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CRITICAL SOBOLEV EXPONENT FOR DEGENERATE ELLIPTIC OPERATORS

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Abstract. Semilinear equations for degenerate elliptic operators are considered. A simple proof of imbedding theorems for appropriate Sobolev spaces are given. Via generalized Pokhozaev identity we prove some non-existence theorems for the equations.

The purpose of this paper is to point out some similar properties for Laplace's equation in $\mathbf{R}^n$ $(n \geq 3)$ and the hypoelliptic equation $\dfrac{\partial^2}{\partial x_1^2} + x_1^{2k}\dfrac{\partial^2}{\partial x_2^2}$ in $\mathbf{R}^2$ $(k \geq 1)$. Similar properties between this degenerate elliptic operators and Laplace's operator were studied by many authors (see, for example, [1-6] and therein references). Let Ω be a bounded domain with a smooth boundary in $\mathbf{R}^2$ and $0 \in \Omega$. We consider the following boundary value problem:

$$L_k u = \frac{\partial^2 u}{\partial x_1^2} + x_1^{2k}\frac{\partial^2 u}{\partial x_2^2} + g(u) = 0 \quad \text{in} \quad \Omega, \tag{1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{2}$$

where $g(0) = 0$ and $g(u) \in C(\mathbf{R})$. Put $G(u) = \int\limits_0^u g(s)\,ds$ and let $\nu = (\nu_1, \nu_2)$ be the outward normal to $\partial\Omega$. By C we will denote a general constant that is independent of functions and may change its value.

Definition 1. A domain Ω is called L_k-starshape with respect to the point 0 if the inequality $(\nu_1^2 + x_1^{2k}\nu_2^2)(x_1\nu_1 + (k+1)x_2\nu_2) > 0$ holds almost everywhere on $\partial\Omega$.

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Example. The unit ball $B_1 = \left\{(x_1, x_2)|x_1^2 + x_2^2 < 1\right\}$ is L_k-starshape for every k.

Lemma. *Let $u(x)$ be a solution of the boundary value problem* (1)-(2)*, which belongs to the class $H^2(\Omega)$. Then the function $u(x)$ satisfies the equation*

$$\begin{aligned}
&\int\limits_{\Omega} \left\{(k+2)G(u) - \frac{k}{2}g(u)u\right\} dx_1dx_2 \\
(3) \qquad &= \frac{1}{2}\int\limits_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 (\nu_1^2 + x_1^{2k}\nu_2^2)\{x_1\nu_1 + (k+1)x_2\nu_2\}\, ds.
\end{aligned}$$

Proof. The Sobolev imbedding theorem for a smooth bounded domain gives $H^2(\Omega) \subset C^{o,\alpha}(\overline{\Omega})$, where $0 < \alpha < 1$. Note that

$$\begin{aligned}
\frac{\partial}{\partial x_1}\big(x_1 G(u)\big) &= G(u) + x_1 g(u)\frac{\partial u}{\partial x_1}\,, \\
\frac{\partial}{\partial x_2}\big(x_2 G(u)\big) &= G(u) + x_2 g(u)\frac{\partial u}{\partial x_2}\,\cdot
\end{aligned}$$

From the Gauss-Ostrogradskii formula we have

$$\int\limits_{\Omega} G(u)\, dx_1dx_2 = -\int\limits_{\Omega} x_1 g(u)\frac{\partial u}{\partial x_1}\, dx_1dx_2$$

and

$$\beta\int\limits_{\Omega} G(u)\, dx_1dx_2 = -\beta\int\limits_{\Omega} x_2 g(u)\frac{\partial u}{\partial x_2}\, dx_1dx_2.$$

Hence

$$\begin{aligned}
(1+\beta)\int\limits_{\Omega} G(u)\, dx_1dx_2 &= -\int\limits_{\Omega} \Big(x_1\frac{\partial u}{\partial x_1} + \beta x_2\frac{\partial u}{\partial x_2}\Big) g(u)\, dx_1dx_2 \\
&= \int\limits_{\Omega} \Big(x_1\frac{\partial u}{\partial x_1} + \beta x_2\frac{\partial u}{\partial x_2}\Big)\Big(\frac{\partial^2 u}{\partial x_1^2} + x_1^{2k}\frac{\partial^2 u}{\partial x_2^2}\Big)\, dx_1dx_2.
\end{aligned}$$

