rangle|^2 \right)^{1/2}.
\]

By [9, Theorem 1] we have:

**Lemma 1.** Let \( X \) be a Banach space of cotype 2 and \( \mu \) a Gaussian measure in \( X \). If \((x_n)\) is a sequence in \( X \) and \( r \) is a real number with \( 1 \leq r < 2 \) such that

\[
\sum_n |\langle x^*, x_n \rangle|^r < \infty, \quad \forall x^* \in X^*,
\]

and the function

\[
\sum_n |\langle x^*, x_n \rangle|^r
\]

is continuous in the topology \( \tau_\mu \), then there is a probability measure \( \nu \) in \( X \) such that

\[
\hat{\nu}(x^*) = \int_X \exp(i \langle x^*, x \rangle) d\nu(x) = \exp(-\sum_n |\langle x^*, x_n \rangle|^r).
\]

**Proof of Theorem 1.**

(a) \( \Rightarrow \) (b). Let \( X \) be a Banach space of cotype 2. We must prove that the random series

(2.1) \[
\sum_n b_n \gamma_n^{(r)} x_n
\]
converges a.s. in $X$ for all $(b_n) \in l_q$ with $1/r = 1/2 + 1/q$ and $(x_n \in X)$ such that

$$\sum_n \gamma_n x_n$$

(2.2) converges a.s. in $X$. Indeed, the a.s. convergence of series (2.2) implies that $(\langle x^*, x_n \rangle) \in l_2$. On the other hand, by the Hölder inequality, we have:

$$\left( \sum_n |\langle x^*, b_n x_n \rangle|^r \right)^{1/r} \leq \left( \sum_n |b_n|^q \right)^{1/q} \left( \sum_n |\langle x^*, x_n \rangle|^2 \right)^{1/2}.$$

This shows that the function

$$\sum_n |\langle x^*, b_n x_n \rangle|^r$$

is continuous in the topology $\tau_\mu$ where $\mu$ is the distribution of the a.s. convergent series (2.2). Clearly, $\mu$ is Gaussian. By Lemma 1, there is a probability measure $\nu$ in $X$ such that

$$\hat{\nu}(x^*) = \int_X \exp(i\langle x^*, x \rangle) d\nu(x) = \exp \left( - \sum_n |\langle x^*, b_n x_n \rangle|^r \right).$$

Consequently, series (2.1) converges a.s. in $X$, by the Ito-Nisio theorem (see [4]),

(a) $\implies$ (b). Let $(x_n)$ be a sequence in $X$ such that the series (2.2) converges a.s. in $X$. To prove that $X$ is of cotype 2 we must show that $(\|x_n\|) \in l_2$. As a matter of fact, by (b), we obtain that the random series

$$\sum_n b_n \gamma_n^{(r)} x_n$$

converges a.s. in $X$ for all $(b_n) \in l_q$ with $1/r = 1/2 + 1/q$. It is well known that any Banach space is of $r$-stable cotype (see [8]), so that we have

$$\sum_n |b_n|^r \|x_n\|^r < \infty$$

for all $(b_n) \in l_q$, $1/r = 1/2 + 1/q$. Therefore, by Landau’s theorem *, this implies that $(\|x_n\|) \in l_2$. The proof of Theorem 1 is complete.

* See G. H. Hardy, J. E. Littlewood, G. Polya, Inequalities, Cambridge Univ. Press, 1988, p. 120.
3. Rademacher series in the Schatten classes

Let \((r_n)\) denote the Rademacher sequence (it is an example of Bernoulli sequences). It was proved in [11] that:

**Proposition 2.** Let \(X\) be a Banach space, \(H\) a separable Hilbert space and \((e_n)\) arbitrary fixed orthonormal basis of \(H\). Consider the following statements:

(a) \(X\) is of a finite cotype (i.e., \(X\) if of cotype \(q\) for some \(q, 2 \leq q < \infty\)).

(b) For a continuous linear operator \(T : H \rightarrow X\) the random series

\[
\sum_n r_n T e_n
\]

converges a.s. in \(X\) if and only if \(T^*\) is absolutely 1-summing.

Then \((b) \implies (a)\) is always true, and if in addition \(X\) is a GL-space, \((a) \implies (b)\) is also true.

We now shall give an example which shows that, in general, \((a) \implies (b)\) is not true if \(X\) is not a GL-space (see Theorem 2 below). To do this let us begin with well-known results on absolutely summing operators (due to Pietsch).

**Definition 2.** Let \(X, Y\) be two Banach spaces and \(T\) a continuous linear operator from \(X\) into \(Y\). \(T\) is said to be an absolutely \(p\)-summing operator, \(1 \leq p < \infty\), if \(\sum_n \|T x_n\|^p < \infty\) for any sequence \((x_n)\) in \(X\) such that \(\sum_n |\langle x_n, x^* \rangle|^p < \infty\), for all \(x^* \in X^*\).

