ACTA MATHEMATICA VIETNAMICA
Volume 23, Number 1, 1998, pp. 35–48

ON THE ALMOST PERIODIC n-COMPETING SPECIES PROBLEM

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ABSTRACT. We consider the n-dimensional Lotka-Volterra competition equations with almost periodic coefficients. Conditions for the existence of a globally asymptotically stable almost periodic solution with positive components are given. This is a generalization of a result in [5].

INTRODUCTION

Consider the Lotka-Volterra equations for n-competing species

$$u_i' = u_i\big[b_i(t) - \sum_{j=1}^{n} a_{ij}(t)u_j\big], \quad 1 \le i \le n, \tag{0.1}$$

where $n \ge 2$ and a_{ij}, $b_i : \mathbf{R} \to \mathbf{R}$ are continuous and bounded above and below by positive constants. Given a bounded function $g(t)$ on $(-\infty, +\infty)$, let g_L and g_M denote $\inf\limits_{t\in R}\{g(t)\}$ and $\sup\limits_{t\in R}\{g(t)\}$, respectively.

In [5] K. Gopalsamy considered the system (0.1) in which a_{ij}, b_i ($1 \le i, j \le n$) are assumed to be almost periodic. He showed that under the conditions

$$b_{i_L} > \sum_{j\in J_i} a_{ij_M}\big(b_{j_M}/a_{jj_L}\big), \quad i = 1, \dots, n, \tag{0.2}$$

and

$$a_{ii_L} > \sum_{j\in J_i} a_{ij_M}, \quad i = 1, \dots, n, \tag{0.3}$$

Received February 3, 1996; in revised form June 11, 1996.
1991 Mathematics Subject Classification. 34D05, 34C27, 92D40
Key words. Almost periodic solution, the Lotka-Volttera equation, the logistic equation egress point, retraction.
This work is financially supported in part by the National Basic Research Program in Natural Sciences KT04 1.3.7.

where $J_i = \{1, \dots, i-1, i+1, \dots, n\}$, the system (0.1) has a unique solution $u^0(t)$ such that each its component is almost periodic and bounded above and below by positive constants. Moreover, if $u(t)$ is a solution of (0.1) such that $u_i(t_0) > 0$ $(1 \le i \le n)$ for some $t_0 \in \mathbf{R}$, then $\lim_{t\to+\infty} \big(u_i(t) - u_i^0(t)\big) = 0$.

In this paper we will show that alone conditions (0.2) imply the assertion of the above mentioned theorem of K. Gopalsamy.

The case of $n = 2$ was treated by S. Ahmad [1]. It is well-known, for example in [2], that for $i = 1, \dots, n$ the logistic equation

$$U' = U\big[b_i(t) - a_{ii}(t)U\big], \tag{0.4i}$$

has a unique solution, say $U_i^0(t)$, defined on $(-\infty, +\infty)$ which is bounded above and below by positive constants.

Our main result is the following: *If*

(i) *There exists a positive number* ε_1 *such that*

$$b_i(t) \ge \sum_{j\in J_i} a_{ij}(t)U_j^0(t) + \varepsilon_1, \quad 1 \le i \le n, \quad t \in \mathbf{R}, \tag{0.5}$$

and

(ii) *There are positive constants* $\varepsilon_2, \alpha_1, \dots, \alpha_n$ *such that*

$$\alpha_i a_{ii}(t) \ge \sum_{j\in J_i} a_{ji}(t)\alpha_j + \varepsilon_2, \quad 1 \le i \le n, \quad t \in \mathbf{R}, \tag{0.6}$$

then the system (0.1) *has a unique solution* $u^0(t) = \big(u_1^0(t), \dots, u_n^0(t)\big)$ *defined on* $(-\infty, +\infty)$, *whose components are bounded above and below by positive constants. However,* $u_i(t) - u_i^0(t) \to 0$ *as* $t \to +\infty$ $(1 \le i \le n)$ *for any solution* $u(t) = \big(u_1(t), \dots, u_n(t)\big)$ *of* (0.1) *with* $u_i(t_0) > 0$ *for some* $t_0 \in \mathbf{R}$. *If, in addition,* a_{ij}, b_i $(1 \le i, j \le n)$ *are almost periodic then the above solution* $u^0(t)$ *is almost periodic.*

The case of $n = 2$ under conditions (0.5) and (0.6) was treated in [8]. The periodic case under conditions (0.5) and (0.6) was considered by A. Tineo and C. Alvarez [7]. The ecological significance of such a system is discussed in [4, 5].