Again from the Gauss-Ostrogradskii formula we have (for details see [7])

$$\begin{aligned}
&\int\limits_{\Omega}\Big(x_1\frac{\partial u}{\partial x_1}+\beta x_2\frac{\partial u}{\partial x_2}\Big)\Big(\frac{\partial^2 u}{\partial x_1^2}+x_1^{2k}\frac{\partial^2 u}{\partial x_2^2}\Big)\,dx_1dx_2\\
&=\frac{\beta-1}{2}\int\limits_{\Omega}\Big(\frac{\partial u}{\partial x_1}\Big)^2 dx_1dx_2+\frac{2k+1-\beta}{2}\int\limits_{\Omega}x_1^{2k}\Big(\frac{\partial u}{\partial x_2}\Big)^2 dx_1dx_2\\
&\quad+\frac{1}{2}\int\limits_{\partial\Omega}\Big(x_1\nu_1\Big)\Big(\frac{\partial u}{\partial x_1}\Big)^2 ds-\frac{1}{2}\int\limits_{\partial\Omega}\Big(x_1^{2k+1}\nu_1\Big)\Big(\frac{\partial u}{\partial x_2}\Big)^2 ds\\
&\quad+\int\limits_{\partial\Omega}\Big(x_1^{2k+1}\nu_2\Big)\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\,ds+\frac{\beta}{2}\int\limits_{\partial\Omega}\Big(x_2x_1^{2k}\nu_2\Big)\Big(\frac{\partial u}{\partial x_2}\Big)^2 ds\\
&\quad-\frac{\beta}{2}\int\limits_{\partial\Omega}\Big(x_2\nu_2\Big)\Big(\frac{\partial u}{\partial x_1}\Big)^2 ds+\beta\int\limits_{\partial\Omega}\Big(x_2\nu_1\Big)\frac{\partial u}{\partial x_2}\frac{\partial u}{\partial x_1}\,ds.
\end{aligned}$$

Finally, choosing $\beta=k+1$, we have

$$\begin{aligned}
&\int\limits_{\Omega}\Big\{(k+2)G(u)-\frac{k}{2}g(u)u\Big\}\,dx_1dx_2\\
&=\frac{1}{2}\int\limits_{\partial\Omega}\Big(x_1\nu_1^3\Big)\Big(\frac{\partial u}{\partial \nu}\Big)^2 ds-\frac{1}{2}\int\limits_{\partial\Omega}\Big(x_1^{2k+1}\nu_1\nu_2^2\Big)\Big(\frac{\partial u}{\partial \nu}\Big)^2 ds\\
&\quad+\int\limits_{\partial\Omega}\Big(x_1^{2k+1}\nu_1\nu_2^2\Big)\Big(\frac{\partial u}{\partial \nu}\Big)^2 ds-\frac{k+1}{2}\int\limits_{\partial\Omega}\Big(x_2\nu_1^2\nu_2\Big)\Big(\frac{\partial u}{\partial \nu}\Big)^2 ds\\
&\quad+\Big(k+1\Big)\int\limits_{\partial\Omega}\Big(x_2\nu_1^2\nu_2\Big)\Big(\frac{\partial u}{\partial \nu}\Big)^2 ds+\frac{k+1}{2}\int\limits_{\partial\Omega}\Big(x_1^{2k}x_2\nu_2^3\Big)\Big(\frac{\partial u}{\partial \nu}\Big)^2 ds\\
&=\frac{1}{2}\int\limits_{\partial\Omega}\Big(\frac{\partial u}{\partial \nu}\Big)^2(\nu_1^2+x_1^{2k}\nu_2^2)\{x_1\nu_1+(k+1)x_2\nu_2\}\,ds.
\end{aligned}$$

□

Note that (3) is similar to the so-called Pohozaev identity for Laplace's operator proven in [8]. The two following theorems are obviously obtained by using the lemma.

Theorem 1. *Let Ω be a L_k-starshape with respect to the point 0 and $(k+2)G(u)-\frac{k}{2}g(u)<0$ when $u\neq 0$. Then there exists no non-trivial solution $u\in H^2(\Omega)$ for the problem* (1)-(2).

