REGULARIZED SOLUTIONS OF A CAUCHY PROBLEM FOR THE LAPLACE EQUATION IN AN IRREGULAR LAYER: A THREE DIMENSIONAL MODEL

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1. Introduction

Consider the problem of finding a function $u$, harmonic in the domain $D$ defined by

$$D = \{(x,y,z) : -\infty < x, y < \infty, 0 < z < \phi(x,y)\}$$

and continuous on $\overline{D}$, given $u, u_x, u_y$ and $u_z$ on the portion of the boundary represented by the surface $z = \phi(x,y)$. Here $\phi$ is of class $C^1$.

This is a Cauchy problem for the Laplace equation and is well known as an ill-posed problem, i.e., solutions of the problem do not always exist and, whenever they do exist, there is no continuous dependence on the given data. The reader is referred to [1, 2, 3, 4, 6, 7, 9, 10] on the earlier literature on the Cauchy problem for the Laplace equation.

For numerical computations, ill-posed problems need to be regularized. A regularized solution is a stable approximate solution. An important question arises as to how close a regularized solution is to an exact solution, especially when the measured data is affected with noise. The problem of regularisation of the Cauchy problem for the Laplace equation in a rather general context was considered, e.g., in [5]; using the method of quasi-reversibility, the authors (loc. cit.) stabilized the problem, but no error estimates are given. We shall take the approach followed in [1] by taking the boundary value $v(x,y) = u(x,y,0)$ as our unknown and we shall show that if the discrepancy between the given values of $u, u_x, u_y, u_z$ on the surface $z = \phi(x,y)$ and their exact values is of the order $\varepsilon$, then, assuming

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the exact solution $v_0(x, y)$ to be smooth (in $H^1(R^2)$), the discrepancy between the regularized solution and the exact solution $v_0(x, y)$ is of the order $(ln1/\varepsilon)^{-1}$ as $\varepsilon \to 0$.

2. INTEGRAL EQUATION FORMULATION AND REGULARIZATION

First, we set some notations:

$$u_x(x, y, \phi(x, y)) = f(x, y)$$
$$u_y(x, y, \phi(x, y)) = g(x, y)$$
$$u_z(x, y, \phi(x, y)) = h(x, y)$$
$$u(x, y, \phi(x, y)) = u_1(x, y)$$

These functions, we recall, are given. Let us put

$$\Gamma(x, y, z; \xi, \eta, \zeta) = \frac{1}{4\pi} \cdot \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}},$$
$$G(x, y, z; \xi, \eta, \zeta) = \Gamma(x, y, z; \xi, \eta, \zeta) - \Gamma(x, y, z; \xi, \eta, -\zeta),$$

where $\Gamma$ is a fundamental solution of the Laplace equation and $G$ is the Green’s function for the Laplacian corresponding to a Dirichlet condition at the boundary $z = 0$.

It is sufficient to determine $u(x, y, 0) = v(x, y)$. Once this is done, $u(x, y, z)$ is known. We shall derive an integral equation in $v$. In order to do this, suppose that

(i) $\frac{\partial \phi}{\partial x}(x, y) = \frac{\partial \phi}{\partial y}(x, y) = 0$ for large $r = \sqrt{x^2 + y^2}$.

(ii) $f(x, y), g(x, y), h(x, y), u_1(x, y)$ tend to 0 sufficiently fast, say as

$$\frac{1}{\sqrt{x^2 + y^2}} \text{ for } \sqrt{x^2 + y^2} \to \infty.$$

(iii) $\sqrt{1 + x^2 + y^2} \cdot v(x, y)$ is in $L^2(R^2)$.