1. Existence

In this section we do not assume the almost periodicity conditions on the coefficients a_{ij} and b_i $(1 \le i, j \le n)$. We shall prove the existence of the solution $u^0(t)$ as mentioned above. The following proposition was given by A. Tineo and C. Alvarez [7].

Proposition 1.1. *Let* $u = (u_1, \dots, u_n)$ *be a solution of* (0.1) *with* $u_i(t_0) > 0$, $i = 1, 2, \dots, n$, *for some* $t_0 \in \mathbf{R}$. *For each* $i = 1, \dots, n$, *let* U_i *be a solution of* (0.4i) *such that* $U_i(t_0) \geq u_i(t_0)$ *(or* $U_i(t_0) \leq u_i(t_0)$*). Then* $U_i(t) > u_i(t)$ *for* $t > t_0$ *(*$U_i(t) < u_i(t)$ *for* $t < t_0$*, respectively).*

We now recall the topological principle of Wazewski (see, for example [6]). Let $f(t, y)$ be a continuous function defined on an open (t, y)-set $\Omega \subset \mathbf{R} \times \mathbf{R}^n$. Let Ω^0 be an open subset of Ω with the boundary $\partial\Omega^0$ and the closure $\overline{\Omega^0}$. Recall that a point $(t_0, y_0) \in \Omega \cap \partial\Omega^0$ is called an *egress point* of Ω^0 with respect to the system

$$y' = f(t, y), \tag{1.1}$$

if for every solution $y = y(t)$ of (1.1) satisfying the initial condition

$$y(t_0) = y_0, \tag{1.2}$$

there is an $\varepsilon > 0$ such that $(t, y(t)) \in \Omega^0$ for $t_0 - \varepsilon \leq t < t_0$. An egress point (t_0, y_0) of Ω^0 is called a *strict egress point* if $(t, y(t)) \notin \overline{\Omega^0}$ for $t_0 < t \leq t_0 + \varepsilon$ for a small $\varepsilon > 0$. The set of egress points of Ω^0 will be denoted by Ω^0_e and the set of strict egress points by Ω^0_{se}.

If X is a topological space, V a subset of X, a continuous mapping $\pi : X \to V$ defined on all of X is called a *retraction* of X onto V if the restriction $\pi\big|_V$ of π to V is the identity. When there exists a retraction of X onto V, V is called a retract of X.

Remark 1.2. For $a_i < b_i$ $(1 \leq i \leq n)$, let X be the n-parallepiped $\{(x_1, \dots, x_n) : a_i \leq x_i \leq b_i, \ \ 1 \leq i \leq n\}$ in the Euclidean space $\mathbf{R}^n$, and V its boundary. Then V is not a retract of X. For if there exists a retraction $\pi : X \to V$, then there exists a continuous map of X into itself

$$(x_1, \dots, x_n) \mapsto \Big(\frac{a_1 + b_1}{2}, \dots, \frac{a_n + b_n}{2}\Big) - \pi(x_1, \dots, x_n),$$

without fixed points, which is impossible by the fixed point theorem of Schauder.

Theorem 1.3 (Topological principle, see [6]). *Let* $f(t, y)$ *be continuous on an open* (t, y)*-set* Ω *with the property that initial values determine unique solution of* (1.1). *Let* Ω^0 *be an open subset of* Ω *satisfying* $\Omega^0_e = \Omega^0_{se}$. *Let*

S be a nonempty subset of $\Omega^0 \cup \Omega_e^0$ such that $S \cap \Omega_e^0$ is not a retract of S but is a retract of Ω_e^0. Then there exists at least one point (t_0, y_0) in $S \cap \Omega^0$ such that the solution $(t, y(t))$ of (1.1), (1.2) *is contained in Ω^0 on its right maximal interval of existence.*