Theorem 2. *Let Ω be a L_k-starshape with respect to the point* 0 *and* $(k+2)G(u) - \frac{k}{2}g(u) < 0$ *when* $u > 0$. *Then there exists no non-trivial positive solution* $u \in H^2(\Omega)$ *for the problem* (1)-(2).

The following theorem provides another non-existence criterion.

Theorem 3. Let Ω be a L_k-starshape with respect to the point 0 and $g(u) = \lambda u + |u|^\gamma u$ with $\lambda \le 0$, $\gamma \ge \frac{4}{k}\cdot$ Then the problem (1)-(2) has no non-trivial solution $u \in H^2(\Omega)$.

Proof. Indeed, in this case $G(u) = \frac{\lambda u^2}{2} + \frac{|u|^{\gamma+2}}{\gamma+2}\cdot$ Putting $G(u)$ and $g(u)$ into (3) yields

$$\int_\Omega \Big\{(k+2)\Big(\frac{\lambda u^2}{2} + \frac{|u|^{\gamma+2}}{\gamma+2}\Big) - \frac{k}{2}(\lambda u^2 + |u|^{\gamma+2})\Big\}\, dx_1dx_2 = \frac{1}{2}\int_{\partial\Omega} \Big(\frac{\partial u}{\partial \nu}\Big)^2 (\nu_1^2 + x_1^{2k}\nu_2^2)\big\{x_1\nu_1 + (k+1)x_2\nu_2\big\}\, ds.$$

That is,

$$\lambda \int_\Omega u^2\, dx_1dx_2 + \int_\Omega |u|^{\gamma+2}\Big(\frac{k+2}{\gamma+2} - \frac{k}{2}\Big)\, dx_1dx_2 = \frac{1}{2}\int_{\partial\Omega} \Big(\frac{\partial u}{\partial \nu}\Big)^2 (\nu_1^2 + x_1^{2k}\nu_2^2)\big\{x_1\nu_1 + (k+1)x_2\nu_2\big\}\, ds.$$

If $\gamma > \frac{4}{k}$ or $\lambda < 0$, we have

$$\lambda \int_\Omega u^2\, dx_1dx_2 + \int_\Omega |u|^{\gamma+2}\Big(\frac{k+2}{\gamma+2} - \frac{k}{2}\Big)\, dx_1dx_2 < 0 \quad (\text{if } u \not\equiv 0),$$

which leads to a contradiction. If $\gamma = \frac{4}{k}$ and $\lambda = 0$, we have

$$\frac{1}{2}\int_{\partial\Omega} \Big(\frac{\partial u}{\partial \nu}\Big)^2 (\nu_1^2 + x_1^{2k}\nu_2^2)\big\{x_1\nu_1 + (k+1)x_2\nu_2\big\}\, ds = 0.$$

Thus, $\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \equiv 0$. From the uniqueness theorem by Aronszain-Cordes it then follows that $u \equiv 0$. The only trouble, when $x_1 = 0$, can be eliminated by using $u \in C^{0,\alpha}(\overline{\Omega})$. □

Remark 1. If $0 \notin \Omega$ Theorem 3 may be not true. In the case when $\Omega \cap \{-\varepsilon < x_1 < \varepsilon\} = \emptyset$ one can prove the existence theorem for any polynomial growth of $g(u)$ by using the usual Sobolev imbedding theorem.

Now we need to introduce some definitions.

Definition 2. By $S_1^p(\Omega)$ $(1 \le p < \infty)$ we denote the set of all functions $u \in L^p(\Omega)$ such that $\dfrac{\partial u}{\partial x_1} \in L^p(\Omega)$ and $x_1^k \dfrac{\partial u}{\partial x_2} \in L^p(\Omega)$. For the norm we take

$$\|u\|_{S^p_{1,0}(\Omega)} = \Big\{ \int\limits_\Omega \Big(|u|^p + \Big|\frac{\partial u}{\partial x_1}\Big|^p + \Big|x_1^k \frac{\partial u}{\partial x_2}\Big|^p \Big)\, dx_1 dx_2 \Big\}^{\frac{1}{p}}.$$

If $p = 2$ we can also define scalar product in $S_1^2(\Omega)$ as follows

$$\big(u, v\big)_{S_1^2(\Omega)} = \big(u, v\big)_{L^2(\Omega)} + \Big(\frac{\partial u}{\partial x_1}, \frac{\partial v}{\partial x_1}\Big)_{L^2(\Omega)} + \Big(x_1^k \frac{\partial u}{\partial x_2}, x_1^k \frac{\partial v}{\partial x_2}\Big)_{L^2(\Omega)}.$$

Theorem 4. *$S_1^p(\Omega)$ is a Banach space. $S_1^2(\Omega)$ is a Hilbert space.*

Proof. It is obvious. For details see [7].

