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CONVEX VECTOR FUNCTIONS AND THEIR SUBDIFFERENTIAL

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Abstract. The continuity of a convex vector function on relative interior points of its domain is studied. As a corollary of this we can see that a convex vector function is Lipschitz near any relative interior point of its domain. A new concept of subdifferential of a convex function is introduced and some its properties similar to those in the scalar case are shown. The inclusive relations between generalized Jacobian and subdifferential, the convexcity of a vector function and the monotonicity of its subdifferential are also established. Further, some neccessary and sufficient conditions for the existence of efficient solutions of vector optimization problems are also proved.

1. Introduction

Convex analysis is now a standard subject and a must in undergraduate and graduate study. It offers several ideas and methods of thinking to approach nonlinear functions from both theoretical and computational points of view. Convex vector functions have interested several authors, in particular those working in vector optimizations. Some results have been obtained on properties of these functions, their relationship with convex scalar functions. Applications have also been made in obtaining optimality conditions for vector problems (see [1]). However, to our knowledge, very few attention was paid to differential of convex vector functions and their characterizations. The aim of the present paper is to investigate convex vector functions and their subdifferential. We do not intend to make direct extension of results of convex analysis to the vector case. We rather use a new approach of nonsmooth analysis to tackle the problem and by this enlighten the structure of vector functions.

The paper is organized as follows. The next section contains some preliminaries which are needed in the sequel. Section 3 is devoted to continuity properties of convex vector functions. Unlike the scalar case, a

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convex vector function is not necessarily continuous in the interior of its domain. Section 4 deals with subdifferential of convex vector functions. We define the subdifferential in a standard way and compare it with generalized Jacobian introduced by Clarke. Calculus rules are provided for the subdifferential of convex vector functions. The last section is about the monotonicity of the subdifferential introduced above. We show that a vector function is convex if and only if its local convex subdifferential admits a monotone selection.

2. Preliminaries

Suppose that a convex cone $C \subseteq R^m$ specifies a partial order '$\succeq_C$' as follows.

$$x, y \in R^m, \quad x \succeq_C y \quad \text{if } x - y \in C.$$

Sometimes we write $\succeq$ instead of $\succeq_C$ if it is clear which cone is under consideration. Let f be a function from a nonempty convex subset $D \subseteq R^n$ to R^m. We recall that f is said to be convex (or more precisely C-convex) on D if for every $x, y \in D$, $\lambda \in (0, 1)$, one has

$$\lambda f(x) + (1 - \lambda) f(y) \succeq f(\lambda x + (1 - \lambda) y).$$

Denote by C' the positive polar cone of C, i.e.

$$C' := \big\{\xi \in L(R^m, R) : \ \xi(c) \ \geq 0, \ \text{for all } \ c \in C\big\},$$

where $L(R^m, R)$ denotes the space of linear functionals on R^m.

Denote by lC the set $C \cap (-C)$. The cone C is said to be pointed if $lC = \{0\}$.

The following result from [1] will be needed in the sequel.

Lemma 2.1. *If C is closed, then f is convex if and only if the composition $\xi \circ f$ is a scalar convex function on D, for every $\xi \in C'$.*

Lemma 2.2. *If f is convex with respect to C, then it is also convex with respect to any cone larger than C.*

Proof. The proof is straightforward. We omit it. □

3. Continuity

It is known that scalar convex functions are Lipschitz near any relative interior point of their domains. A natural question arises whether this is true for vector functions. We shall show that if the cone C satisfies certain properties, then the answer is affirmative.

Theorem 3.1. *Assume that the closure clC of the cone C is pointed and f is a convex vector function from a convex subset $D \subseteq R^n$ to R^m. Then f is locally Lipschitz on the relative interior of D.*

Proof. From Remark 1.6 and Proposition 1.10, Chapter 1 in [1], it follows that clC has a convex compact base and $\operatorname{int}(clC)' \neq \emptyset$. Then there are m linearly independent vectors $\xi_1, \xi_2, \ldots, \xi_m \in (clC)'$. By Lemma 2.2, f is convex with respect to the cone clC and by Lemma 2.1, $\xi_i \circ f$ is a scalar convex function, for every $i = 1, 2, \ldots, m$. Hence, $\xi_i \circ f$ is locally Lipschitz on riD, for every $i = 1, 2, \ldots, m$. Let $\xi \in L(R^m, R)$ be arbitrary, we can represent ξ as $\xi = \sum_{i=1}^{m} \alpha_i \xi_i$ for some $\alpha_1, \alpha_2, \cdots, \alpha_m \in R$. Then $\xi f = \sum_{i=1}^{m} \alpha_i(\xi_i f)$. Hence $\xi \circ f$ is locally Lipschitz on riD. Since this is true for all $\xi \in L(R^m, R)$ then f is locally Lipschitz on riD. □

It should be noted that if the closure of the cone C is not pointed, then Theorem 3.1 is not valid. To see this, let f_1 be a convex function, f_2 be a discontinuous function from R to R. Then $f = (f_1, f_2)$ is a convex vector function from R to R^2 with respect to the cone $C := \{(x, y) \in R^2 : x \geq 0\}$. Of course this function is not continuous. Hence, it is not locally Lipschitz on R. Observe that $clC = \{(x, y) : \ x \geq 0\}$ is not pointed.

4. Subdifferential

Throughout this section we assume that $C \subseteq R^m$ is a closed pointed convex cone. Let f be a convex function from a convex subset $D \subseteq R^n$ to R^m and $x_0 \in D$. We define the subdifferential of f at x_0 as the set

$$\partial f(x_0) := \{A \in L(R^n, R^m) : \ f(x) - f(x_0) \succeq A(x - x_0), \text{ for all } x \in D\},$$

where $L(R^n, R^m)$ denotes the space of linear continuous mappings from R^n into R^m which is also considered as the space of $(m \times n)$-matrices.

First, we consider a relation between the generalized Jacobian and the subdifferential of the convex vector function f. Assume that $\operatorname{int} D \neq \emptyset$. Let $x_0 \in \operatorname{int} D$. By Theorem 3.1, f is Lipschitz near x_0. By Rademacher's Theorem f is differentiable almost everywhere on some neighborhood of x_0. The generalized Jacobian $Jf(x_0)$ of f at x_0 in Clarke's sense [2] is defined as the convex hull of all $(m \times n)$ matrices obtained as the limits of sequences of the form $\big(Df(x_i)\big)_i$, where $\big(x_i\big)_i$ converges to x_0 and the classical Jacobian matrix $Df(x_i)$ of f at x_i exists. It is known that for $m = 1$ one always has equality $\partial f(x) = Jf(x)$, $x \in \operatorname{int} D$. For $m > 1$ this is not true in general. However, the inclusion $Jf(x) \subseteq \partial f(x)$, $x \in \operatorname{int} D$ is still valid.