Theorem 1.4. *If conditions* (0.5) *hold, then the system* (0.1) *has at least a solution $u^0(t)$ defined on $(-\infty, +\infty)$ satisfying*

$$\eta_i \le u_i^0(t) \le U_i^0(t), \quad 1 \le i \le n,$$

where η_i is a positive number such that

$$\eta_i < \min\Big\{\varepsilon_1/a_{iiM},\ \inf_{t\in R} U_i^0(t)\Big\}.$$

Proof. Consider the system

$$v_i' = v_i\Big[-b_i(-t) + \sum_{j=1}^n a_{ij}(-t)v_j\Big], \quad 1 \le i \le n. \tag{1.3}$$

Set $\Omega^0 = \Big\{(t, v_1, \ldots, v_n) : -\infty < t < +\infty,\ \eta_i < v_i < U_i^0(-t),\ 1 \le i \le n\Big\}$, $\Omega = \Big\{(t, v_1, \ldots, v_n) : -\infty < t < +\infty,\ v_i > 0,\ 1 \le i \le n\Big\}$. By Proposition 1.1, any point $(t, v_1, \ldots, v_n)$ in

$$A = \bigcup_{i=1}^n \Big\{t, v_1, \ldots, v_n) \in \overline{\Omega^0} : v_i = U_i^0(-t),\ -\infty < t < +\infty\Big\}$$

is a strict egress point of Ω^0. By (0.5) and the definition of η_i $(1 \le i \le n)$, it follows that any point $(t, v_1, \ldots, v_n)$ in

$$B = \bigcup_{i=1}^n \Big\{(t, v_1, \ldots, v_n) \in \overline{\Omega^0} : v_i = \eta_i\Big\}$$

is a strict egress point of Ω^0. Therefore $\Omega_e^0 = \Omega_{se}^0 = A \cup B$. Let us take $S = \Big\{(0, v_1, \ldots, v_n) : \eta_i \le v_i \le U_i^0(0),\ 1 \le i \le n\Big\}$. Then S is a parallepiped. By Remark 1.2, $S \cap \Omega_e^0$ is not a retract of S.

Define

$$\pi : \Omega_e^0 \to S \cap \Omega_e^0,$$

$$(t, v_1, \dots, v_n) \mapsto \Big(0, \eta_1 + \frac{v_1 - \eta_1}{U_1^0(t) - \eta_1}\big(U_1^0(0) - \eta_1\big), \dots,$$
$$\eta_n + \frac{v_n - \eta_n}{U_n^0(t) - \eta_n}\big(U_n^0(0) - \eta_n\big)\Big).$$

The map π is clearly continuous with respect to the subtopologies on Ω_e^0 and $S \cap \Omega_e^0$ of the Euclidean space $\mathbf{R}^{n+1}$, and its restriction to $S \cap \Omega_e^0$ is the identity. Therefore $S \cap \Omega_e^0$ is a retract of Ω_e^0. By Theorem 1.3, the system (1.3) has at least a solution $v^0(t)$ satisfying $\eta_i < v_i^0(t) < U_i^0(-t)$ for $t \geq 0$. In fact, $u^*(t) = v^0(-t)$ is a solution of (0.1) for $t \leq 0$. By Proposition 1.1, conditions (0.5) and the definition of η_i $(1 \leq i \leq n)$, it follows that the solution $\overline{u}(t)$ of (0.1) with $\overline{u}(0) = v^0(0)$ satisfies

$$\eta_i \leq \overline{u}_i(t) \leq U_i^0(t), \quad \text{for} \quad t \geq 0 \text{ and } 1 \leq i \leq n.$$

Let

$$u^0(t) = \begin{cases} u^*(t), & t \leq 0, \\ \overline{u}(t), & t > 0. \end{cases}$$

Then $u^0(t)$ is a solution of (0.1) satisfying $\eta_i \leq u_i^0(t) \leq U_i^0(t)$ $(t \in \mathbf{R},\ 1 \leq i \leq n)$. The theorem is proved.