Definition 3. The space $S^p_{1,0}(\Omega)$ is defined as a closure of $C_0^1(\Omega)$ in the space $S_1^p(\Omega)$.

The following theorem is due to Rothschild and Stein. For a definition of $L_s^p(\Omega)$ see [3].

Theorem 5. *Assume $1 \le p < \infty$, then $S^p_{1,0}(\Omega) \subset L^p_{\frac{1}{k+1}}(\Omega)$.*

From this theorem we conclude that the imbedding map $S^p_{1,0}(\Omega) \subset L^p(\Omega)$ is compact, because we can always shrink our domain to $\widetilde{\Omega} \supset \Omega$.

Theorem 6. *Assume $1 \le p < k+2$. Then $S^p_{1,0}(\Omega) \subset L^{\frac{(k+2)p}{k+2-p}-\tau}(\Omega)$ for every positive small τ.*

Proof. We will act in the spirit of Gagliardo and Nirenberg (see [9], [10] and [11]). It suffices to prove the following inequality:

$$\|u\|_{L^{\frac{(k+2)p}{k+2-p}-\tau}(\Omega)} \le C\|u\|_{S^p_{1,0}(\Omega)} \quad \text{for every} \quad u \in C_0^1(\Omega), \tag{4}$$

where $C_0^1(\Omega)$ denotes the set of functions in $C^1(\Omega)$ with a compact support in Ω. First we prove the estimate (4) for $p = 1$. Choose M sufficiently large such that the square $[-M, +M] \times [-M, +M]$ contains Ω. We have

$$u(x_1, x_2) = \int\limits_{-M}^{x_1} \frac{\partial u}{\partial t}(t, x_2)\, dt, \quad \text{for every} \quad (x_1, x_2) \in \Omega.$$

Therefore

$$|u(x_1, x_2)| \leq \int\limits_{-M}^{+M} \Big|\frac{\partial u}{\partial t}(t, x_2)\Big|\, dt, \quad \text{for every} \quad (x_1, x_2) \in \Omega. \tag{5}$$

Analogously

$$|u(x_1, x_2)| \leq \int\limits_{-M}^{+M} \Big|\frac{\partial u}{\partial t}(x_1, t)\Big|\, dt, \quad \text{for every} \quad (x_1, x_2) \in \Omega.$$

Hence

$$|u(x_1, x_2)|^\delta \leq \Big(\int\limits_{-M}^{+M} \Big|\frac{\partial u}{\partial t}(x_1, t)\Big|\, dt\Big)^\delta, \tag{6}$$

for every $(x_1, x_2) \in \Omega$ and a positive δ. Multiplying (5) by (6) and integrating over the square $[-M, +M] \times [-M, +M]$ gives

$$\begin{aligned}
&\int\limits_\Omega |u|^{1+\delta}\, dx_1 dx_2 \leq \\
&\quad \leq \int\limits_{-M}^{+M} \int\limits_{-M}^{+M} \Big\{ \Big(\int\limits_{-M}^{+M} \Big|\frac{\partial u}{\partial t}(x_1, t)\Big|\, dt\Big)^\delta \Big(\int\limits_{-M}^{+M} \Big|\frac{\partial u}{\partial t}(t, x_2)\Big|\, dt\Big)\Big\}\, dx_1 dx_2 \\
&\quad = \int\limits_{-M}^{+M} \Big(\int\limits_{-M}^{+M} \Big|\frac{\partial u}{\partial t}(x_1, t)\Big|\, dt\Big)^\delta \Big\{ \int\limits_{-M}^{+M} \Big(\int\limits_{-M}^{+M} \Big|\frac{\partial u}{\partial t}(t, x_2)\Big|\, dt\Big)\, dx_2 \Big\}\, dx_1 \\
&\qquad\qquad = \int\limits_{-M}^{+M} \int\limits_{-M}^{+M} \Big|\frac{\partial u}{\partial x_1}\Big|\, dx_1 dx_2 \int\limits_{-M}^{+M} \Big(\int\limits_{-M}^{+M} \Big|\frac{\partial u}{\partial x_2}(x_1, x_2)\Big|\, dx_2\Big)^\delta\, dx_1
\end{aligned}$$

