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RELATION BETWEEN THE SPECTRUM OF OPERATORS AND LYAPUNOV EXPONENTS

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Abstract. The article concerned with the problem of regarding Lyapunov exponents of a random difference equation as the spectrum of an operator acting on a suitable space. Let L be the set of all sequences of random variables having finite p^{th}−moments for some p small, endowed with a certain topology. From the difference equation $X(n+1)=A(n)X(n)$; $X(0)=x\in R^d$, where $(A(n),n\in Z)$ is an i.i.d. sequence of random variables, we construct an operator T acting on the space L. It is proved that the spectrum of the operator T is contained in the set of sample Lyapunov exponents of this random dynamical system.

1. Introduction

As is known, the sample Lyapunov exponents are a useful tool to describe the growth of stochastic dynamical systems, especially for linear systems. Therefore, there are many works dealling with the Lyapunov exponents of difference and differential equations (see [Ar], [Ku]). It is proved that if the top Lyapunov exponents of linear system is negative then the system is stable in probability. Thus studying Lyapunov exponents is important both in theory and practice, although it is sometime not easy to calculate them. On the other hand, in the deterministic cases there is another approach to study non autonomous systems by enlarging the phase spaces and relating the stability of a system to the spectrum of the associated operator. But the relations between the spectrum of this operator and the Lyapunov exponents of the original system is still not established, except for some special cases under a strong assumption on the coefficients (see, for example, [Ar]). In our opinion, this relation is very important, because one can use functional analysis approaches to study the problem. We will show that, in a certain context, the Lyapunov exponents of the product of i.i.d. random matrices or of the solution of Ito'equations are indeed the spectrum of an operator T acting on a suitable

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space. Unfortunately the space constructed by us is a linear topology space only. Thus, the spectrum of operators acting on this space must be understood in a wider sense. The idea of our construction is based on a relation between the sample Lyapunov exponents and the $p^{\text{th}}-$mean exponents, when p is small.

The article is organised as follows. In section 2 we recall the Multiplicative Ergodic Theorem of linear systems and make some remarks. Section 3 devote to the main theorem. It gives a necessary condition for which a complex number λ belongs to the spectrum $\sigma(T)$ of the operator T acting on the space of the union of the spaces L_p as p small. It is proved that if $\lambda \in \sigma(T)$ then $\ln|\lambda|$ is a Lyapunov exponent of this linear system. Unfortunately, we don't know yet if the inverse relation is true or not, i.e. whether or not the spectrum of T is exactly the Lyapunov spectrum.

2. Cocycles and multiplicative ergodic theorem

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and $\theta : \Omega \to \Omega$ a random transformation of $(\Omega, \mathcal{F}, \mu)$ preserving the measure μ, i.e, $\mu(\theta^{-1}A) = \mu(A)$ for any $A \in \mathcal{F}$. Let $Gl(d, R)$ denote the general linear group of $d \times d$-matrices. We endow $Gl(d, R)$ with its σ-Borel field. Let $A(\cdot)$ be a random variable defined on $(\Omega, \mathcal{F}, \mu)$ and having values in $Gl(d, R)$. Saying that θ is a random transformation means that $(A(\theta^n(\omega)),\ n \in Z)$ is an independent, identically distributed sequence where Z is the set of all integer numbers, and (θ^n) is defined by $\theta^{n+1} = \theta \cdot \theta^n$. We consider the difference equation

$$\begin{cases} X(n+1) &= A(\theta^n \omega)X(n), \quad n \in Z, \\ X(0) &= x \in R^d. \end{cases} \tag{2.1}$$

For the sack of the simplicity, we suppose that there is a number $\alpha_0 > 0$ such that

$$E\left[|A^{-1}(\omega)|^{\alpha_0} + |A(\omega)|^{\alpha_0}\right] < \infty. \tag{2.2}$$

The difference equation (2.1) generates a linear cocycle, i.e. a random dynamical system over the dynamical system $(\Omega, \mathcal{F}, \mu)$ (see, for example, [AC]), by the following formula

$$\Phi(n, \omega) = \begin{cases} A(\theta^{n-1}\omega) \cdot A(\theta^{n-2}\omega) \cdots A(\omega) & \text{if } n > 0, \\ Id & \text{if } n = 0, \\ A^{-1}(\theta^n\omega) \cdot A^{-1}(\theta^{n+1}\omega) \cdots A^{-1}\theta^{-1}\omega) & \text{if } n < 0. \end{cases}$$