2. Uniqueness and asymptoticity

In this section we also do not assume the almost periodicity conditions on a_{ij} and b_i $(1 \leq i, j \leq n)$. From now on, $\mathbf{R}_+^n$ denotes the set of points $x = (x_1, \dots, x_n)$ in $\mathbf{R}^n$ such that $x_i > 0$, $1 \leq i \leq n$. Moreover $u(t, t_0, x) := (u_1(t, t_0, x), \dots, u_n(t, t_0, x))$ denotes the solution of (0.1) defined by the initial condition $u(t_0, t_0, x) = x$. Remember that $u(t, t_0, x)$ is defined on $[t_0, +\infty)$ and $u(t, t_0, x) \in \mathbf{R}_+^n$ for $t \in [t_0, +\infty)$ if $x \in \mathbf{R}_+^n$. We shall prove in this section that with (0.6) the conditions in Theorem 1.4 give a unique solution $u^0(t)$ and $u_i^0(t) - u_i(t, t_0, x) \to 0$ as $t \to +\infty$ $(1 \leq i \leq n)$ for any solution $u(t, t_0, x)$ with $u(t_0, t_0, x) = x \in \mathbf{R}_+^n$. To do this we need the following theorem by A. Tineo and C. Alvarez [7].

Theorem 2.1. *Suppose that there are positive constants $\alpha_1, \dots, \alpha_n$, δ_1 such that*

$$\alpha_i a_{ii}(t) > \delta_1 + \sum_{j \in J_i} \alpha_j a_{ji}(t),\ t > 0,\ 1 \leq i \leq n. \tag{2.1}$$

If $K \subset \mathbf{R}_+^n$ is a convex set and there are positive constants ε_K, M_K such that $\varepsilon_K \leq u_i(t, 0, x) \leq M_K$ for $t \geq 0$ and $x \in K$, $1 \leq i \leq n$, then there are positive constants δ, k depending on δ_1, $\alpha_1, \dots, \alpha_n$, ε_K, M_K such that

$$\|u(t, 0, x) - u(t, 0, y)\| \leq k e^{-\delta t} \|x - y\|, \text{ for } t \geq 0 \text{ and } x, y \in K,$$

where $\|\cdot\|$ *is the usual Euclidean norm of* $\mathbf{R}^n$*.*

Theorem 2.2. *Suppose that the system* (0.1) *satisfies conditions* (0.5)-(0.6)*. Then the system* (0.1) *has a unique solution defined on* $(-\infty,+\infty)$ *whose components are bounded above and below by positive constants.*

Proof. The existence follows from Theorem 1.4. We now prove the uniqueness. Suppose by contradiction that the system (0.1) has two different solutions defined on $(-\infty,+\infty)$, say $u^1(t)$ and $u^2(t)$, such that $0 < u^\ell_{iL} \le u^\ell_{iM} < +\infty$ $(1 \le i \le n,\ \ell = 1,2)$. We claim that

$$\eta_i \le u^\ell_i(t) \le U^0_i(t) \quad (t \in \mathbf{R},\ 1 \le i \le n,\ \ell = 1,2) \tag{2.2}$$

where η_i is as in Theorem 1.4. If it is false for some $\ell \in \{1,2\}$, then one of the following alternatives occurs:

(i) There exist $t_1 \in \mathbf{R}$ and $i_0 \in \{1,\dots,n\}$ such that $u^\ell_{i_0}(t_1) > U^0_{i_0}(t_1)$, or

(ii) $u^\ell_i(t) \le U^0_i(t)$ $(1 \le i \le n,\ t \in \mathbf{R})$ and there exist $t_1 \in \mathbf{R}$ and $i_0 \in \{1,\dots,n\}$ such that $u^\ell_{i_0}(t_1) < \eta_{i_0}$.

If (i) holds, then by Proposition 1.1 we have $u^\ell_{i_0}(t) \ge U_{i_0}(t,t_1,Z)$, for $t < t_1$, where $Z = u^\ell(t_1)$ and $U_{i_0}(t,t_1,Z)$ is the solution of (0.4i$_0$) with $U_{i_0}(t_1,t_1,Z) = Z_{i_0}$. By the uniqueness of the solution $U^0_{i_0}$ of (0.4i$_0$), it is not hard to prove that $U^0_{i_0}(t,t_1,Z) \to +\infty$ as $t \to t_2$ for some $t_2 \in [-\infty, t_1)$. Hence $u^\ell_{i_0}(t) \to +\infty$ as $t \to t_2$, which contradicts the boundedness of u^ℓ.