Lemma 4.1. *For every $x \in D$, $\partial f(x)$ is a closed convex set.*

Proof. From the definition of subdifferential we have immediately the convexity of $\partial f(x)$. Now, we verify the closedness of $\partial f(x)$. Assume that a sequence $\big(A_i\big)_i \subseteq \partial f(x)$ converges to some $A \in L(R^n, R^m)$. For every $y \in D$, one has

$$f(y) - f(x) - A_i(y - x) \in C.$$

Taking $i \to \infty$, by the closedness of C, we have

$$f(y) - f(x) - A(y - x) \in C.$$

Hence, $A \in \partial f(x)$. Thus, $\partial f(x)$ is closed. □

Lemma 4.2. *If f is differentiable at $x \in \operatorname{int} D$, then $Df(x) \in \partial f(x)$.*

Proof. Since f is differentiable at x, then $\xi \circ f$ is also differentiable at x, for every $\xi \in C'$. By Theorem 25.1 in [3], $\xi \circ Df(x) = D(\xi \circ f)(x) \in \partial(\xi \circ f)(x)$. By the definition of subdifferentials, for every $\xi \in C'$, $y \in D$, we have

$$(\xi \circ f)(y) - (\xi \circ f)(x) - (\xi \circ Df(x))(y - x) \geq 0,$$

or

$$\xi\left(f(y) - f(x) - Df(x)(y - x)\right) \geq 0.$$

Hence,

$$f(y) - f(x) - Df(x)(y - x) \succeq 0.$$

Thus, $Df(x) \in \partial f(x)$. □

Lemma 4.3. *The set-valued map ∂f from D into $L(R^n, R^m)$ is closed at any point $x \in D$ at which f is continuous.*

Proof. Assume f is continuous at $x \in D$. Let $(x_i, A_i)_{i \in N}$ be a sequence which converges to (x, A), for some $A \in L(R^n, R^m)$, where $A_i \in \partial f(x_i)$. For every $y \in D$, one has

$$f(y) - f(x_i) - A_i(y - x_i) \in C.$$

Taking $i \to \infty$, since f is continuous at x and since C is closed, we have

$$f(y) - f(x) - A(y - x) \in C.$$

It follows that $A \in \partial f(x)$. This completes the proof of the lemma. □

Theorem 4.4. *Let f be a convex function from a convex subset $D \subseteq R^n$ to R^m with $\operatorname{int} D \neq \emptyset$. Then $Jf(x) \subseteq \partial f(x)$, for all $x \in \operatorname{int} D$.*

Proof. Let A be the limit of a sequence of the form $(Df(x_i))_{i\in N}$, where $(x_i)_{i\in N}$ converges to x and the classical Jacobian matrix $Df(x_i)$ of f at x_i exists. By Lemma 4.2, $Df(x_i) \in \partial f(x_i)$. By Theorem 3.1, f is continuous at x. From Lemma 4.3 one has $A \in \partial f(x)$. Since $\partial f(x)$ is convex, $Jf(x) \subseteq \partial f(x)$. □

It should be noted that in general the inclusion of Theorem 4.4 is strict. For instance, let $f_1(x) = f_2(x) = |x|, x \in R$. The vector function $f = (f_1, f_2)$ is convex with respect to the cone R^2_+. It is not difficult to see that

$$\partial f(0) = \partial f_1(0) \times \partial f_2(0) = [-1, 1] \times [-1, 1]$$

and

$$Jf(0) = [(-1, -1), (1, 1)],$$

where $[(-1, -1), (1, 1)]$ denotes the line segment $\{\lambda(-1, -1)+(1-\lambda)(1, 1) : \lambda \in [0, 1]\} \subset R^2$. Hence, $Jf(0) \neq \partial f(0)$.

Let $\mathcal{A} \subseteq L(R^n, R^m)$, $\xi \in L(R^m, R)$ be arbitrary. Let us denote by $\xi\mathcal{A}$ the set $\{\xi \circ A : \ A \in \mathcal{A}\}$.

Corollary 4.5. *Let f be a convex function from a convex subset $D \subseteq R^n$ to R^m with $\operatorname{int} D \neq \emptyset$. If $x \in \operatorname{int} D$, then for every $\xi \in C'$, $\xi Jf(x) = \xi\partial f(x)$ (i.e. the projections of $\partial f(x)$ and $Jf(x)$ on every direction $\xi \in C'$ coincide).*

Proof. Let $x \in \operatorname{int} D$. By Theorem 2.6.6 in [2], $\xi Jf(x) = J(\xi \circ f)(x)$, for all $\xi \in C'$. Since $\xi \circ f$ is scalar convex then $J(\xi \circ f)(x) = \partial(\xi \circ f)(x)$. From the definition of subdifferentials we have $\partial(\xi \circ f)(x) \supseteq \xi\partial f(x)$. Hence, $\xi Jf(x) \supseteq \xi\partial f(x)$. The converse inclusion is immediate from Theorem 4.4. The proof is complete. □

Now, we shall present some calculus rules for subdifferentials of convex vector functions.

Theorem 4.6. *Let f be a convex function from a convex subset $D \subseteq R^n$ to R^m, $x \in D$ and $\xi \in C'$. If one of the following conditions holds*

i) $\operatorname{int} D \neq \emptyset, \ x \in \operatorname{int} D,$

ii) $\operatorname{int} D = \emptyset, \ x \in riD, \ \xi \neq 0,$

then

$$\partial(\xi \circ f)(x) = \xi\partial f(x). \tag{1}$$

Proof. i) Let $x \in \operatorname{int} D$. For every $\xi \in C'$, since $\xi \circ f$ is a scalar convex function, we have

$$\partial(\xi \circ f)(x) = J(\xi \circ f)(x).$$

By Theorem 2.6.6 in [2],

$$J(\xi \circ f)(x) = \xi J f(x).$$

By Corollary 4.5,

$$\xi J f(x) = \xi \partial f(x).$$

Hence,

$$\partial(\xi \circ f)(x) = \xi \partial f(x).$$

ii) By a translation we may assume $0 \in D$. Without loss of the generality we may also assume that the subspace generated by D is R^k for some $k < n$. Define a function $\bar{f} : D \subseteq R^k \to R^m$ as follows

$$\bar{f}(y) = f(y),$$

for all $y \in D$. Obviously, $\bar{f}$ is convex. Since x is a relative interior point of D in R^n, then x is an interior point of D in the subspace R^k. Let $\xi \in C' \setminus \{0\}$ and $A \in \partial(\xi \circ f)(x)$. Denote by A' the restriction of A to R^k, then $A' \in \partial(\xi \circ \bar{f})(x)$. By i), one has

$$\partial(\xi \circ \bar{f})(x) = \xi \partial \bar{f}(x).$$

Then there is some $B' \in \partial \bar{f}(x)$ such that $A' = \xi \circ B'$. Let $\{e_1, e_2, \cdots, e_k\}$ be a base of R^k and $\{e_1, \cdots, e_k, e_{k+1}, \cdots, e_n\}$ be a base of R^n. Since $\xi \neq 0$, then for every $i = k+1, \cdots, n$, there exists a vector $y_i \in R^m$ such that $\xi(y_i) = A(e_i)$. Define a linear map $B : R^n \to R^m$ as follows

$$B(e_i) = \begin{cases} B'(e_i) & i = 1, 2, \cdots, k, \\ y_i & i = k+1, \cdots, n. \end{cases}$$

Then $B \in \partial f(x)$ and $A = \xi \circ B \in \xi \partial f(x)$. Hence, $\partial(\xi \circ f)(x) \subseteq \xi \partial f(x)$. The converse inclusion is trivial. This completes the proof of the theorem. □

It can be easily seen that if $\operatorname{int} D \neq \emptyset$ and $x \notin \operatorname{int} D$, then in general, (1) does not hold. For instance, let f be any convex function from [0,1]$\subseteq R$

to R. Set $\xi = 0$, $x = 0$. Then $\partial(\xi \circ f)(x) = (-\infty, 0]$ while $\xi\partial f(x) \subseteq \{0\}$. for $x = 0 \notin \operatorname{int}[0,1]$.