In the other words,

$$\begin{cases} \Phi(n+1) & = A(\theta^n \omega)\Phi(n,\omega), \quad n \in Z, \\ \Phi(0,\omega) & = Id. \end{cases}$$

The cocycle property means that

$$\Phi(n+m,\omega) = \Phi(n,\theta^m\omega) \cdot \Phi(m,\omega),$$

or equivalently,

$$\Phi(n,\omega) \cdot \Phi^{-1}(m,\omega) = \Phi(n-m,\theta^m\omega), \tag{2.3}$$

for any $m, n \in Z$. The solution starting from x at $n = 0$ of the equation (2.1) is then $\Phi(n,\omega)x$. Applying the Multiplicative Ergodic Theorem for the case of products of an i.i.d sequence or Ito' stochastic differential equations (see [FK]) to the cocycle $\Phi(n,\omega)$.

Theorem 2.1 (Multiplicative Ergodic Theorem). *We get*

a) *There exists the limit*

$$\lambda[x] = \lim_{n\to\pm\infty} \frac{1}{n} \log|\Phi(n,\omega)x|, \tag{2.4}$$

for each $x \neq 0$, which takes non-random values

$$\lambda_1 < \lambda_2 < \ldots < \lambda_r, \tag{2.5}$$

called Lyapunov spectrum of $A(\cdot)$

b) *There is a sequence of linear subspaces of R^d (or Lyapunov filtration)*

$$\{0\} = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \ldots \subset \mathcal{V}_r = R^d \tag{2.6}$$

associated with (2.5), such that

$$\lambda[x] = \lim_{n\to\infty} \frac{1}{n} \log|\Phi(n,\omega)x| \le \lambda_i, \quad \textit{for any} \quad x \in \mathcal{V}_i, \quad i = 1, 2, \ldots, r,$$

and a sequence of linear subspaces of R^d

$$\{0\} = \mathcal{L}_r \subset \mathcal{L}_{r-1} \subset \ldots \subset \mathcal{L}_0 = R^d, \tag{2.6'}$$

such that

$$\lambda[x] = \lim_{n\to-\infty} \frac{1}{|n|} \log|\Phi(n,\omega)x| \le -\lambda_{i+1}, \quad \textit{for any} \quad x \in \mathcal{L}_i, \quad i = 0, 1, 2, \ldots, r-1,$$

It was shown in [KF] that the linear subspaces $\mathcal{V}_i$ and $\mathcal{L}_i$ are μ - invariant, i.e. $\mu\{\omega : A(\omega)\mathcal{V}_i = \mathcal{V}_i\} = 1$ and $\mu\{\omega : A(\omega)\mathcal{L}_i = \mathcal{L}_i\} = 1$. Moreover, for any $i = 0, 1, \ldots, r$, we have

$$\dim \mathcal{V}_i + \dim \mathcal{L}_i = d. \tag{2.6”}$$

The difference between the Oseledets's decomposition and the Furstenberg-Kiffer one is that (2.6) and (2.6') are non random filtrations

Beside the sample Lyapunov exponents $\lambda[x]$ defined by (2.4), we consider also the so-called $\alpha^{th}-$mean exponents

$$g_{\pm}(\alpha, x) = \limsup_{n\to\pm\infty} \frac{1}{n} \log E|\Phi(n,\omega)|^{\alpha}. \tag{2.7}$$

By Hypothesis (2.2), it follows that $g(\alpha, x)$ is finite for all $|\alpha| < \alpha_0$. Moreover, it was proved that

Theorem 2.2 (see [DN] and [Ar]). *The function $g(\alpha, x)$ is convex, differentiable with respect to α at $\alpha = 0$ for any fixed $x \neq 0$ in R^d, and*

$$g'_{\pm}(\alpha, x)\big|_{\alpha=0} = \lambda[x]. \tag{2.8}$$

3. Relation between the Lyapunov exponents of (2.1) and the spectrum of operators

It would be important if the Lyapunov exponents of an dynamical system would form the spectrum of an operator defined on a topological linear space, because we could then use the operator theory to solve many problems. Let us begin with a simple example, when A is not random, i.e. $A \in Gl(R, d)$. It is easy to see that the equation

$$\begin{cases} X(n+1) &= A \cdot X(n), \quad n \in Z, \\ X(0) &= x \in R^d, \end{cases}$$