Suppose that (ii) holds. It is easy to see that if $u^\ell_{i_0}(t) < \eta_{i_0}$ then

$$b_{i_0}(t) - \sum_{j=1}^n a_{i_0 j}(t) u^\ell_j(t) \ge b_{i_0}(t) - \sum_{j \in J_{i_0}} a_{ij}(t) U^0_j(t) - a_{i_0 i_0}(t)\eta_{i_0}.$$

It follows by (0.5) that

$$b_{i_0}(t) - \sum_{j=1}^n a_{i_0 j}(t) u^\ell_j(t) \ge \varepsilon_1 - a_{i_0 i_0}(t)\eta_{i_0} \ge \varepsilon_1 - a_{i_0 i_0 M}\eta_{i_0} > 0.$$

It implies from classical arguments that $u^\ell_{i_0}(t) \to 0$ as $t \to -\infty$, which contradicts $u^\ell_{i_0 L} > 0$. Since (i) and (ii) are exhaustive, the claim is proved.

For $t \in \mathbf{R}$, set $K_t = \{x \in \mathbf{R}^n : \eta_i \le x_i \le U^0_i(t),\ 1 \le i \le n\}$. It is easy to see that K_t is a compact convex subset of $\mathbf{R}^n_+$. If we set $\varepsilon_0 = \min\limits_{1 \le i \le n} \eta_i$

and $M_0 = \sup_{\substack{1\le i\le n\\ -\infty<t<+\infty}} U_i^0(t)$, then $0 < \varepsilon_0 \le M_0 < +\infty$. We know that $\overline{u}(t,0,x) = u(t+t_0,t_0,x)$ is the solution of

$$\overline{u}_i' = \overline{u}_i\big[b_i(t+t_0) - \sum_{j=1}^{n} a_{ij}(t+t_0)\overline{u}_j\big], \quad 1\le i\le n, \tag{2.3}$$

with $\overline{u}(0,0,x) = u(t_0,t_0,x) = x$, $t_0\in \mathbf{R}$. Furthermore,

$$\varepsilon_0 \le \overline{u}_i(t,0,x) \le M_0 \quad \text{for } t\ge 0,\ 1\le i\le n \text{ and } x\in K_{t_0}.$$

From conditions (0.6) and Theorem 2.1, it follows that there exist positive constants δ_0, k_0 depending on ε_0, M_0, ε_2, $\alpha_1,\dots,\alpha_n$ such that

$$\|\overline{u}(t,0,x) - \overline{u}(t,0,y)\| \le k_0 e^{-\delta_0 t}\|x-y\|,\ t\ge 0,\ x,y\in K_{t_0}. \tag{2.4}$$

Hence

$$\|u(t,t_0,x) - u(t,t_0,y)\| \le k_0 e^{-\delta_0(t-t_0)}\|x-y\|,\ t\ge t_0,\ x,y\in K_{t_0}. \tag{2.5}$$

It follows from (2.2) and (2.5) that

$$\|u^1(t_1) - u^2(t_1)\| \le k_0.e^{-\delta_0(t_1-t_0)}\|u^1(t_0)-u^2(t_0)\|,\ t_1,t_0\in\mathbf{R},\ t_1\ge t_0. \tag{2.6}$$

Hence

$$\|u^1(t_0) - u^2(t_0)\| \ge k_0^{-1} e^{\delta_0(t_1-t_0)}\|u^1(t_1)-u^2(t_1)\|,\ t_1,t_0\in\mathbf{R},\ t_1\ge t_0. \tag{2.7}$$

If $t_1 = 0$ and $t_0 = -\dfrac{1}{\delta_0}\ln\dfrac{(d+1)p}{\|u^1(0)-u^2(0)\|}$, where $d = \sup_{t\in\mathbf{R}}\Big\{\sup_{x,y\in K_t}\|x-y\|\Big\}$ and $p\ge \max\{1,k_0\}$, then by (2.7) we have

$$\|u^1(t_0) - u^2(t_0)\| \ge k_0^{-1}p(d+1) \ge d+1. \tag{2.8}$$

On the other hand, we have $u^1(t_0)$, $u^2(t_0)\in K_{t_0}$. The definition of d then implies $\|u^1(t_0)-u^2(t_0)\| \le d$, which contradicts (2.8). This proves the theorem.