If $\operatorname{int} D = \emptyset$, $x \in riD$ and $\xi = 0$, then in general, (1) also does not hold. For instance, set $D = [(-1,0),(1,0)] \subseteq R^2$. Consider the function

$$f : x \in D \subseteq R^2 \to 0 \in R.$$

Let $\xi = 0$, we have $\xi\partial f(0) = \{0\}$ and $\partial(\xi \circ f)(0) = \{(0,t) : \ t \in R\}$. Hence, $\partial(\xi \circ f)(0) \neq \xi\partial f(0)$.

Now, let g be another convex function from another convex subset $D' \subseteq R^n$ to R^m. We shall establish the subdifferential of the sum $f + g$.

Theorem 4.7. *For every $x \in D \cap D'$ we have*

$$\partial(f+g)(x) \supseteq \partial f(x) + \partial g(x).$$

If the cone C is generated by linearly independent vectors and $riD \cap riD' \neq \emptyset$, then equality holds for every $x \in D \cap D'$.

We need four following lemmata.

Lemma 4.8. *Let X be a convex subset of R^m with nonempty interior and let $h : X \subseteq R^n \to R$ be affine. Then h can be affinely extended on the whole space.*

Proof. By a translation we may assume $0 \in \operatorname{int} X$ and $h(0) = 0$. Then $x \in X$ if $\|x\| = \varepsilon$ for some $\varepsilon > 0$. Define the function $H : R^n \to R$ by

$$H(x) = \begin{cases} \dfrac{\|x\|}{\varepsilon} h\Big(\dfrac{\varepsilon x}{\|x\|}\Big) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

A direct verification shows that H is affine. Moreover, $H(x) = h(x)$, for all $x \in X$. This completes the proof of the lemma. □

Let X be a subset of R^m. Let us denote by $\langle X \rangle$ the subspace generated by X. Let $x \in \langle X \rangle$ and let ε be a positive number. Denote by $\bar{B}_{\langle X \rangle}(x,\varepsilon)$ the ball $\{y \in \langle X \rangle : \ \|y - x\| \leq \varepsilon\}$.

Lemma 4.9. *Let X, Y be subsets of R^m. If $riX \cap riY \neq \emptyset$ and $0 \in X \cap Y$, then*

$$\langle X \rangle > \cap \langle Y \rangle = \langle X \cap Y \rangle.$$

Proof. First, we consider the case $0 \in riX \cap riY$. We can find a positive number ε such that $\bar{B}_{\langle X \rangle}(0,\varepsilon) \subseteq X$, $\ \bar{B}_{\langle Y \rangle}(0,\varepsilon) \subseteq Y$. Then $\bar{B}_{\langle X \rangle}(0,\varepsilon) \cap$

$\bar{B}_{\langle Y\rangle}(0,\varepsilon) \subseteq X \cap Y$. Let $z \in \langle X\rangle \cap \langle Y\rangle$ with $z \neq 0$. Then $\dfrac{\varepsilon z}{\|z\|} \in \bar{B}_{\langle X\rangle}(0,\varepsilon) \cap \bar{B}_{\langle Y\rangle}(0,\varepsilon)$. Hence, $z \in \langle X \cap Y\rangle$. Thus, $\langle X\rangle \cap \langle Y\rangle \subseteq \langle X \cap Y\rangle$. The converse inclusion is obvious.

Now, assume $riX \cap riY \neq \emptyset$ and $0 \in X \cap Y$. Let $x_0 \in riX \cap riY$. Then $0 \in ri(X - x_0) \cap ri(Y - x_0)$. From the proof above we have

$$\langle (X - x_0)\rangle \cap \langle (Y - x_0)\rangle = \langle ((X - x_0) \cap (Y - x_0)\rangle.$$

Since $0 \in X \cap Y$, then $\langle (X - x_0)\rangle = \langle X\rangle$, $\langle (Y - x_0)\rangle = \langle Y\rangle$, $\langle (X - x_0) \cap (Y - x_0)\rangle = \langle X \cap Y - x_0\rangle = \langle X \cap Y\rangle$. Hence,

$$\langle X\rangle \cap \langle Y\rangle = \langle X \cap Y\rangle.$$

This completes the proof of the lemma. □

Lemma 4.10. *Assume that three functions $h : X \subseteq R^n \to R$, $k : Y \subseteq R^n \to R$, $A : R^n \to R$ are given and that*

a) *X, Y are convex , $riX \cap riY \neq \emptyset$,*
b) *$0 \in X \cap Y$, $h(0) = k(0) = 0$,*
c) *h, k are affine on X, Y,*
d) *A is linear and $A(x) = h(x) + k(x)$, for all $x \in X \cap Y$.*

Then h, k have linear extensions H, K on R^n such that $A = H + K$.

Proof. Assume that $\{e_1, e_2, \cdots, e_r\}$, $\{e_1, e_2, \cdots, e_r, u_1, u_2, \cdots, u_s\}$, $\{e_1, e_2, \cdots, e_r, v_1, v_2, \cdots, v_t\}$ are bases of $\langle X \cap Y\rangle$, $\langle X\rangle$, $\langle Y\rangle$, respectively. By Lemma 4.9, we have

$$\dim(\langle X\rangle + \langle Y\rangle) = \dim\langle X\rangle + \dim\langle Y\rangle - \dim\langle X \cap Y\rangle = r + s + t.$$

Hence, $e_1, \cdots, e_r, u_1, \cdots, u_s, v_1, \cdots, v_t$ are linearly independent. Then we can find $w_1, w_2, \cdots, w_q \in R^n$ such that $e_1, \cdots, e_r, u_1, \cdots, u_s, v_1, \cdots, v_t$, $w_1, \cdots, w_q$ is a base of R^n. It follows from Lemma 4.8 and $h(0) = k(0) = 0$ that h, k have linear extensions H_1, K_1 on the subspaces $\langle X\rangle$, $\langle Y\rangle$, respectively. Define the linear maps $H : R^n \to R$, $K : R^n \to R$ as follows