$$\le \int_{-M}^{+M}\int_{-M}^{+M} \Big|\frac{\partial u}{\partial x_1}\Big|\, dx_1 dx_2 \Big(\int_{-M}^{+M} |x_1|^{\frac{k\delta}{1-\delta}}\ dx_1\Big)^{1-\delta}.$$

$$\cdot\Big\{\int_{-M}^{+M} |x_1|^k \Big(\int_{-M}^{+M} \Big|\frac{\partial u}{\partial x_2}(x_1,x_2)\Big|\, dx_2\Big)\, dx_1\Big\}^{\delta}.$$

We choose $0<\delta<\dfrac{1}{k+1}$, then the last term is finite and hence

$$\int_\Omega |u|^{1+\delta}\, dx_1 dx_2 \le C\Big\|x_1^k \frac{\partial u}{\partial x_2}\Big\|^{\delta}_{L^1(\Omega)} \Big\|\frac{\partial u}{\partial x_1}\Big\|^{1}_{L^1(\Omega)}.$$

Thus

$$\text{(7)}\quad \|u\|_{L^{1+\delta}(\Omega)} \le C\Big\|x_1^k \frac{\partial u}{\partial x_2}\Big\|^{\frac{\delta}{1+\delta}}_{L^1(\Omega)} \Big\|\frac{\partial u}{\partial x_1}\Big\|^{\frac{1}{1+\delta}}_{L^1(\Omega)}$$
$$\le C\Big(\Big\|x_1^k \frac{\partial u}{\partial x_2}\Big\|_{L^1(\Omega)} + \Big\|\frac{\partial u}{\partial x_1}\Big\|_{L^1(\Omega)}\Big).$$

For an arbitrary p, we put $|u|^\gamma (\gamma>1)$ into (7) and obtain

$$\||u|^\gamma\|_{L^{1+\delta}(\Omega)} \le C\Big(\Big\|x_1^k |u|^{\gamma-1}\frac{\partial u}{\partial x_2}\Big\|_{L^1(\Omega)} + \Big\||u|^{\gamma-1}\frac{\partial u}{\partial x_1}\Big\|_{L^1(\Omega)}\Big) \le$$
$$\le C\big(\||u|^{\gamma-1}\|_{L^{p'}(\Omega)}\big)\Big(\Big\|x_1^k \frac{\partial u}{\partial x_2}\Big\|_{L^p(\Omega)} + \Big\|\frac{\partial u}{\partial x_1}\Big\|_{L^p(\Omega)}\Big).$$

Choosing $\gamma = \dfrac{p}{1-\delta p+\delta}$ gives

$$\|u\|_{L^{\frac{(1+\delta)p}{1-p\delta+\delta}}(\Omega)} \le C\Big(\Big\|x_1^k \frac{\partial u}{\partial x_2}\Big\|_{L^p(\Omega)} + \Big\|\frac{\partial u}{\partial x_1}\Big\|_{L^p(\Omega)}\Big).$$

This completes the proof of the theorem, since we can choose δ sufficiently near $\dfrac{1}{1+k}$. □

Remark 2. The above proof is applied to imbedding theorems for wider classes of energy spaces associated with hypoelliptic operators. For example, we can prove imbedding theorems for energy spaces associated with the operator $\Delta_x + \lambda(x)\Delta_y$ where $x \in \mathbf{R}^n, y \in \mathbf{R}^m$ or with the operator $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + x^2 \dfrac{\partial^2}{\partial z^2}$ in $\mathbf{R}^3$.

Remark 3. It is obvious from the proof of Theorem 6 that the two norms

$$\|u\|_{S^p_{1,0}(\Omega)} \quad \text{and} \quad \|u\|_{\tilde{S}^p_{1,0}(\Omega)} = \Big\{ \int\limits_\Omega \Big(\Big| \frac{\partial u}{\partial x_1} \Big|^p + \Big| x_1^k \frac{\partial u}{\partial x_2} \Big|^p \Big) \, dx_1 dx_2 \Big\}^{\frac{1}{p}}$$

are equivalent in $S^p_{1,0}(\Omega)$.