has a bounded solution starting from $x \neq 0$ if and only if A has an eigenvalue λ belonging to the unit circle $S(0, 1) = \{z : |z| = 1\}$ on the complex plane and x is in the invariant subspace generated by this eigenvalue. On the other hand, let M be the Banach space of all bounded sequences $(v_n)_{n\in Z}$ equipped by the supremum norm: if $v = (v_n) \in M$ then $|v| = \sup\limits_{n\in Z} |v_n|$. We define an operator T acting on M into itself by the formula:

$$(Tv)_{k+1} = Av_k, \quad k \in Z.$$

Then, a complex number λ is in the spectrum of T iff $\lambda \in S(0, |\lambda_0|) = \{z : |z| = |\lambda_0|\}$, where λ_0 is an eigenvalue of A. Thus, the spectrum of the operator T can be regarded as the spectrum of the matrix A. This simple result suggests us to use the operator T to describe the Lyapunov spectrum of the linear difference equation (2.1).

The main difficulty, when we want to extend this result to random matrix products or Ito's equation, is that the sample trajectory of the map $n \to A(\theta^n\omega)$ is, in general, not bounded, i.e. $\limsup |A(\theta^n\omega)| = \infty$ a.s. Hence, it is impossible to use the same space M. To overcome this difficulty, we can use the so-called random or Lyapunov norms considered by many authors (see, for example, [Ar]). However this norm depends strictly on the considered cocycle $\Phi(n, \omega)$ and sometime it is not convenient. On the other hand, if we replace M by the metric linear space L_p of all random sequences having finite p^{th}- moment for some fixed p, then we can regard $A(\cdot)$ as a constant matrix. Unfortunately, in this case $A(\cdot)$ does not act from L_p into L_p and it seems that there is no relation between the spectrum of T and the sample Lyapunov exponents of the difference equation (2.1).

Note that the relation (2.8) between Lyapunov sample exponents and α^{th}-mean exponents says that the α^{th}-mean exponents is closely related to $\lambda[x]$, when α is small enough. Based on this relation we introduce the following space.

Let L_α be the metric linear space of random sequences (v_k) where for any $k \in Z$, v_k is a random variable taking values in C^d (C denotes the complex plane) such that

$$\sup_{k \in Z} E|v_k|^\alpha < \infty.$$

A sequence $(v^{(n)})$ of elements of L_α is said to be *convergent to* 0 if

$$\sup_{k \in Z} E|v_k^{(n)}|^\alpha \to 0 \quad \text{as} \quad n \to \infty.$$

We denote by $\mathcal{B}_\alpha$ the topology of the type of this convergence on L_α and put

$$L = \bigvee_{0 < \alpha \leq \alpha_0} L_\alpha. \tag{3.2}$$

Suppose that L is endowed with the topology $\mathcal{B}$ of the following convergence: a sequence $(v^{(n)}) \in L$ is said to converge to $v \in L$, if there is a sequence of positive numbers (p_n) such that: for any $\epsilon > 0$

$$\lim_{n\to\infty} \sup_{k\in Z} E|v_k^{(n)} - v_k|^{p_n} \cdot 1_{\{|v^{(n)} - v_k| > \varepsilon\}} = 0. \tag{3.3}$$

It is easy to show that if the sequence $(v^{(n)})$ converges to 0 in this sense, then it is convergent in probability to 0, i.e.:

$$\lim_{n\to\infty} \sup_{k\in Z} P\{|v_k^{(n)}| > \varepsilon\} = 0, \quad \text{for any} \quad \varepsilon > 0.$$

By virtue of the inequality

$$|x+y| \cdot 1_{\{|x+y|>\epsilon\}} \le 2[|x| 1_{\{|x|>\frac{\varepsilon}{2}\}} + |y| \cdot 1_{\{|y|>\frac{\varepsilon}{2}\}}],$$

it follows that $(L, \mathcal{B})$ is a linear topological space and the topology $\mathcal{B}$ is stronger than the one induced by the family $\{\mathcal{B}_\alpha, 0 < \alpha \le \alpha_0\}$. Moreover for any $0 < \alpha \le \alpha_0$, the space $(Ł_\alpha, \mathcal{B}_\alpha)$ is dense in $(L, \mathcal{B})$. On the other hand, the space L is constructed from the view point of random dynamical systems and we do not require that every element of $(L, \mathcal{B})$ is adapted with respect to a filtration. The picture is quite different if we consider the problem with the last requirement, i.e. from the view point of stochastic analysis: our conclusions are no more true.