$H(e_i) = H_1(e_i)$,
$H(u_i) = H_1(u_i)$,
$H(v_i) = A(v_i) - K_1(v_i)$,
$H(w_i) = \dfrac{A(w_i)}{2}$ ·
$K(e_i) = K_1(e_i)$,
$K(v_i) = K_1(v_i)$,
$K(u_i) = A(u_i) - H_1(u_i)$,

$K(w_i) = \dfrac{A(w_i)}{2} \cdot$

Then H, K are linear extensions of h, k on R^n and $A = H + K$. This completes the proof of the lemma. □

Now, assume that the cone C is generated by the convex hull of some linearly independent vectors, i.e. $C = \text{cone}(\text{co}\{c_1, c_2, \cdots, c_k\})$, where $c_1, c_2, \cdots, c_k$ are linearly independent vectors in R^m. We can find $m - k$ vectors $c_{k+1}, \cdots, c_m$ such that $\{c_1, c_2, \cdots, c_m\}$ is a base of R^m. For every $x \in R^m$, we can represent $x = \sum\limits_{i=1}^{m} \alpha_i c_i$, for some $\alpha_1, \alpha_2, \cdots, \alpha_m \in R$. It is easily seen that $x \in C$ if and only if

$$\begin{cases} \alpha_1, \alpha_2, \cdots, \alpha_k \geq 0, \\ \alpha_{k+1} = \cdots = \alpha_m = 0. \end{cases}$$

Let $h : D \subseteq R^n \to R^m$. Represent h as

$$h(x) = \sum_{i=1}^{m} h_i(x) c_i,$$

Lemma 4.11. *h is C-convex if and only if $h_1, h_2, \cdots, h_k$ are scalar convex and $h_{k+1}, \cdots, h_m$ are affine.*

Proof. For every $x, y \in D$, $\lambda \in [0, 1]$, one has (h is convex)

$$\begin{aligned}
&\Leftrightarrow h(\lambda x + (1-\lambda)y) \preceq_C \lambda h(x) + (1-\lambda)h(y) \\
&\Leftrightarrow \sum_{i=1}^{m} [\lambda h_i(x) + (1-\lambda)h_i(y) - h_i(\lambda x + (1-\lambda)y)] c_i \in C \\
&\Leftrightarrow \begin{cases} \lambda h_i(x) + (1-\lambda)h_i(y) - h_i(\lambda x + (1-\lambda)y) \geq 0, & i = 1, 2, \cdots, k, \\ \lambda h_i(x) + (1-\lambda)h_i(y) - h_i(\lambda x + (1-\lambda)y) = 0, & i = k+1, \cdots, m. \end{cases} \\
&\Leftrightarrow \begin{cases} h_1, h_2, \cdots, h_k & \text{are convex,} \\ h_{k+1}, \cdots, h_m & \text{are affine.} \end{cases}
\end{aligned}$$

The proof is complete. □

Proof of Theorem 4.7. Let $x \in D \cap D'$ be arbitrary. The inclusion $\partial(f+g)(x) \supseteq \partial f(x) + \partial g(x)$ is obviously true without any additional condition. Now assume that C is generated by linearly independent vectors, i.e. $C = \text{cone}\,(\text{co}\,\{c_1, c_2, \cdots, c_k\})$, where $c_1, c_2, \cdots, c_k$ are linearly independent vectors in R^m. We show the inclusion $\partial(f+g)(x) \subseteq \partial f(x)+\partial g(x)$. Indeed, we can find $m-k$ vectors $c_{k+1}, \cdots, c_m$ such that $\{c_1, c_2, \cdots, c_m\}$ is a base of R^m. For $A \in \partial(f+g)(x)$, we represent A, f, g, $f+g$ as

$$\begin{aligned} A(y) &= \sum_{i=1}^{m} A_i(y)c_i, \\ f(y) &= \sum_{i=1}^{m} f_i(y)c_i, \\ g(y) &= \sum_{i=1}^{m} g_i(y)c_i, \\ (f+g)(y) &= \sum_{i=1}^{m} (f+g)_i(y)c_i. \end{aligned}$$

Obviously, $A_1, A_2, \cdots, A_m$ are linear and $(f+g)_i = f_i+g_i$, $i = 1, 2, \cdots, m$. By Lemma 4.11, $f_1, \cdots, f_k$, $g_1, \cdots, g_k$ are convex and $f_{k+1}, \cdots, f_m$, g_{k+1}, $\cdots, g_m$ are affine. By the definition of subdifferentials, we have

$$(f+g)(y) - (f+g)(x) - A(y-x) \in C,$$

for all $y \in D \cap D'$. It follows that

$$\sum_{i=1}^{m} [(f+g)_i(y) - (f+g)_i(x) - A_i(y-x)]c_i \in C,$$

or

$$(f+g)_i(y) - (f+g)_i(x) - A_i(y-x) \geq 0, \quad i = 1, 2, \cdots, k, \tag{2}$$

$$(f+g)_i(y) - (f+g)_i(x) - A_i(y-x) = 0, \quad i = k+1, \cdots, m. \tag{3}$$

From (2) one has $A_i \in \partial(f_i+g_i)(x)$, $i = 1, 2, \cdots, k$. Then by Theorem 23.8, Chapter 5 in [3], $A_i \in \partial f_i(x) + \partial g_i(x)$, $i = 1, 2, \cdots, k$. Hence for any i we can find two linear maps $F_i \in \partial f_i(x)$, $G_i \in \partial g_i(x)$ such that

$$A_i = F_i + G_i, \quad i = 1, 2, \cdots, k.$$

For every $i = k+1, \cdots, m$, we define the functions $f_i' : D - \{x\} \to R$, $g_i' : D' - \{x\} \to R$ as follows

$$f'_i(z) := f_i(y) - f_i(x), \text{ where } z = y - x, \ y \in D.$$

$$g'_i(z) := g_i(y) - g_i(x), \text{ where } z = y - x, \ y \in D'.$$

Since f_i, g_i are affine on D, D', respectively, then f'_i, g'_i are affine on $D - \{x\}$, $D' - \{x\}$, respectively. It is clear that $f'_i(0) = g'_i(0) = 0$. By (3), $A_i(z) = f'_i(z) + g'_i(z)$, for all $z \in (D - \{x\}) \cap (D' - \{x\})$. Then by Lemma 4.10, f'_i, g'_i have linear extensions F_i, G_i on R^n such that $A_i = F_i + G_i$. Set $F = \sum_{i=1}^{m} F_i c_i$ and $G = \sum_{i=1}^{m} G_i c_i$. A direct verification shows that $F \in \partial f(x)$, $G \in \partial g(x)$ and $A = F + G$. Hence, $\partial(f+g)(x) \subseteq \partial f(x) + \partial g(x)$. This completes the proof of the theorem. □

Now, we show that if the cone C is not generated by linearly independent vectors, then in general the equality $\partial(f+g)(x) = \partial f(x) + \partial g(x)$ doesn't hold.