Integrating Green’s identity on $D_\varepsilon, \varepsilon > 0$, where $D_\varepsilon = D \setminus D'_\varepsilon$ and $D'_\varepsilon$ is the closed ball in $D$ of radius $\varepsilon$ centered at $(x, y, z)$, and let $\varepsilon \to 0$, we then have, after some rearrangements

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{zv(\xi, \eta)}{((x-\xi)^2 + (y-\eta)^2 + z^2)^{3/2}} d\xi d\eta =$$
\begin{equation}
\begin{aligned}
= u(x, y, z) - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y, z; \xi, \eta, \phi(\xi, \eta)) f_1(\xi, \eta) d\xi d\eta \\
+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_1(x, y, z; \xi, \eta, \phi(\xi, \eta)) u_1(\xi, \eta) d\xi d\eta,
\end{aligned}
\end{equation}

where $-\infty < x, y < \infty$, $0 < z < \phi(x, y)$,

\[ f_1(\xi, \eta) = h(\xi, \eta) - f(\xi, \eta) \frac{\partial}{\partial \xi} \phi(\xi, \eta) - g(\xi, \eta) \frac{\partial}{\partial \eta} \phi(\xi, \eta) \]

and

\[ G_1(x, y, z; \xi, \eta, \phi(\xi, \eta)) = G_\zeta(x, y, z; \xi, \eta, \phi(\xi, \eta)) \]

\[ - G_\xi(x, y, z; \xi, \eta, \phi(\xi, \eta)) \frac{\partial}{\partial \xi} \phi(\xi, \eta) - G_\eta(x, y, z; \xi, \eta, \phi(\xi, \eta)) \frac{\partial}{\partial \eta} \phi(\xi, \eta). \]

Letting $z \to \phi(x, y)$ in (3), we have (see [8])

\begin{equation}
\begin{aligned}
\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\phi(x, y)v(\xi, \eta)}{((x - \xi)^2 + (y - \eta)^2 + \phi^2(x, y))^{3/2}} d\xi d\eta = \\
= \frac{1}{2} u_1(x, y) - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y, \phi(x, y); \xi, \eta, \phi(\xi, \eta)) f_1(\xi, \eta) d\xi d\eta \\
+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_1(x, y, \phi(x, y); \xi, \eta, \phi(\xi, \eta)) u_1(\xi, \eta) d\xi d\eta,
\end{aligned}
\end{equation}

which is an integral equation in $v(x, y)$. We shall convert (5) into a convolution equation.

We note that the function

\[ H(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{zv(\xi, \eta)}{((x - \xi)^2 + (y - \eta)^2 + z^2)^{3/2}} d\xi d\eta \]
is harmonic in the upper half space \( z > 0 \). The value \( H(x, y, \phi(x, y)) \) is then the right hand side of (5). Furthermore, we can calculate \( \frac{\partial H}{\partial n}(x, y, \phi(x, y)) \) as the limit from below of the directional derivative of the right hand side of (3) when \((x, y, z) \to (x, y, \phi(x, y))\), \( \vec{n} \) being the inner unit normal to the surface \( z = \phi(x, y) \).

Let \( \lambda(x, y) = H(x, y, \phi(x, y)) \), \( \mu(x, y) = \frac{\partial H}{\partial n}(x, y, \phi(x, y)) \)

Then \( \lambda(x, y) \) and \( \mu(x, y) \) are defined on \( \mathbb{R}^2 \), and depend continuously on \( \phi(x, y) \), \( \frac{\partial \phi}{\partial x}(x, y) \), \( \frac{\partial \phi}{\partial y}(x, y) \), \( u_1(x, y) \), \( \frac{\partial u_1}{\partial x}(x, y) \), \( \frac{\partial u_1}{\partial y}(x, y) \), \( f(x, y) \), \( g(x, y) \) and \( h(x, y) \) in the \( L^2 \)-sense. Furthermore, \( H(x, y, z) \) can be represented as a potential with densities \( \lambda, \mu \) on the domain \( z > \phi(x, y) \). In fact, integrating Green’s identity in the domain

\[
D_R = \{(x, y, z) : x^2 + y^2 < R^2, \phi(x, y) < z < R\}
\]

and letting \( R \to \infty \), we get

\[
H(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma(x, y, z; \xi, \eta, \phi(\xi, \eta)) \mu(\xi, \eta) d\xi d\eta
- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma_1(x, y, z; \xi, \eta, \phi(\xi, \eta)) \lambda(\xi, \eta) d\xi d\eta
\]

for \(-\infty < x, y < \infty, z > \phi(x, y)\), where

\[
\Gamma_1(x, y, z; \xi, \eta, \phi(\xi, \eta)) = \Gamma_{\xi}(x, y, z; \xi, \eta, \phi(\xi, \eta)) \frac{\partial \phi}{\partial \xi}(\xi, \eta)
+ \Gamma_{\eta}(x, y, z; \xi, \eta, \phi(\xi, \eta)) \frac{\partial \phi}{\partial \eta}(\xi, \eta) - \Gamma_{\xi}(x, y, z; \xi, \eta, \phi(\xi, \eta)).
\]