We need an inequality which will be used many times in the sequel. Let ξ and η be two elements of L. Then there exist two constants α_1 and α_2 such that

$$\xi = (\xi_k) \in L_{\alpha_1} \ ; \quad \eta = (\eta_k) \in L_{\alpha_2}.$$

Choose $\alpha = \max\{\alpha_1, \alpha_2\}$. Then

$$\big(E|\xi_k \cdot \eta_k|^{\frac{\alpha}{2}}\big)^2 \le E|\xi_k|^\alpha \cdot E|\eta_k|^\alpha, \tag{3.4}$$

and

$$E|\xi_k + \eta_k|^\alpha \le Ee|\xi_k|^\alpha + E|\eta_k|^\alpha, \tag{3.5}$$

for any $k \in Z$. (Hölder's inequality, see [HS]). This means that $(\xi_k \cdot \eta_k) \in L$ and $(\xi_k + \eta_k) \in L$.

We now turn to our problem. We define an operator T acting on L by

$$(Tv)_{k+1} = A(\theta^k \omega) \cdot v_k, \quad k \in Z, \tag{3.6}$$

for any $v = (v_k) \in L$. By virtue of Hölder's inequality, it is easy to show that the operator T acts from L into itself. Indeed, for any $v \in L_\alpha$ we have

$$E\big|(Tv)_{k+1}\big|^{\frac{\alpha}{2}} = E\big|A(\theta^k)v_k\big|^{\frac{\alpha}{2}} \le \sqrt{E|A(\theta^k)|^\alpha \cdot E|v_k|^\alpha}, \quad k \in Z.$$

Since

$$E|A(\theta^k\omega)|^\alpha = E|A(\omega)|^\alpha \le \big(E|A|^{\alpha_0}\big)^{\frac{\alpha}{\alpha_0}} \le \max(1,\ E|A(\omega)|^{\alpha_0}),$$

it then follows that

$$E\big|(Tv)_{k+1}\big|^{\frac{\alpha}{2}} \le \max(1,\ E|A(\omega)|^{\alpha_0}) \cdot \sqrt{E|v_k|^\alpha}\ . \tag{3.7}$$

This implies that $Tv \in L_{\frac{\alpha}{2}}$, i.e. $Tv \in L$.

Theorem 3.2. *T is continuous in L.*

Proof. It follows immediately from (3.2), (3.3) and Relation (3.7). $\diamondsuit$

Let $\sigma(T)$ denote the spectrum of T, i.e. the subset of C such that the operator

$$(\lambda I - T)^{-1}$$

exists for any $\lambda \neq \sigma(T)$. Suppose that

$$\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_r\},$$

where

$$-\infty = \lambda_0 < \lambda_1 < \dots < \lambda_r < \lambda_{r+1} = \infty,$$

is the sample Lyapunov exponents of the system (2.1) associated with the filtrations of linear subspaces

$$\{0\} = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \dots \subset \mathcal{V}_r = R^d,$$

and

$$\{0\} = \mathcal{L}_r \subset \mathcal{L}_{r-1} \subset \dots \subset \mathcal{L}_0 = R^d.$$

From (2.6") we get

$$R^d = \mathcal{V}_i \oplus \mathcal{L}_i.$$

Theorem 3.3. *If λ is a complex number such that $\ln|\lambda| \notin \Lambda$, then $\lambda \notin \sigma(T)$.*

Proof. Suppose that

$$\lambda_i < \ln|\lambda| < \lambda_{i+1}, \tag{3.8}$$

for an i, $0 \leq i \leq r$. To prove that $\lambda \notin \sigma(T)$ we have to show that the equation

$$(\lambda I - T)\, v = f$$

has a unique solution in L for any $f \in L$, and the map $f \to v \ \ (= v(f))$ is continuous. In other words:

$$v_{k+1} = \lambda^{-1} A(\theta^k \omega) \cdot v_k + f_k \tag{3.9}$$

has a unique solution $v = (v_k) \in L$. Put

$$\varepsilon = \min\left\{ \frac{|\lambda| - e^{\lambda_i}}{2}\, ;\, \frac{e^{\lambda_{i+1}} - |\lambda|}{2} \right\}.$$