For this we consider functions f, $g : [-1,1] \to R^3$ defined by

$$f(x) = \begin{cases} (x, 0, -x) & x \in [-1, 0], \\ (0, -x, x) & x \in [0, 1]. \end{cases}$$

$$g(x) = \begin{cases} (-x, 0, -x) & x \in [-1, 0], \\ (0, x, x) & x \in [0, 1]. \end{cases}$$

Hence,

$$(f+g)(x) = \begin{cases} (0, 0, -2x) & x \in [-1, 0], \\ (0, 0, 2x) & x \in [0, 1]. \end{cases}$$

The space R^3 is ordered by the cone C which is generated by the set $D = \{(x, y, 1) : \ x, y \in [-1, 1]\}$. It is easily seen that C is not generated by linearly independent vectors. Let $(x, y, z) \in R^3$. By the definition of C, we have $(x, y, z) \in C$ if and only if

$$(x = y = z = 0) \text{ or } \begin{cases} z > 0, \\ -1 \leq \dfrac{x}{z} \leq 1, \\ -1 \leq \dfrac{y}{z} \leq 1. \end{cases} \tag{4}$$

A direct verification shows that f, g are C-convex.

The sets $\partial f(x_0)$, $\partial g(x_0)$, $\partial(f+g)(x_0)$, $\partial f(x_0) + \partial g(x_0)$ are defined as follows

a) Let $A = (k_1, k_2, k_3) \in L(R, R^3)$ be arbitrary. Then

$$
\begin{aligned}
& A \in \partial f(0) \\
& \Leftrightarrow f(x) - A(x) \in C, \quad (\forall x \in [-1,1]). \\
& \Leftrightarrow \begin{cases} (x,0,-x) - (k_1x, k_2x, k_3x) \in C, & x \in [-1,0), \\ (0,-x,x) - (k_1x, k_2x, k_3x) \in C, & x \in (0,1]. \end{cases} \\
& \Leftrightarrow \begin{cases} x(1-k_1, -k_2, -1-k_3) \in C, & x \in [-1,0), \\ x(-k_1, -1-k_2, 1-k_3) \in C, & x \in (0,1]. \end{cases} \\
& \Leftrightarrow \begin{cases} (-1+k_1, k_2, 1+k_3) \in C, \\ (-k_1, -1-k_2, 1-k_3) \in C. \end{cases} \\
& \Leftrightarrow \begin{cases} (-1+k_1 = k_2 = 1+k_3 = 0) \ \text{ or } \begin{cases} 1+k_3 > 0, \\ -1 \le \dfrac{-1+k_1}{1+k_3} \le 1, \\ -1 \le \dfrac{k_2}{1+k_3} \le 1, \end{cases} \\ (-k_1 = -1-k_2 = 1-k_3 = 0) \ \text{ or } \begin{cases} 1-k_3 > 0, \\ -1 \le \dfrac{-k_1}{1-k_3} \le 1, \\ -1 \le \dfrac{-1-k_2}{1-k_3} \le 1. \end{cases} \end{cases}
\end{aligned}
$$

Solving this system, $\partial f(0)$ is defined as a convex polyhedron whose vertices are $A_1(1,0,-1)$, $A_2\Big(-\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}\Big)$, $A_3\Big(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\Big)$, $A_4\Big(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\Big)$, $A_5(0,-1,1)$, $A_6\Big(\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}\Big)$, $A_7\Big(\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}\Big)$, $A_8\Big(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}\Big)$.

b) Let $A = (k_1, k_2, k_3) \in L(R, R^3)$ be arbitrary. Then

$$
\begin{aligned}
& A \in \partial g(0) \\
& \Leftrightarrow g(x) - A(x) \in C, \quad (\forall x \in [-1,1]). \\
& \Leftrightarrow \begin{cases} (-x,0,-x) - (k_1x, k_2x, k_3x) \in C, & x \in [-1,0), \\ (0,x,x) - (k_1x, k_2x, k_3x) \in C, & x \in (0,1]. \end{cases} \\
& \Leftrightarrow \begin{cases} x(-1-k_1, -k_2, -1-k_3) \in C, & x \in [-1,0), \\ x(-k_1, 1-k_2, 1-k_3) \in C, & x \in (0,1]. \end{cases} \\
& \Leftrightarrow \begin{cases} (1+k_1, k_2, 1+k_3) \in C, \\ (-k_1, 1-k_2, 1-k_3) \in C. \end{cases}
\end{aligned}
$$

$$\Leftrightarrow \begin{cases} (1+k_1 = k_2 = 1+k_3 = 0) \ \text{ or } \begin{cases} 1+k_3 > 0, \\ -1 \le \dfrac{1+k_1}{1+k_3} \le 1, \\ -1 \le \dfrac{k_2}{1+k_3} \le 1, \end{cases} \\ (-k_1 = 1-k_2 = 1-k_3 = 0) \ \text{ or } \begin{cases} 1-k_3 > 0, \\ -1 \le \dfrac{-k_1}{1-k_3} \le 1, \\ -1 \le \dfrac{1-k_2}{1-k_3} \le 1. \end{cases} \end{cases}$$

Solving this system, we can see that $\partial g(0)$ is defined as a convex polyhedron whose vertices are $B_1(-1,0,-1)$, $B_2\Big(-\dfrac{3}{2},-\dfrac{1}{2},-\dfrac{1}{2}\Big)$, $B_3\Big(-\dfrac{1}{2},-\dfrac{1}{2},-\dfrac{1}{2}\Big)$, $B_4(-\dfrac{3}{2},\dfrac{1}{2},-\dfrac{1}{2}\Big)$, $B_5(0,1,1)$, $B_6\Big(-\dfrac{1}{2},\dfrac{3}{2},\dfrac{1}{2}\Big)$, $B_7\Big(\dfrac{1}{2},\dfrac{3}{2},\dfrac{1}{2}\Big)$, $B_8\Big(\dfrac{1}{2},\dfrac{1}{2},\dfrac{1}{2}\Big)$.

c) Let $A=(k_1,k_2,k_3)\in L(R,R^3)$ be arbitrary. Then

$$\begin{aligned} & A \in \partial(f+g)(0) \\ & \Leftrightarrow (f+g)(x) - A(x) \in C, \ \ (\forall x \in [-1,1]). \\ & \Leftrightarrow \begin{cases} (0,0,-2x) - (k_1x,k_2x,k_3x) \in C, & x \in [-1,0), \\ (0,0,2x) - (k_1x,k_2x,k_3x) \in C, & x \in (0,1]. \end{cases} \\ & \Leftrightarrow \begin{cases} x(-k_1,-k_2,-2-k_3) \in C, & x \in [-1,0), \\ x(-k_1,-k_2,2-k_3) \in C, & x \in (0,1]. \end{cases} \\ & \Leftrightarrow \begin{cases} (k_1,k_2,2+k_3) \in C, \\ (-k_1,-k_2,2-k_3) \in C. \end{cases} \\ & \Leftrightarrow \begin{cases} (k_1 = k_2 = 2+k_3 = 0) \ \text{ or } \begin{cases} 2+k_3 > 0, \\ -1 \le \dfrac{k_1}{2+k_3} \le 1, \\ -1 \le \dfrac{k_2}{2+k_3} \le 1 \end{cases} \\ (k_1 = k_2 = 2-k_3 = 0) \ \text{ or } \begin{cases} 2-k_3 > 0, \\ -1 \le \dfrac{-k_1}{2-k_3} \le 1, \\ -1 \le \dfrac{-k_2}{2-k_3} \le 1. \end{cases} \end{cases} \end{aligned}$$