Note that as \( R \to \infty \), the integral on

\[
C_R = \{(x, y, z) : x^2 + y^2 = R^2, \phi(x, y) < z < R\}
\cup \{(x, y, R) : x^2 + y^2 < R^2\}
\]

tends to 0 as a consequence of our assumption on \( v \) (i.e., \( \sqrt{1 + x^2 + y^2} = v(x, y) \) is in \( L^2(R^2) \)).
Evaluating $H(x, y, z)$ at $(x, y, k)$ where $k$ is a fixed number greater than $\phi(x, y)$ for all $(x, y)$ in $\mathbb{R}^2$, we have by (6)

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{kv(\xi, \eta)}{((x - \xi)^2 + (y - \eta)^2 + k^2)^{3/2}} d\xi d\eta =$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma(x, y, k; \xi, \eta, \phi(\xi, \eta)) \mu(\xi, \eta) d\xi d\eta$$

$$- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma_1(x, y, k; \xi, \eta, \phi(\xi, \eta)) \lambda(\xi, \eta) d\xi d\eta ,$$

Let

$$F(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma(x, y, k; \xi, \eta, \phi(\xi, \eta)) v(\xi, \eta) d\xi d\eta$$

$$- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma_1(x, y, k; \xi, \eta, \phi(\xi, \eta)) \lambda(\xi, \eta) d\xi d\eta .$$

Then we have a convolution integral equation in $v(\xi, \eta)$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{kv(\xi, \eta)}{((x - \xi)^2 + (y - \eta)^2 + k^2)^{3/2}} = F(x, y), \quad \forall (x, y) \in \mathbb{R}^2,$$

which is an integral equation of first kind, and we know that this problem is ill-posed. We shall construct a family $(v_\beta)$, $\beta > 0$, of regularized solutions (see [11]), and we pick a regularized solution that is “close” to the exact solution. We recall that, by regularized solution we mean a function that is stable with respect to variations in the right hand side of (7).

We now state and prove our main result.

**Theorem.** Suppose the exact solution $v_0$ of (7) in the right hand side is in $H^1(\mathbb{R}^2)$ and let

$$|F_0 - F|_2 < \varepsilon , \quad |. |_2 = L^2(\mathbb{R}^2) - \text{norm}.$$
Then there exists a regularized solution \( v_\varepsilon \) of (7) such that

\[
|v_\varepsilon - v_0|_2 \leq K \left( \ln \left( \frac{1}{\varepsilon} \right) \right)^{-1} \quad \text{for } \varepsilon \to 0,
\]

where \( K \) is a constant depending only on the \( H^1 \)-norm of \( v_0 \).

**Proof.** Letting

\[
G(x, y) = \frac{k}{(x^2 + y^2 + k^2)^{3/2}},
\]

we have

\[
\hat{G}(s, t) = \exp \left( -k \sqrt{s^2 + t^2} \right),
\]

where

\[
\hat{G}(s, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y)e^{-i(xs+yt)}dxdy.
\]

For \( v \) in \( L^2(R^2) \), we then have by (7)

\[
\hat{G}(s, t) \hat{v}(s, t) = \hat{F}(s, t).
\]

Now let \( v_0 \in H^1(R^2) \) be the exact solution of the equation

(8) \quad \hat{G}(s, t) \hat{v}_0(s, t) = \hat{F}_0(s, t), \quad \forall (s, t) \in R^2,

with \( F \) and \( F_0 \) in \( L^2(R^2) \) such that

(9) \quad |F - F_0|_2 < \varepsilon.