We denote by P the projection operator on $\mathcal{V}_i$. Then $I - P$ is the projection on $\mathcal{L}_i$. By the definition of Lyapunov exponents, we have

$$\lim_{n\to+\infty} \frac{1}{n} \log|\Phi(n)Px| \leq \lambda_i \qquad \text{a.s.},$$
$$\lim_{n\to-\infty} \frac{1}{n} \log|\Phi(n)(I-P)x| \geq \lambda_{i+1} \qquad \text{a.s.},$$

for any $x \in R^d$. Hence, it follows that

$$b(\omega) = \sup_{0\leq n<\infty} |e^{-(\lambda_i+\varepsilon)\, n} \Phi(n,\omega)P| < \infty \quad \text{a.s.}, \tag{3.10}$$

$$c(\omega) = \sup_{-\infty<n\leq 0} |e^{-(\lambda_{i+1}-\varepsilon)\, n} \Phi(n,\omega)(I-P)| < \infty \quad \text{a.s.}. \tag{3.11}$$

From (3.10) and (3.11) it follows that

$$|\Phi(n,\omega)P| \leq b(\omega) \cdot \exp(\lambda_i + \varepsilon)\, n, \quad \text{for } n > 0,$$
$$|\Phi(n,\omega)(I-P)| \leq c(\omega) \exp(\lambda_{i+1} - \varepsilon)\, n \quad \text{for } n < 0.$$

Therefore

$$|\lambda^{-n} \Phi(n,\omega)P| \leq b(\omega) \cdot \exp(-\varepsilon\, n), \quad \text{for } n > 0, \tag{3.12}$$

$$|\lambda^{-n}\Phi(n,\omega)(I-P)| \le c(\omega)\cdot \exp(\varepsilon\, n), \quad \text{for } n<0. \tag{3.13}$$

Let $f=(f_k)$ be an element of L. Then there is β_0 such that $(f_k)\in L_{\beta_0}$. Put

$$\begin{aligned} v_n(\omega) = &\sum_{m=-\infty}^{n} \lambda^{m-n}\Phi(n-m,\theta^m\omega)\cdot Pf_m(\omega) \\ &- \sum_{m=n+1}^{\infty} \lambda^{m-n}\Phi(n-m,\theta^m\omega)(I-P)f_m(\omega). \end{aligned}$$

We show that $(v_n)\in L$, i.e there exists a constant $\alpha>0$ such that

$$\sup_{n\in Z} E|v_n|^\alpha < \infty.$$

Using (3.12), we get

$$\begin{aligned} \Big|\sum_{m=-\infty}^{n} \lambda^{m-n}\Phi(n-m,\theta^m\omega)Pf_m| &\le \sum_{m=-\infty}^{n} |\lambda^{m-n}\Phi(n-m,\theta^m\omega)Pf_m| \\ &\le \sum_{m=-\infty}^{n} b(\theta^m\omega)\cdot \exp\{-\varepsilon(n-m)\}\cdot |Pf_m|. \end{aligned}$$

By Lemma 3.4 below, there is a number β such that $E|b(\omega)|^\beta<\infty$. Let $2\alpha=\min(\beta,\beta_0)$. Then

$$\begin{aligned} &E\Big|\sum_{m=-\infty}^{n} \lambda^{m-n}\Phi(n-m,\theta^m\omega)Pf_m\Big|^\alpha \\ &\le \sum_{m=-\infty}^{n} \exp\{-\epsilon\alpha(n-m)\} E|b(\theta^m\omega)Pf_m\Big|^\alpha \\ &\le \big[E|b(\cdot)|^\beta\big]^{\frac{\alpha}{2\beta}}\cdot \sup_{m\in Z}\big(E|Pf_m|^{\beta_0}\big)^{\frac{\alpha}{2\beta_0}} \sum_{m=-\infty}^{n}\exp\{-\epsilon\alpha(n-m)\} \\ &\le \frac{K}{1-\exp\{-\epsilon\alpha\}}\ , \end{aligned} \tag{3.14}$$

where K is a certain constant. Similarly, using again Lemma 3.4 we get

$$E\Big|\sum_{m=n+1}^{\infty} \lambda^{m-n}\Phi(n-m,\theta^m)(I-P)f_m\Big|^\alpha \le \frac{K}{1-\exp\{-\varepsilon\alpha\}}\ , \tag{3.15}$$

which implies that

$$\sup_{n\in Z} E|v_n|^\alpha < \infty.$$

This means $(v_n) \in L_\alpha \subset L$.