Solving this system we can see that $\partial(f+g)(0)$ is defined as a convex polyhedron whose vertices are $E_1(0,0,2)$, $E_2(-2,-2,0)$, $E_3(-2,2,0)$,

$E_4(0,0,-2)$, $E_5(2,-2,0)$, $E_6(2,2,0)$.

d) To define $\partial f(0) + \partial g(0)$, we recall that if X is a real linear space and A, B be convex polyhedrons in X whose vertices are $\{a_1, a_2, \cdots, a_k\}$, $\{b_1, b_2, \cdots, b_\ell\}$, respectively, then $A+B$ is also a convex polyhedron generated by $\{a_i + b_j : \ i = 1, 2, \cdots, k; j = 1, 2, \cdots, \ell\}$. By this, $\partial f(0) + \partial g(0)$ is defined as the convex polyhedron generated by $\{A_i + B_j : i, j = 1, 2, \cdots, 8\}$.

We see that the vertices $E_3(-2,2,0)$, $E_5(2,-2,0)$ of the convex polyhedron $\partial(f+g)(0)$ are not in the set $\{A_i + B_j : i, j = 1, 2, \cdots, 8\}$. Hence,

$$\partial(f+g)(0) \neq \partial f(0) + \partial g(0).$$

The rest of this section is devoted to the study of the structure of subdifferentials.

Theorem 4.12. *Let f be a convex function from a convex subset $D \subseteq R^n$ to R^m. Then*

i) *For every $x \in riD$, $\partial f(x)$ is a closed convex nonempty set.*

ii) *Furthermore, $x \in intD$ if and only if $\partial f(x)$ is nonempty and bounded. Particularly, if $x \in intD$, then $\partial f(x)$ is a nonempty convex compact set.*

Proof. i) Since the cone C is closed and pointed, then $C' \setminus \{0\}$ is nonempty. By the fact that the subdifferential of a scalar convex function at any relative interior point of its domain is nonempty, then i) is implied from Lemma 4.1 and Theorem 4.6.

ii) By a translation we may assume that $x = 0$, $f(x) = 0$.

For the "only if" part ,by Theorem 3.1, f is continuous at 0. Then we can find a positive number $\delta < 1$ such that $\|f(y)\| < 1$, for all $y \in \bar{B}(0,\delta)$, where $\bar{B}(0,\delta)$ denotes the closed ball $\{y \in R^n : \ \|y\| \leq \delta\}$. Suppose in contrary that $\partial f(0)$ is not bounded, then for every natural number k, there is $A_k \in \partial f(0)$ with $\|A_k\| > k$. We have

$$\|A_k\| = \sup_{y \in \bar{B}(0,1)} \|A_k(y)\| = \frac{1}{\delta} \sup_{y \in \bar{B}(0,\delta)} \|A_k(y)\|.$$

Hence, for every natural number k, there is $x_k \in \bar{B}(0,\delta)$ with $\|A_k(x_k)\| > k\delta$. Put

$$z_k = \frac{f(x_k) - A_k(x_k)}{\|f(x_k) - A_k(x_k)\|}, \quad z'_k = \frac{f(-x_k) - A_k(-x_k)}{\|f(-x_k) - A_k(-x_k)\|},$$

with $k > \dfrac{1}{\delta}$, then $z_k, z'_k \in C$ and $\|z_k\| = \|z'_k\| = 1$. Without loss of generality, we may assume that (z_k), (z'_k) converge to some unit vectors z, z', respectively. Then

$$z,\ z' \in C, \tag{5}$$

and

$$z_k + z'_k \to z + z.' \tag{6}$$

Set $v_k = z_k + z'_k$. One has

$$\begin{aligned}
\|v_k\| &= \Big\| \frac{f(x_k) - A_k(x_k)}{\|f(x_k) - A_k(x_k)\|} + \frac{f(-x_k) + A_k(x_k)}{\|f(-x_k) + A_k(x_k)\|} \Big\| \\
&\le \Big\| \frac{f(x_k)}{\|f(x_k) - A_k(x_k)\|} + \frac{f(-x_k)}{\|f(-x_k) + A_k(x_k)\|} \Big\| + \\
&\quad + \|A_k(x_k)\| \Big| \frac{1}{\|f(-x_k) + A_k(x_k)\|} - \frac{1}{\|f(x_k) - A_k(x_k)\|} \Big| \\
&\le \frac{2}{k\delta - 1} + \|A_k(x_k)\| \Big| \frac{\|f(x_k) - A_k(x_k)\| - \|f(-x_k) + A_k(x_k)\|}{\|f(-x_k) + A_k(x_k)\|.\|f(x_k) - A_k(x_k)\|} \Big| \\
&\le \frac{2}{k\delta - 1} + \|A_k(x_k)\| \frac{\|f(x_k) + f(-x_k)\|}{\|f(-x_k) + A_k(x_k)\|.\|f(x_k) - A_k(x_k)\|} \\
&\le \frac{2}{k\delta - 1} + \frac{2}{\left(1 - \frac{1}{k\delta}\right)(k\delta - 1)}.
\end{aligned}$$

It follows that $\lim_{k\to\infty} v_k = 0$. By (6), we have $z + z' = 0$. Together with (5) it implies that the unitary vector $z \in lC$ contradicting the pointedness of C.

For the "if" part, we first show that $\operatorname{int} D \neq \emptyset$. Indeed, if $\operatorname{int} D = \emptyset$, we denote by X the subspace generated by D then $X \neq R^n$. Assume that $\{e_1, e_2, \cdots, e_k\}$ is a base of X. We can find some vectors $e_{k+1}, \cdots, e_n \in R^n$ such that $\{e_1, \cdots, e_k, e_{k+1}, \cdots, e_n\}$ is a base of R^n. Let $A \in \partial f(0)$. For any vectors $y_{k+1}, \cdots, y_n \in R^m$, define the linear map $B : R^n \to R^m$ by

$$B(e_i) = \begin{cases} A(e_i) & i = 1, \cdots, k. \\ y_i & i = k+1, \cdots, n. \end{cases}$$

Then $B \in \partial f(0)$. Hence, $\partial f(0)$ is not bounded contradicting the hypothesis.