Theorem 2.3. *Suppose that the system* (0.1) *satisfies conditions* (0.5) - (0.6). *Then* $u_i(t,t_0,x) - u_i^0(t) \to 0$ *as* $t\to+\infty$ $(1\le i\le n)$ *for any solution* $u(t,t_0,x)$ *with* $x\in\mathbf{R}_+^n$, *where* $u^0(t)$ *is the solution given by Theorem 2.2.*

Proof. Let K be a convex compact subset of $\mathbf{R}^n_+$. It is enough to show that $u_i(t,0,x) - u_i^0(t) \to 0$ as $t \to +\infty$, for $1 \le i \le n$ and $x \in K$. For each $i = 1,\dots,n$, denote by $U_i(t,t_0,x)$ the solution of (0.4i) with $U_i(t,t_0,x) = x_i$. From (0.5) it follows that there exists a $\gamma > 0$ such that

$$b_i(t) - \gamma a_{ii}(t) - \sum_{j\in J_i} a_{ij}(t)\big(U_j^0(t) + \gamma\big) > 0,\ 1 \le i \le n,\ t \in \mathbf{R}. \tag{2.9}$$

It is not hard to prove that $U_i(t,0,x) - U_i^0(t) \to 0$ as $t \to +\infty$ uniformly for $x \in K$, $1 \le i \le n$. Consequently, there is a $t_0 \ge 0$ such that

$$U_i(t,0,x) \le U_i^0(t) + \gamma, \quad t \ge t_0,\ x \in K,\ 1 \le i \le n. \tag{2.10}$$

We claim that

$$u_i(t,0,x) \ge \gamma_i = \min\Big\{u_i(t_0,0,x), \gamma\Big\},\ t \ge t_0,\ 1 \le i \le n. \tag{2.11}$$

Suppose that it is false. For each $i = 1,\dots,n$ let us define $g_i(t) = \gamma_i - u_i(t,0,x)$. Then there exists $i \in \{1,\dots,n\}$ and $t_1 > t_0$ such that $g_i(t_1) > 0$. Since $g_i(t_0) \le 0$, there exists $t_2 > t_0$ such that $g_i(t_2) > 0$ and $g_i'(t_2) > 0$. It implies

$$0 < -b_i(t_2) + a_{ii}(t_2)u_i(t_2,0,x) + \sum_{j\in J_i} a_{ij}(t_2)u_j(t_2,0,x).$$

Hence

$$0 < -b_i(t_2) + a_{ii}(t_2)\gamma + \sum_{j\in J_i} a_{ij}(t_2)u_j(t_2,0,x). \tag{2.12}$$

By Proposition 1.1, it follows that

$$u_i(t,0,x) < U_i(t,0,x),\ t > 0. \tag{2.13}$$

From (2.10), (2.12) and (2.13) we have

$$0 < -b_i(t_2) + a_{ii}(t_2)\gamma + \sum_{j\in J_i} a_{ij}(t_2)\big(U_j^0(t_2) + \gamma\big).$$

which contradicts (2.9). Hence our claim is proved.

It follows from (2.10), (2.11), Proposition 1.1 and the definition of $U_i^0(t)$ that there exist positive constants $\overline{\varepsilon}_K$ and $\overline{M}_K$ such that $\overline{\varepsilon}_K \leq u_i(t,0,x) \leq \overline{M}_K$ for $x \in K$, $t \geq t_0$, $1 \leq i \leq n$. Consequently, since K is compact, there exist positive numbers ε_K, M_K such that $\varepsilon_K \leq u_i(t,0,x) \leq M_K$ for $t \geq 0$, $1 \leq i \leq n$, $x \in K$. The proof follows now from Theorem 2.1.