We now prove that T is an one to one map. For this purpose we show that for any random variable x such that $E|x|^\alpha < \infty$ for some $0 < \alpha \le \alpha_0$, we have

$$\sup_{n\in Z} E|\lambda^{-n}\Phi(n,\omega)x|^\beta = \infty,$$

for any $0 < \beta \le \alpha_0$. Without loss of generality, suppose that $(I-P)x \neq 0$ with a positive probability. Then, by using the Fatou's lemma, it yields

$$\limsup_{n\to\infty} E|\lambda^{-n}\Phi(n,\omega)(I-P)x|^\beta \ge E\liminf_{n\to\infty} |\lambda^{-n}\Phi(n,\omega)(I-P)x|^\beta = \infty.$$

Combining above results, we get $\lambda \notin \sigma(T)$. Theorem 3.3 is proved $\diamondsuit$.

In proof of the above theorem we have used the following lemma.

Lemma 3.4. *There exists a constant* $0 < \beta < \alpha_0$ *such that* $E|b(\omega)|^\beta < \infty$ *and* $E|c(\omega)|^\beta < \infty$.

Proof. We only show that $E|b(\omega)|^\beta < \infty$. The proof of $E|c(\omega)|^\beta < \infty$ is similar. Putting

$$b_n(\omega) = \sup_{0\le k<n} |e^{-(\lambda_i+\varepsilon)\,k}\Phi(k,\omega)P|.$$

Then $b_n(\omega) \uparrow b(\omega)$ as $n \to \infty$. On the other hand, by (2.8) we have

$$\frac{d}{d\alpha}\Big(\limsup \frac{1}{n}\log E\big[e^{-(\lambda_i+\varepsilon)\,n}|\Phi(n,\omega)\,P|\big]^\alpha\Big)\Big|_{\alpha=0} \le -(\lambda_i+\varepsilon)+\lambda_i = -\varepsilon < 0.$$

Therefore, there exists a constant $\beta > 0$ such that

$$\limsup \frac{1}{n}\log E\left[e^{-(\lambda_i+\varepsilon)\,n}|\Phi(n,\omega)\,P|\right]^\beta \le \frac{-\varepsilon\beta}{2},$$

or equivalently, there is a constant M such that

$$E\left[e^{-(\lambda_i+\varepsilon)\,n}|\Phi(n,\omega)\,P|\right]^\beta \le M\cdot\exp\Big\{\frac{-\varepsilon\beta\cdot n}{2}\Big\}, \quad \forall n \ge 0.$$

This implies that

$$
\begin{aligned}
&E|b_n(\omega)|^\beta \\
&\le E\left[\sum_{k=1}^{n} e^{-(\lambda_i+\varepsilon)\,k}|\Phi(k,\omega)\,P|\right]^\beta \le \sum_{k=1}^{n} E\left[e^{-(\lambda_i+\varepsilon)\,k}|\Phi(k,\omega)\,P|\right]^\beta \\
&\le M\sum_{k=1}^{n}\exp\{\frac{-\varepsilon\beta k}{2}\} \qquad\qquad \le \frac{M}{1-\exp\left\{\dfrac{-\varepsilon\beta}{2}\right\}}\,.
\end{aligned}
$$

Using a theorem for monotonous sequences, we get

$$E|b(\omega)|^\beta = E\lim|b_n|^\beta = \lim\, E|b_n|^\beta \le \frac{M}{1-\exp\left\{\frac{-\varepsilon\beta}{2}\right\}}\,.$$

Hence, the statement follows. $\diamondsuit$

Theorem 3.4. *If $\lambda \notin \sigma(T)$, then $e^{i\phi}\lambda \notin \sigma(T)$, for any $0 \le \phi \le 2\pi$.*

Proof. It follows immediately from the definition. $\diamondsuit$

Remark 3.5. From (3.14) and (3.15), it follows that the map

$$T_\lambda : f \to T_\lambda f,$$

where $T_\lambda f$ is the solution of (3.9) associated with λ, satisfies the condition

$$\sup_{k\in Z} E|(T_\lambda f)_k|^\alpha \le K\cdot\sup_{k\in Z} E|f_k|^{2\alpha}.$$

Therefore, by virtue of (3.2) and (3.3), it follows that the map $f \to T_\lambda f$ is continuous.

Open Problem. Is it true that

$$\sigma(T) = \cup_{i=1}^{r} S(0,\,\exp\{\lambda_i\})?$$

Remark. It can be proved that the picture is quite different when we require that every element of L is adapted to the filtration generated by the sequence $(A(\theta^n), n = 1, 2, \dots)$.

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