Now, assume in contrary that $0 \notin \operatorname{int} D$. By a separating theorem, there exists a nonzero functional $\lambda \in L(R^n, R)$ with $\lambda(x) \geq 0$, for all $x \in D$. In the hyperplane $\{x \in R^n : \ \lambda(x) = 1\}$ one can find n linearly independent vectors $e_1, e_2, \cdots, e_n$. Then for every $x \in R^n$, we can represent x as $x = \sum_{i=1}^{n} \alpha_i e_i$, for some $\alpha_1, \cdots, \alpha_n \in R$. It is easily seen that $x \in \{x \in R^n : \ \lambda(x) \geq 0\}$ if and only if $\sum_{i=1}^{n} \alpha_i \geq 0$. Let $A \in \partial f(0)$. For every $c \in C$, define the linear map $B_c : R^n \to R^m$ as follows

$$B_c(e_i) = A(e_i) - c, \ \ i = 1, 2, \cdots, n.$$

Let $x \in D$ be arbitrary. Represent x as $x = \sum_{i=1}^{n} \alpha_i e_i$, for some $\alpha_i \in R$. We have

$$f(x) - B_c(x) = f(x) - A(x) + \Big(\sum_{i=1}^{n} \alpha_i\Big) c \succeq f(x) - A(x) \succeq 0.$$

Therefore, $B_c \in \partial f(0)$. Since this is true for all $c \in C$, then $\partial f(0)$ is not bounded contradicting the hypothesis. Thus, $0 \in \operatorname{int} D$. This completes the proof of the theorem. □

Theorem 4.13. *Let f be a convex function from a convex subset $D \subseteq R^n$ to R^m with $\operatorname{int} D \neq \emptyset$ and let $x_0 \in \operatorname{int} D$. Then f is differentiable at x_0 if and only if $\partial f(x_0)$ reduces to a singleton. In this case $\partial f(x_0) = Jf(x_0) = \{Df(x_0)\}$.*

Proof. For the "only if" part, let $A \in \partial f(x_0)$ be arbitary. Set $B = A - Df(x_0)$. By Theorem 4.6, one has $\xi \circ A \in \partial(\xi \circ f)(x_0)$, for every $\xi \in C'$. Since the scalar convex function $\xi \circ f$ is differentiable at x_0 then by Theorem 25.1 in [3], we obtain

$$\xi \circ A = \xi \circ Df(x_0).$$

Hence, $\xi \circ B = 0$, for all $\xi \in C'$. Since C is closed, pointed then $intC' \neq \emptyset$. Therefore, there are m linearly independent vectors $\xi_1, \cdots, \xi_m \in C'$. For every $\xi \in L(R^m, R)$, we can represent ξ as

$$\xi = \sum_{i=1}^{m} \alpha_i \xi_i,$$

for some $\alpha_1, \alpha_2, \cdots, \alpha_m \in R$. This follows $\xi \circ B = 0$ for all $\xi \in L(R^m, R)$ and then $B = 0$. Hence, $\partial f(x_0) = \{Df(x_0)\}$.

For the "if" part, let $\xi \in C'$. By Theorem 4.6, $\partial(\xi \circ f)(x_0) = \xi \partial f(x_0)$. Therefore, $\partial(\xi \circ f)(x_0)$ reduces to a singleton. By Theorem 25.1 in [3], the scalar convex function $\xi \circ f$ is differentiable at x_0. Since C is closed, pointed then $\operatorname{int} C' \neq \emptyset$. Therefore, there are m linearly independent vectors $\xi_1, \cdots, \xi_m \in C'$. Let A be the $m \times m$ matrix whose rows are $\xi_1, \cdots, \xi_m$. Then A is nonsingular and $A \circ f$ is differentiable at x_0. It follows that $f = A^{-1} \circ (A \circ f)$ is differentiable at x_0.

The equalities $\partial f(x_0) = Jf(x_0) = \{Df(x_0)\}$ are obvious. This completes the proof of the theorem. $\square$

5. Monotonicity of the subdifferential

Let F be a set_valued map from a subset $D \subseteq R^n$ to $L(R^n, R^m)$. The space R^m is ordered by a convex cone C which throughout this section is assumed to be closed and pointed. We say that F is monotone if for every $x, y \in D$, $A \in F(x)$, $B \in F(y)$, one has

$$(A - B)(x - y) \succeq 0 \ \text{(or equivalently, } (A - B)(x - y) \in C). \tag{7}$$

For every $\xi \in R^m$, the set-valued map ξF is defined on $D \subseteq R^n$ and takes values in $L(R^n, R)$. The classical monotonicity of this map means that for every $x, y \in D$, $A \in F(x)$, $B \in F(y)$, inequality

$$(\xi \circ A - \xi \circ B)(x - y) \geq 0 \tag{8}$$

holds with respect to the usual order of real numbers.

The following lemma will be needed in the sequel.

Lemma 5.1. *F is monotone if and only if ξF is monotone in the classical sence, for every $\xi \in C'$.*

Proof. This follows from the fact that (7) holds if and only if (8) holds, for every $\xi \in C'$. $\square$

Theorem 5.2. *Let f be a convex function from an open convex subset $D \subseteq R^n$ to R^m. Then ∂f is a nonvoid-valued maximal monotone map from D to $L(R^n, R^m)$.*

Proof. The monotonicity of ∂f follows immediately from the definition. By Theorem 4.12, ∂f is nonvoid. To complete the proof it remains to show that if $A \notin \partial f(x)$, then there are $y \in D$, $B \in \partial f(y)$ such that

$$(B - A)(y - x) \notin C.$$

By Theorem 3.1, there is an open convex neighbourhood V of x such that f is Lipschitz on V. Since $A \notin \partial f(x)$, then there is $\bar{y} \in D$ such that

$$f(\bar{y}) - f(x) \not\succeq A(\bar{y} - x),$$

or

$$(f - A)(\bar{y}) - (f - A)(x) \notin C.$$

Without loss of the generality we may assume $\bar{y} \in V$. By a separating theorem, one can find $\xi \in C' \setminus \{0\}$ such that

$$(\xi \circ (f - A))(\bar{y}) - (\xi \circ (f - A))(x) < 0.$$

Hence, in view of the Mean Value Theorem, there exist $y \in (x, \bar{y})$, $\gamma \in \partial\left(\xi \circ (f - A)\right)(y)$ such that $\gamma(y - x) < 0$. By Theorem 4.6, we can represent γ as $\gamma = \xi \circ (B - A)$, for some $B \in \partial f(y)$. Then $\left(\xi \circ (B - A)\right)(y - x) < 0$, or $(B - A)(y - x) \notin C$. This completes the proof of the theorem. □

Remark 5.3. Let f be a vector function not necessarily convex from a subset $D \subseteq R^n$ to R^m. We define the subdifferential of f at $x \in D$ as the set

$$\partial f(x) := \{A \in L(R^n, R^m) : \ A(y - x) \preceq f(y) - f(x), \text{ for all } y \in D\}.$$

It is easily seen that if D is convex and $\partial f(x) \neq \emptyset$, for all $x \in D$, then f is convex.