The following is a corollary of Pietsch’s domination theorem (see [12, 17.3.7, p. 234], and also [2, p. 44]).

**Lemma 2.** \(T \in \Pi_2(l_2, Y)\) if and only if \(T\) factors through \(l_2\) in the form: \(T = BA\) where \(A\) is a Hilbert-Schmidt operator from \(l_2\) into \(l_2\) and \(B\) is a linear continuous operator from \(l_2\) into \(Y\).

The following is (probably) known and we give the proof for the sake of completeness.

**Lemma 3.** Let \(T : l_2 \rightarrow l_2\). Then \(T\) is a Hilbert-Schmidt operator if and only if \(T\) factors through \(l_1\).

*Proof.* It is known that \(T\) is a Hilbert-Schmidt operator if and only if for all \(h \in l_2\) we have

\[
Th = \sum_n \lambda_n \langle h, g_n \rangle f_n
\]
where \((g_n)\) and \((f_n)\) are orthonormal bases in \(l_2\), \((\lambda_n)\) is a sequence of real numbers with \(0 \leq \lambda_{n+1} \leq \lambda_n\) and
\[
\sum_n |\lambda_n|^2 < \infty.
\]

Put
\[
A : l_2 \longrightarrow l_1
\]
\[
Ah = \left(\lambda_n \langle h, g_n \rangle\right),
\]
\[
B : l_1 \longrightarrow l_2
\]
\[
B(a_n) = \sum_n a_n f_n, \quad \forall (a_n) \in l_1.
\]

Evidently, \(T = BA\). The converse follows from the Grothendieck theorem which says that every continuous linear operator from \(l_1\) into \(l_2\) is absolutely 1-summing (see [2, p. 15]).

The following notion is taken from [13] (see also [2, p. 154 and 350]).

**Definition 3.** A Banach space \(X\) is said to be a GL-space if each absolutely 1-summing operator \(A\) from \(X\) into \(l_2\) is absolutely 1-factorable, i.e. \(A = CB\) where \(B\) is a continuous linear operator from \(X\) into some \(L_1\) and \(C\) is a continuous linear operator form \(L_1\) into \(l_2\).

It is known that any Banach space with local unconditional structure is a GL-space (see [2]). In particular, Banach spaces with Schauder unconditional basis, Banach lattices are examples of GL-spaces. It is also shown that a Banach space \(X\) is a GL-space if and only if so is \(X^*\).

We now come to give examples of Banach spaces which are not GL-spaces. Let \(S_p, 1 \leq p < \infty\), denote the Schatten class in \(l_2\). Recall that a continuous linear operator \(A : l_2 \longrightarrow l_2\) belongs to \(S_p\) if and only if it can be represented in the form
\[
Ah = \sum_n \lambda_n \langle h, g_n \rangle f_n,
\]
where \((g_n)\) and \((f_n)\) are orthonormal bases in \(l_2\), \((\lambda_n)\) is a sequence of real numbers with \(0 \leq \lambda_{n+1} \leq \lambda_n\) and
\[
\sum_n |\lambda_n|^p < \infty
\]
(see [2, p. 80]).

Put

$$\sigma_p(A) = \left( \sum_n |\lambda_n|^p \right)^{1/p}.$$  

We need the following properties of $S_p$.

**Lemma 4.** 1) Schatten’s Theorem (see [14] or [2, p. 80-81]): $(S_p, \sigma_p)$ is a Banach space for all $p$, $1 \leq p < \infty$ and the dual space of $S_p$ is $S_q$ with $1/p + 1/q = 1$.

2) Tomczak-Jaegermann’s Theorem (see [15]): $S_p$ is of type $\min(2, p)$ and of cotype $\max(2, p)$.

3) Pisier’s Theorem (see [13] or [2, Theorem 17.24, p. 363]): If $p \neq 2$, $S_p$ is not a GL-space.

We now are going to prove the following:

**Theorem 2.** For $1 < p < 2$ there exists a continuous linear operator $T : l_2 \to S_p$ such that $T^*$ is absolutely 1-summing, but the random series

$$\sum_n r_n T e_n$$

does not converge a.s. in $S_p$, where $(e_n)$ is the natural basis of $l_2$.