For every \( \beta > 0 \), the function

(10) \quad \psi(s, t) = \frac{\hat{G}(s, t) \hat{F}(s, t)}{\beta + \hat{G}^2(s, t)}

is in \( L^2(R^2) \). Let

\[
v_\beta(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(s, t)e^{i(xs+yt)}dxdt.
\]
Then \( v_\beta \in L^2(\mathbb{R}^2) \) and, by (10), \( v_\beta \) satisfies the equation

\[
\beta \dot{v}_\beta(s, t) + \dot{G}^2(s, t)\dot{v}_\beta(s, t) = \dot{G}(s, t)\dot{F}(s, t), \quad \forall (s, t) \in \mathbb{R}^2,
\]

and depends continuously on \( F(s, t) \).

We now derive error estimates. From (8) and (11), we have

\[
\beta(\dot{v}_\beta(s, t) - \dot{v}_0(s, t)) + \dot{G}^2(s, t)(\dot{v}_\beta(s, t) - \dot{v}_0(s, t)) =
\]

\[
- \beta\dot{v}_0(s, t) + \dot{G}(s, t)(\dot{F}(s, t) - \dot{F}_0(s, t)), \quad \forall (s, t) \in \mathbb{R}^2.
\]

We multiply both sides of (12) by \( \dot{v}_\beta(s, t) - \dot{v}_0(s, t) \) and then integrate on \( \mathbb{R}^2 \). Then we have

\[
\beta |\dot{v}_\beta - \dot{v}_0|^2 + |\dot{G}(\dot{v}_\beta - \dot{v}_0)|^2
\]

\[
= - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \beta\dot{v}_0(s, t) \left( \dot{v}_\beta(s, t) - \dot{v}_0(s, t) \right) dsdt
\]

\[
+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dot{G}(s, t)(\dot{F}(s, t) - \dot{F}_0(s, t)) \left( \dot{v}_\beta(s, t) - \dot{v}_0(s, t) \right) dsdt
\]

\[
(13)
\]

\[
\leq \beta |\dot{v}_0|^2 |\dot{v}_\beta - \dot{v}_0|^2 + |\dot{F} - \dot{F}_0|^2 |\dot{v}_\beta - \dot{v}_0|^2.
\]

Let \( \beta = \varepsilon \) and note that \( |\dot{F} - \dot{F}_0|_2 = |F - F_0|_2 < \varepsilon \), we have

\[
\varepsilon |\dot{v}_\varepsilon - \dot{v}_0|^2 + |\dot{G}(\dot{v}_\varepsilon - \dot{v}_0)|^2 \leq \varepsilon (|\dot{v}_0|^2 + 1) |\dot{v}_\varepsilon - \dot{v}_0|^2.
\]

In particular

\[
|\dot{v}_\varepsilon - \dot{v}_0|^2 \leq (|\dot{v}_0|^2 + 1).
\]

Similarly, letting \( \beta = \varepsilon \) in (12) and multiplying both sides by \( (s^2 + t^2) \left( \dot{v}_\varepsilon(s, t) - \dot{v}_0(s, t) \right) \) and then integrating over \( \mathbb{R}^2 \), we have
\[
\varepsilon |\sqrt{s^2 + t^2} (\dot{v}_\varepsilon - \dot{v}_0) |^2 + |\hat{G} \sqrt{s^2 + t^2} (\dot{v}_\varepsilon - \dot{v}_0) |^2 \\
= - \int \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varepsilon \tilde{v}_0(s, t) (s^2 + t^2) (\dot{v}_\varepsilon(s, t) - \dot{v}_0(s, t)) \, ds \, dt \\
+ \int \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{s^2 + t^2} \cdot \hat{G}(s, t)(\dot{F}(s, t) \\
- \hat{F}_0(s, t)) \sqrt{s^2 + t^2} (\dot{v}_\varepsilon(s, t) - \dot{v}_0(s, t)) \, ds \, dt \\
\leq \varepsilon |D\tilde{v}_0|_2 |\sqrt{s^2 + t^2} (\dot{v}_\varepsilon - \dot{v}_0) |_2 \\
+ \frac{1}{k\varepsilon} |\hat{F} - \hat{F}_0|_2 |\sqrt{s^2 + t^2} (\dot{v}_\varepsilon - \dot{v}_0) |_2 \\
(16) \leq \varepsilon \left( |D\tilde{v}_0|_2 + \frac{1}{k\varepsilon} \right) |\sqrt{s^2 + t^2} (\dot{v}_\varepsilon - \dot{v}_0) |_2,
\]

where
\[
|D\tilde{v}_0|_2^2 \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (s^2 + t^2) \tilde{v}_0(s, t) ds \, dt.
\]