Theorem 5.4. *Let f be a locally Lipschitz function from an open convex subset $D \subseteq R^n$ to R^m. Then f is convex on D if and only if the generalized Jacobian Jf of f is monotone.*

Proof. It is clear that f is convex if and only if $\xi \circ f$ is convex, for every $\xi \in C'$. By Proposition 2.2.9 in [2], $\xi \circ f$ is convex if and only if $J(\xi \circ f)$ is monotone. From Proposition 2.6.6 in [2] one has $J(\xi \circ f) = \xi Jf$. To complete the proof it remains to apply Lemma 5.1. □

Now, let f be a vector function from a subset $D \subseteq R^n$ to R^m. We define the locally subdifferential of f at $x \in D$ as the set

$$\partial_L f(x) := \{A \in L(R^n, R^m) : \text{ there exists a neighborhood } V_A \text{ of } x \text{ such that } A(y - x) \preceq f(y) - f(x), \text{ for all } y \in V_A \cap D\}.$$

Lemma 5.5. *If f is convex, then $\partial f(x) = \partial_L f(x)$, for all $x \in D$.*

Proof. The inclusion $\partial f(x) \subseteq \partial_L f(x)$ is obvious.

Conversely, suppose in contrary that there is a point $x \in D$ such that $\partial_L f(x) \not\subseteq \partial f(x)$. Then one can find a matrix $A \in \partial_L f(x)$ with $A \notin \partial f(x)$. This means that there is a point $y \in D$ such that

$$f(y) - f(x) \notin A(y - x) + C.$$

Therefore, for every $\lambda \in [0, 1)$, one has

$$(1 - \lambda)[f(y) - f(x)] \notin (1 - \lambda)A(y - x) + C,$$

or

$$\lambda f(x) + (1 - \lambda)f(y) \notin (1 - \lambda)A(y - x) + f(x) + C.$$

Hence,

$$(\lambda f(x) + (1 - \lambda)f(y) - C) \cap ((1 - \lambda)A(y - x) + f(x) + C) = \emptyset. \tag{9}$$

Since f is convex, then $f(\lambda x + (1 - \lambda)y) \in \lambda f(x) + (1 - \lambda)f(y) - C$. It implies from (9) that

$$f(\lambda x + (1 - \lambda)y) \notin (1 - \lambda)A(y - x) + f(x) + C.$$

Set $z_\lambda = \lambda x + (1 - \lambda)y$, we obtain

$$f(z_\lambda) - f(x) \not\geq A(z_\lambda - x),$$

for all $\lambda \in [0, 1)$. This contradicts $A \in \partial_L f(x)$ and the lemma is proved. $\square$

We note that in general the converse conclusion of Lemma 5.5 is not true. For instance, consider the Dirichlet function defined by

$$f(x) = \begin{cases} 0 & x \in Q, \\ 1 & x \in R \setminus Q, \end{cases}$$

where Q is the set of rational numbers. It is easily seen that

$$\partial f(x) = \partial_L f(x) = \begin{cases} \{0\} & x \in Q, \\ \emptyset & x \in R \setminus Q, \end{cases}$$

and f is not convex.

Now, let f be a vector function from a subset $D \subseteq R^n$ to R^m. We say that f is upper semicontinuous (with respect to the cone C) at $x \in D$ if for every neighborhood W of $f(x)$, there exists a neighborhood V of x such that $y \in V \cap D$ implies $f(y) \in W - C$.

It is clear that if f is continuous at $x \in D$ then f is upper semicontinuous at x. The converse is not true generally.

We recall that a single valued map $\rho : D \to R^n$ is said to be a selection of a set valued map $F : D \rightrightarrows R^n$ if $\rho(x) \in F(x)$, for all $x \in D$.

Theorem 5.6. *Let f be a upper semicontinuous function from an open convex subset $D \subseteq R^n$ to R^m. Then f is convex if and only if $\partial_L f$ admits a monotone selection.*

Proof. The "only if" part follows immediately from Theorem 5.2 and Lemma 5.5 and from the fact that any selection of a monotone map is monotone.

For the "if" part, let ρ be a monotone selection of $\partial_L f$. We show that it is also a selection of ∂f. Then by Remark 5.3, f is convex.

Suppose to the contrary that there is a point $x \in D$ such that $\rho(x) \notin \partial f(x)$. Then there exists a point $y \in D$ such that $f(y) - f(x) \not\succeq \rho(x)(y-x)$. Without loss of the generality, we may assume that $x = 0$, $f(x) = 0$. Set $t_0 = \inf\{t > 0 : \ f(ty) \not\succeq t\rho(0)(y)\}$. Since $\rho(0) \in \partial_L f(0)$, then $t_0 > 0$. We shall prove that $f(t_0 y) \succeq t_0\rho(0)(y)$. Indeed, if $f(t_0 y) \not\succeq t_0\rho(0)(y)$, then $f(t_0 y) \notin t_0\rho(0)(y) + C$. Since C is closed then there is a positive number ϵ such that $\bar{B}(f(t_0 y), \varepsilon) \cap (t_0\rho(0)(y) + C) = \emptyset$. This implies $t_0\rho(0)(y) \notin \bar{B}(f(t_0 y), \varepsilon) - C$. Since $\bar{B}(f(t_0 y), \varepsilon)$ is compact and C is closed, then $\bar{B}(f(t_0 y), \varepsilon) - C$ is closed. Hence there is a positive number ε' such that $B(t_0\rho(0)(y), \epsilon') \cap (\bar{B}(f(t_0 y), \varepsilon) - C) = \emptyset$. Therefore,

$$(B(t_0\rho(0)(y), \varepsilon') + C) \cap (\bar{B}(f(t_0 y), \varepsilon) - C) = \emptyset. \tag{10}$$

From the continuity of $\rho(0)$ and the upper semicontinuity of f there exists a positive number δ such that $f(z) \in \bar{B}(f(t_0 y), \varepsilon) - C$ and $\rho(0)(z) \in B(t_0\rho(0)(y), \varepsilon')$, for all $z \in B(t_0 y, \delta)$. Choose $t \in (0, t_0)$ with $ty \in B(t_0 y, \delta)$ then $f(ty) \in \rho(0)(ty) + C$. Hence, $f(ty) \in (B(t_0\rho(0)(y), \varepsilon') + C) \cap (\bar{B}(f(t_0 y), \varepsilon) - C)$. This contradicts to (10).

Since ρ is monotone, then $(\rho(t_0 y) - \rho(0))(t_0 y) \succeq 0$. Hence, $\rho(t_0 y)(y) \succeq \rho(0)(y)$. Since $\rho(t_0 y) \in \partial_L f(t_0 y)$, then there exists a positive number γ such that $f(z) - f(t_0 y) - \rho(t_0 y)(z - t_0 y) \succeq 0$, for all $z \in B(t_0 y, \gamma)$. From the definition of t_0 we can find $t > t_0$ with

$$ty \in B(t_0 y, \gamma), \ \ f(ty) \not\succeq \rho(0)(ty). \tag{11}$$

Since $ty \in B(t_0 y, \gamma)$, we have $f(ty) - f(t_0 y) \succeq (t - t_0)\rho(t_0 y)(y)$. Hence,

$$f(ty) \succeq f(t_0 y) + (t - t_0)\rho(t_0 y)(y) \succeq \rho(0)(t_0 y) + (t - t_0)\rho(0)(y) = t\rho(0)(y).$$

This contradicts to (11). The proof is complete. $\square$

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