**Proof.** Let $p, q$ real numbers with $1 < p < 2$, $2 < q < \infty$ and $1/p + 1/q = 1$.

By the Lemma 4, $S_p$ is of cotype 2 and $S_q$ is not a GL-space for any $1 < p < 2$. Therefore, by the definition of GL-spaces, there is an operator $T : l_2 \to S_p$ such that $T^* \in \Pi_1(S_q, l_2)$, but $T^*$ can not factors through $l_1$. It is known that the random series

$$\sum_n r_n T e_n$$

converges a.s. in $S_p$ if and only if $T \in \Pi_2(l_2, S_p)$, since $S_p$ is of cotype 2 (see [11] or [2, Proposition 12.29, p. 251]). By Lemma 2, this implies that the operator $T$ can be written in the form: $T = BA$, where $A : l_2 \to l_2$ is a Hilbert-Schmidt operator and $B : l_2 \to S_p$ is a linear continuous operator. Consequently, we have: $T^* = A^*B^*$, where $B^* : S_q \to l_2$ and $A^* : l_2 \to l_2$ is also a Hilbert-Schmidt operator. Therefore, $A^*$ factors through $l_1$ (by Lemma 3), so that $T^*$ factors through $l_1$ as well. Thus, we have got a contradiction. This ends the proof of Theorem 2.

**Remark.** The idea of the above proof is borrowed from [6].
4. A Characterization of Finite Dimensional Banach Spaces

Let us begin with the following

**Definition 4.** Let $X$, $Y$ be Banach spaces and $p$ a number with $1 \leq p < \infty$. A continuous linear operator $T : X \to Y$ is called almost $p$-summing if the random series

$$\sum_n r_n Tx_n$$

converges a.s. in $Y$ for all sequences $(x_n)$ in $X$ with

$$\sum_n |\langle x^*, x_n \rangle|^p < \infty, \quad \forall x^* \in X^*.$$

By the closed graph theorem, it is easy to prove that:

**Lemma 5.** If a continuous linear operator $T : X \to Y$ is almost $p$-summing, then there is a number $K > 0$ such that for all $n = 1, 2, \ldots$ and for all $x_1, x_2, \ldots, x_n \in X$ the following inequality holds true

$$\left( \mathbb{E} \left\| \sum_{k=1}^n r_k Tx_k \right\|^2 \right)^{1/2} \leq K \sup \left\{ \left( \sum_{k=1}^n |\langle x_k, x^* \rangle|^p \right)^{1/p} \parallel x^* \parallel \leq 1 \right\}.$$ 

Note that it was shown in [11] that:

**Proposition 3.** For a Banach space $Y$ the following statements are equivalent:

(a) $Y$ contains no subspace isomorphic to $c_0$;
(b) A continuous linear operator $T$ from $l_2$ into $Y$ is almost 2-summing if and only if there is a number $K > 0$ such that for all $n = 1, 2, \ldots$ and for all $h_1, h_2, \ldots, h_n \in l_2$ the following inequality holds true

$$\mathbb{E} \left\| \sum_{k=1}^n r_k Th_k \right\|^2 \leq K^2 \sup \left\{ \left( \sum_{k=1}^n |\langle h_k, h \rangle|^2 \right)^{1/2} \parallel h \parallel \leq 1 \right\}.$$ 

**Remark.** The identity of $X$ is not almost $p$-summing for any $2 < p < \infty$. In fact, choose $x_k = x$, $k = 1, 2, \ldots$ with $x \in X$, $\parallel x \parallel = 1$. By the inequality in Lemma 5, for all $n = 1, 2, \ldots$ we have

$$n^{1/2} \leq K n^{1/p}.$$
This is impossible for \( p > 2 \).

It was proved in [2, Remark 12.8, p. 235]) that a Banach space \( X \) is finite dimensional if and only if the identity of \( X \) is almost 2-summing. We are now going to study the case \( 1 \leq p \leq 2 \). Consider first the case \( p = 1 \). It can be checked easily that the identity of the space \( c_0 \) is not almost 1-summing. On the other hand, the identity of the space \( l_1 \) is almost 1-summing, by the celebrated Schur theorem (see [2, p. 6]). More generally, we have:

**Proposition 4.** Let \( X \) be a Banach space. The following statements are equivalent:

(a) \( X \) contains no subspace isomorphic to \( c_0 \).

(b) The identity of \( X \) is almost 1-summing.

This proposition is an easy consequence of the following result due to Bessaga-Pelczynsky (see [1] or [2, p. 22]).

**Lemma 6.** Let \( X \) be a Banach space. The following statements are equivalent:

(a) \( X \) contains no subspace isomorphic to \( c_0 \).

(b) The series \( \sum_n x_n \) unconditionally converges for any sequence \( (x_n) \) in \( X \) such that

\[
\sum_n |\langle x^*, x_n \rangle| < \infty, \quad \forall x^* \in X^*.
\]

However, for \( 1 < p \leq 2 \) we have:

**Theorem 3.** Let \( X \) be a Banach space and \( p \) a real number with \( 1 < p \leq 2 \). The following statements are equivalent:

(a) \( \dim X < \infty \).