Counterexample. In the case when $p < k+2$, the imbedding $S^p_{1,0}(\Omega) \subset L^{\frac{(k+2)p}{k+2-p}+\tau}(\Omega)$ is no longer true for any positive τ. Indeed, denote $\dfrac{(k+2)p}{k+2-p}$ $+\tau$ by $p(\tau)$. Take $0 \not\equiv \phi(x_1, x_2) \in C_0^\infty(\Omega)$. Choose Θ sufficiently large such that $\phi_\theta(x_1, x_2)\phi(\theta x_1, \theta^{k+1} x_2) \in C_0^\infty(\Omega)$ for all $\theta \geq \Theta$. We shall compare the ratio

$$A_\theta = \frac{\|\phi_\theta\|_{L^{p(\tau)}(\Omega)}}{\Big\{ \int\limits_\Omega \Big(\Big| \frac{\partial \phi_\theta}{\partial x_1} \Big|^p + \Big| x_1^k \frac{\partial \phi_\theta}{\partial x_2} \Big|^p \Big) \, dx_1 dx_2 \Big\}^{\frac{1}{p}}}$$

with

$$A = \frac{\|\phi\|_{L^{p(\tau)}(\Omega)}}{\Big\{ \int\limits_\Omega \Big(\Big| \frac{\partial \phi}{\partial x_1} \Big|^p + \Big| x_1^k \frac{\partial \phi}{\partial x_2} \Big|^p \Big) \, dx_1 dx_2 \Big\}^{\frac{1}{p}}} \, .$$

We have

$$(8) \quad \|\phi_\theta\|_{L^{p(\tau)}(\Omega)} = \Big\{ \int\limits_\Omega |\phi_\theta(x_1, x_2)|^{p(\tau)} \, dx_1 dx_2 \Big\}^{\frac{1}{p(\tau)}}$$

$$= \Big\{ \int\limits_\Omega \frac{1}{\theta^{k+2}} \, |\phi(x_1, x_2)|^{p(\tau)} \, dx_1 dx_2 \Big\}^{\frac{1}{p(\tau)}} = \frac{1}{\theta^{\frac{k+2}{p(\tau)}}} \|\phi\|_{L^{p(\tau)}(\Omega)}.$$

On the other hand

$$(9) \quad \Big\{ \int\limits_\Omega \Big(\Big| \frac{\partial \phi_\theta}{\partial x_1} \Big|^p + \Big| x_1^k \frac{\partial \phi_\theta}{\partial x_2} \Big|^p \Big) \, dx_1 dx_2 \Big\}^{\frac{1}{p}} = \Big\{ \int\limits_\Omega \Big(\Big| \theta \frac{\partial \phi(\theta x_1, \theta^{k+1} x_2)}{\partial x_1} \Big|^p$$

$$+ \Big| (\theta x_1)^k \cdot \frac{1}{\theta^k} \cdot \theta^{k+1} \frac{\partial \phi(\theta x_1, \theta^{k+1} x_2)}{\partial x_2} \Big|^p \Big) \, dx_1 dx_2 \Big\}^{\frac{1}{p}} =$$

$$= \Big\{ \int\limits_\Omega \theta^{p-(k+2)} \Big(\Big|\frac{\partial \phi}{\partial x_1}\Big|^p + \Big|x_1^k \frac{\partial \phi}{\partial x_2}\Big|^p \Big) \, dx_1 dx_2 \Big\}^{\frac{1}{p}}$$

$$= \theta^{1-\frac{k+2}{p}} \Big\{ \int\limits_\Omega \Big(\Big|\frac{\partial \phi}{\partial x_1}\Big|^p + \Big|x_1^k \frac{\partial \phi}{\partial x_2}\Big|^p \Big) \, dx_1 dx_2 \Big\}^{\frac{1}{p}}.$$

Combining (8) and (9) yields

$$A_\theta = \frac{\theta^{-\frac{k+2}{p(\tau)}} \, \|\phi\|_{L^{p(\tau)}(\Omega)}}{\theta^{1-\frac{k+2}{p}} \Big\{ \int\limits_\Omega \Big(\Big|\frac{\partial \phi}{\partial x_1}\Big|^p + \Big|x_1^k \frac{\partial \phi}{\partial x_2}\Big|^p \Big) \, dx_1 dx_2 \Big\}^{\frac{1}{p}}} = \theta^{\frac{k+2}{p} - 1 - \frac{k+2}{p(\tau)}} A.$$