In particular,
\[
(17) \quad |\sqrt{s^2 + t^2} (\dot{v}_\varepsilon - \dot{v}_0) |_2 \leq \left( |D\tilde{v}_0|_2 + \frac{1}{k\varepsilon} \right).
\]

Since
\[
(18) \quad |v_\varepsilon - v_0|_2 = |\dot{v}_\varepsilon - \dot{v}_0|_2
\]

and
\[
|Dv_\varepsilon - Dv_0|_2 = |\sqrt{s^2 + t^2} (\dot{v}_\varepsilon - \dot{v}_0) |_2,
\]

from (15) and (17) we get
\[
(19) \quad \|v_\varepsilon - v_0\|_{H^1(R^2)} = |v_\varepsilon - v_0|_2 + |Dv_\varepsilon - Dv_0|_2 \leq K_1,
\]

where
\[
K_1 = |v_0|_2 + |Dv_0|_2 + 1 + \frac{1}{k\varepsilon} = \|v_0\|_{H^1(R^2)} + 1 + \frac{1}{k\varepsilon}.
\]
Now, for any \( t_\varepsilon > 0 \),
\[
\int_{s^2 + t^2 \leq t_\varepsilon^2} \int |\hat{v}_\varepsilon (s, t) - \hat{v}_0 (s, t)|^2 ds \, dt \\
\leq \int_{s^2 + t^2 \leq t_\varepsilon^2} \int \hat{G}^2 (s, t) |\hat{v}_\varepsilon (s, t) - \hat{v}_0 (s, t)|^2 ds \, dt \\
\leq e^{2kt_\varepsilon} \int \int \hat{G}^2 (s, t) |\hat{v}_\varepsilon (s, t) - \hat{v}_0 (s, t)|^2 ds \, dt \\
= e^{2kt_\varepsilon} \hat{G} (\hat{v}_\varepsilon - \hat{v}_0)^2 \\
\leq e^{2kt_\varepsilon} K_1 \varepsilon (|\hat{v}_0|_2 + 1) \\
\equiv K_2 \varepsilon e^{2kt_\varepsilon},
\]
(20)

\[
\int_{s^2 + t^2 > t_\varepsilon^2} \int |\hat{v}_\varepsilon (s, t) - \hat{v}_0 (s, t)|^2 ds \, dt \\
\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (s^2 + t^2) t_\varepsilon^{-2} |\hat{v}_\varepsilon (s, t) - \hat{v}_0 (s, t)|^2 ds \, dt \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{s^2 + t^2} (\hat{v}_\varepsilon (s, t) - \hat{v}_0 (s, t))^2 ds \, dt \\
\leq K_1 t_\varepsilon^{-2} \\
\leq K_2 t_\varepsilon^{-2},
\]
(21)

where \( K_2 = K_1 (|\hat{v}_0|_2 + 1) \).

Next consider the equation
\[
y^2 e^{2ky} = \frac{1}{\varepsilon}.
\]
(22)

The function \( h(y) = y^2 e^{2ky} \) is strictly increasing for \( y > 0 \) and \( h(R^+) = R^+ \). Then the equation (22) has a unique solution \( t_\varepsilon \) and \( t_\varepsilon \to \infty \) as \( \varepsilon \to 0 \). Hence, we have
\[
2(1 + k)t_\varepsilon \geq 2 \ln t_\varepsilon + 2kt_\varepsilon = \ln \frac{1}{\varepsilon}.
\]
Letting $\varepsilon < 1$, we have

\begin{equation}
(23) \quad t_{\varepsilon}^{-1} \leq 2(1 + k) \left( \ln \frac{1}{\varepsilon} \right) \left( \ln \frac{1}{\varepsilon} \right)^{-1}.
\end{equation}

By (20), (21) and (23) we have

\[ |v_{\varepsilon} - v_0|^2 \leq 2K_2t_{\varepsilon}^2 \leq K^2 \left( \ln \frac{1}{\varepsilon} \right)^{-2}, \]

where

\[ K^2 = 8(1 + k)^2K_2, \]

as desired. This completes the proof of the theorem.

REFERENCES


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