(b) The identity of \( X \) is almost \( p \)-summing.

To prove this theorem we need the following notion (due to James, see [2, p. 175]).

**Definition 5.** Let \( X, Y \) be Banach spaces. It is said that \( X \) is finitely representable in \( Y \) if given any \( \varepsilon > 0 \) and \( E \) is a finite dimensional subspace of \( X \) there are a finite dimensional subspace \( F \) of \( Y \) and an isomorphism \( u : E \to F \) such that \( \|u\| \cdot \|u^{-1}\| \leq 1 + \varepsilon \).

The following is the heart of the proof of the preceding theorem.

**Lemma 7.** (Dvorezky Theorem [2, p. 396]). The space \( l_2 \) is finitely representable in any Banach space \( X \) with \( \dim(X) = \infty \).
Proof of Theorem 3.

(a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (a) is a consequence of the following lemmas. First, note that, by Lemma 5, we have:

Lemma 8. If the identity of a Banach space $X$ is almost $p$-summing, then there is a number $K > 0$ such that for all $n = 1, 2, \ldots$ and for all $x_1, x_2, \ldots, x_n \in X$ the following inequality holds true

$$
(4.1) \quad \left( E \left\| \sum_{k=1}^{n} r_k x_k \right\|^{2} \right)^{1/2} \leq K \sup \left\{ \left( \sum_{k=1}^{n} |\langle x_k, x^* \rangle|^p \right)^{1/p} \left\| x^* \right\| \leq 1 \right\}.
$$

By definition, it is obvious to see that:

Lemma 9. If the inequality (4.1) holds true for a Banach space $Y$, so does for any Banach space $X$ which is finitely representable in $Y$.

The following is the key to our purpose.

Lemma 10. The inequality (4.1) does not hold true for $l_2$, for any $p$ with $1 < p \leq 2$.

Proof. Let $q$ denote the real number such that $1/p = 1/q + 1/2$. As $1 < p \leq 2$, it easy to check that $2 < q \leq \infty$ ($q = \infty$ iff $p = 2$). Choose $x_n = \frac{1}{\sqrt{n}} e_n$, $n = 1, 2, \ldots$, where $(e_n)$ is the natural basis in $l_2$. If the inequality (4.1) does hold true for $l_2$, then, by the Hölder inequality, for any $n = 1, 2, \ldots$ we have

$$
\sum_{k=1}^{n} \frac{1}{k} = \left( E \left\| \sum_{k=1}^{n} r_k x_k \right\|^{2} \right)^{1/2}
\leq K \sup \left\{ \left( \sum_{k=1}^{n} |\langle x_k, x^* \rangle|^p \right)^{1/p} \left\| x^* \right\| \leq 1 \right\}
= K \sup \left\{ \left( \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < e_k, x^* > |p \right)^{1/p} \left\| x^* \right\| \leq 1 \right\}
\leq K \sup \left\{ \left( \sum_{k=1}^{n} \left( \frac{1}{k} \right)^{q/2} \right)^{1/q} \left( \sum_{k=1}^{n} |\langle e_k, x^* \rangle|^2 \right)^{1/2} \left\| x^* \right\| \leq 1 \right\}
= K \left( \sum_{k=1}^{n} \left( \frac{1}{k} \right)^{q/2} \right)^{1/q}.
$$
This is impossible for $q > 2$, since
\[\sum_{k=1}^{\infty} \frac{1}{k} = \infty\]
and
\[\sum_{k=1}^{\infty} \frac{1}{k^{q/2}} < \infty.\]

By the Dvoretsky theorem and the above lemmas we obtain:

**Lemma 11.** Let $\mathbf{X}$ be a Banach space. If $\dim \mathbf{X} = \infty$, then the inequality (4.1) does not hold true for $\mathbf{X}$, for all $p$ with $1 < p \leq 2$.

The proof of Theorem 3 is complete.

**References**

6. T. Kühn, $\gamma$-summing operators in Banach spaces of type $p$, 1 < $p \leq 2$, and cotype $q$, 2 < $q \leq \infty$, *Theory Probab. Appl.* 26 (1981), 118-129.


Universidad de vigo
departamento de matematica aplicada
36200 Vigo, España
E-mail adress: mjchasco@dma.uvigo.es

Faculty of Mathematics, University of Hanoi
90 Nguyen Trai, Hanoi, Vietnam
E-mail adress: ndtien@it-hu.ac.vn

Muskhelishvili Institute of Computational Mathematics
Georgian Academy of Sciences
Tbilisi-93, Republic of Georgia
E-mail adress: tar@compmath.acnet.ge