Since $\dfrac{k+2}{p} - 1 - \dfrac{k+2}{p(\tau)} > 0$, we obtain $A_\theta \longrightarrow \infty$ as $\theta \longrightarrow \infty$. This leads to a contradiction, as the expression $\Big\{ \int\limits_\Omega \Big(\Big|\frac{\partial \phi}{\partial x_1}\Big|^p + \Big|x_1^k \frac{\partial \phi}{\partial x_2}\Big|^p \Big) \, dx_1 dx_2 \Big\}^{\frac{1}{p}}$ is equivalent to $\|\phi\|_{S^p_{1,0}(\Omega)}$.

Theorem 7. *Assume* $1 \le p < k+2$. *Then the imbedding map* $S^p_{1,0}(\Omega)$ *into* $L^{\frac{(k+2)p}{k+2-p} - \tau}(\Omega)$ *is compact for every positive small* τ.

Proof. Combining Theorem 6, Theorem 5 and the interpolation inequality for $L^p(\Omega)$ gives the statement. For details see [7].

Theorem 8. *Assume* $p > k+2$. *Then* $S^p_{1,0}(\Omega) \subset C^0(\overline{\Omega})$.

Proof. It suffices to prove the following estimate

$$\sup_{x \in \Omega} |u| \le C \|u\|^p_{S^p_{1,0}(\Omega)}, \quad \text{for every} \quad u \in C^1_0(\Omega).$$

First, we assume $\mathrm{Vol}(\Omega) = 1$. By the inequality (7) we have

$$\begin{aligned} \||u|^\gamma\|_{L^{\frac{k+2}{k+1} - \tau}(\Omega)} &\le C \, \||u|^\gamma\|_{S^1_1(\Omega)} \\ &= C\gamma \int\limits_\Omega |u|^{\gamma-1} . \Big(\Big|\frac{\partial u}{\partial x_1}\Big| + \Big|x_1^k \frac{\partial u}{\partial x_2}\Big| \Big) \, dx_1 dx_2 \\ &\le C\gamma \|u\|_{S^p_{1,0}(\Omega)} . \, \big\||u|^{\gamma-1}\big\|_{L^{\frac{p}{p-1}}(\Omega)}, \end{aligned}$$

for $\gamma \ge 1$ and a small positive τ.

That is

$$\|u\|_{L^{\gamma\left(\frac{k+2}{k+1}-\tau\right)}(\Omega)} \leq \left(C\gamma\right)^{\frac{1}{\gamma}} \|u\|^{\frac{1}{\gamma}}_{S^p_{1,0}(\Omega)}. \| u\|^{1-\frac{1}{\gamma}}_{L^{\frac{p(\gamma-1)}{p-1}}(\Omega)} \leq$$
$$\leq \left(C\gamma\right)^{\frac{1}{\gamma}} \|u\|^{\frac{1}{\gamma}}_{S^p_{1,0}(\Omega)}. \| u\|_{L^{\frac{p\gamma}{p-1}}(\Omega)}, \quad \text{since } \mathrm{Vol}(\Omega) = 1.$$

Therefore we get

$$\Big\|\frac{u}{\|u\|_{S^p_{1,0}(\Omega)}}\Big\|_{L^{\gamma\left(\frac{k+2}{k+1}-\tau\right)}(\Omega)} \leq \left(C\gamma\right)^{\frac{1}{\gamma}} \Big\| \frac{u}{\|u\|_{S^p_{1,0}(\Omega)}}\Big\|^{1-\frac{1}{\gamma}}_{L^{\frac{p\gamma}{p-1}}(\Omega)}.$$

Choose τ such that $\dfrac{k+2}{k+1} - \tau > \dfrac{p}{p-1}\cdot$ Let us substitude the value ξ^ρ $(\rho = 1, 2, \dots)$ to γ, where $\xi = \Big(\dfrac{k+2}{k+1} - \tau\Big)\dfrac{p-1}{p} > 1$ by the hypotheses of the theorem. Then we have

$$\Big\|\frac{u}{\|u\|_{S^p_{1,0}(\Omega)}}\Big\|_{L^{\left(\frac{k+2}{k+1}-\tau\right).\xi^\rho}(\Omega)} \leq \left(C\xi^\rho\right)^{\frac{1}{\xi^\rho}} \Big\| \frac{u}{\|u\|_{S^p_{1,0}(\Omega)}}\Big\|^{1-\xi^{-\rho}}_{L^{\left(\frac{k+2}{k+1}-\tau\right).\xi^{\rho-1}}(\Omega)},$$

for $\rho = 1, 2, \dots$. Iteracting from $\rho = 1$ and using the fact that

$$\Big\|\frac{u}{\|u\|_{S^p_{1,0}(\Omega)}}\Big\|_{L^{\left(\frac{k+2}{k+1}-\tau\right).\xi}(\Omega)} \leq 1,$$

we obtain

$$\Big\|\frac{u}{\|u\|_{S^p_{1,0}(\Omega)}}\Big\|_{L^{\xi^\rho}(\Omega)} \leq \Big\|\frac{u}{\|u\|_{S^p_{1,0}(\Omega)}}\Big\|_{L^{\left(\frac{k+2}{k+1}-\tau\right).\xi^\rho}(\Omega)} \leq \left(C\xi\right)^{\Sigma\rho.\xi^{-\rho}} = C.$$

Consequently, as $\rho \longrightarrow \infty$ we have

$$\sup_{x\in\Omega}\Big|\frac{u}{\|u\|_{S^p_{1,0}(\Omega)}}\Big| \leq C,$$

and hence $\sup\limits_{x\in\Omega} u \leq C\|u\|_{S^p_{1,0}(\Omega)}$. To eliminate the restriction $\mathrm{Vol}\,(\Omega) = 1$, we consider the transformation

$$y_1 = \Big\{\mathrm{Vol}(\Omega)\Big\}^{-\frac{1}{k+2}} x_1 \quad \text{and} \quad y_2 = \Big\{\mathrm{Vol}(\Omega)\Big\}^{-\frac{k+1}{k+2}} x_2.$$

This leads to

$$\sup_{x\in\Omega}|u(x)| = \sup_{y\in\tilde{\Omega}}|u(y)| \le C\,\|u(y)\|_{S^p_{1,0}\left(\tilde{\Omega}\right)} \le C\|u(x)\|_{S^p_{1,0}(\Omega)}. \qquad \square$$

From now on we suppose $g(u)$ has only polynomial growth.

Definition 5. A function $u \in S^2_{1,0}(\Omega)$ is called a weak solution of the problem (1)-(2), if the identity

$$\int\limits_{\Omega} \frac{\partial u}{\partial x_1}\cdot\frac{\partial \varphi}{\partial x_1}\,dx_1dx_2 + \int\limits_{\Omega} x_1^{2k}\frac{\partial u}{\partial x_2}\cdot\frac{\partial \varphi}{\partial x_2}\,dx_1dx_2 - \int\limits_{\Omega} g(u)\varphi\,dx_1dx_2 = 0$$

holds for every $\varphi \in C_0^\infty(\Omega)$.

Now we can state our existence theorem.

Theorem 9. *Assume that $g(u)$ satisfies the following conditions*

(1) $g \in C^{0,\alpha}_{loc}(\mathbf{R})$,
(2) $|g(u)| \le C(1+|u|^m)$ *with* $1 < m < \dfrac{k+4}{k}$,
(3) $g(u) = \bar{\bar{o}}(u)$ *as* $u \longrightarrow 0$,
(4) *There exists an A such that for $|u| \ge A, G(u) \le \mu g(u)u$, where* $\mu \in [0, \frac{1}{2})$.

Then the problem (1)-(2) *always has a weak non-trivial solution.*

Proof. Consider the following functional in $S^2_{1,0}(\Omega)$

$$\Phi(u) = \frac{1}{2}\int\limits_{\Omega}\left(\left|\frac{\partial u}{\partial x_1}\right|^2 + \left|x_1^k\frac{\partial u}{\partial x_2}\right|^2\right)dx_1dx_2 - \int\limits_{\Omega} G(u)\,dx_1dx_2.$$

From the conditions on $g(u)$ one concludes that Φ satisfies the hypotheses (I_1), (I_2), (I_3) in the paper [12]. So, Φ has a non-trivial critical point, which will be a weak solution for our problem. $\square$

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