3. Almost periodicity

In this section we assume in addition that a_{ij}, b_i $(1 \leq i,j \leq n)$ are almost periodic. Suppose that $f = (f^1,\dots,f^n) : \mathbf{R} \to \mathbf{R}^n$ $(n \geq 1)$ is continuous. Recall that f is almost periodic if for each $\varepsilon > 0$ there exists a positive number $\ell = \ell(\varepsilon)$ such that each interval $(\alpha, \alpha+\ell)$, $\alpha \in \mathbf{R}$, contains at least a number $\tau = \tau(\varepsilon)$ satisfying $\sup\limits_{t\in\mathbf{R}} \|f(t+\tau) - f(t)\|_\infty \leq \varepsilon$, where $\|f(t)\|_\infty = \max\limits_{1\leq i\leq n} \{|f^i(t)|\}$. We recall Bochner's criterion for the almost periodicity: *$f(t)$ is almost periodic if and only if for every sequence of numbers $\{\tau_m\}_1^\infty$, there exists a subsequence $\{\tau_{m_k}\}_{k=1}^\infty$ such that the sequence of translates $\{f(t+\tau_{m_k})\}_{k=1}^\infty$ converges uniformly on $(-\infty,+\infty)$* (see, for example [3]).

Proposition 3.1. *For each $i = 1,\dots,n$, the solution $U_i^0(t)$ of* (0.4i) *is almost periodic.*

Proof. Let us fix $i = 1,\dots,n$. Take $\varepsilon > 0$. By Bochner's criterion, it follows that $(b_i(t), a_{ii}(t))$ is almost periodic. Therefore there exists a positive number $\ell = \ell(\varepsilon)$ such that each interval $(\alpha, \alpha+\ell)$; $\alpha \in \mathbf{R}$, contains at least a number $\tau = \tau(\varepsilon)$ such that

$$\sup_{t\in\mathbf{R}} |b_i(t+\tau) - b_i(t)| \leq \varepsilon \text{ and } \sup_{t\in\mathbf{R}} |a_{ii}(t+\tau) - a_{ii}(t)| \leq \varepsilon. \tag{3.1}$$

Take an arbitrary τ as above. Define $W_i(t) = \dfrac{1}{U_i^0(t)}$. From (0.4i) it follows that

$$\begin{aligned}\frac{d}{dt}\big[W_i(t) - W_i(t+\tau)\big] = -b_i(t)\big[W_i(t) - W_i(t+\tau)\big] + \big[b_i(t+\tau) \\ - b_i(t)\big]W_i(t+\tau) + a_{ii}(t) - a_{ii}(t+\tau).\end{aligned} \tag{3.2}$$

Consider the following equation

$$Z' = a_{ii}(t) - a_{ii}(t+\tau) + \big[b_i(t+\tau) - b_i(t)\big]W_i(t+\tau) - b_i(t)Z. \tag{3.3}$$

Since $b_{iL} > 0$, it is not hard to see that if $Z(t)$ is a bounded solution of (3.3) defined on $(-\infty, +\infty)$, then

$$\inf_{t\in\mathbf{R}}\Big\{\frac{a_{ii}(t)-a_{ii}(t+\tau)+\big(b_i(t+\tau)-b_i(t)\big)W_i(t+\tau)}{b_i(t)}\Big\} \le Z(t)$$
$$\le \sup_{t\in\mathbf{R}}\Big\{\frac{a_{ii}(t)-a_{ii}(t+\tau)+\big(b_i(t+\tau)-b_i(t)\big)W_i(t+\tau)}{b_i(t)}\Big\}, \quad t \in \mathbf{R}.$$

Therefore, from (3.1) it follows

$$|Z(t)| \le \frac{\varepsilon\Big(1+\dfrac{1}{U^0_{iL}}\Big)}{b_{iL}}, \quad \text{for any} \quad t \in \mathbf{R}.$$

Since $\dfrac{1}{U^0_i(t)} - \dfrac{1}{U^0_i(t+\tau)}$ is a bounded solution of (3.3), we have

$$\Big|\frac{1}{U^0_i(t)} - \frac{1}{U^0_i(t+\tau)}\Big| \le \varepsilon\,\frac{1+\dfrac{1}{U^0_{iL}}}{b_{iL}}\,.$$

Consequently,

$$|U^0_i(t) - U^0_i(t+\tau)| \le \varepsilon\frac{\Big(1+\dfrac{1}{U^0_{iL}}\Big)\big(U^0_{iM}\big)^2}{b_{iL}}\,.$$

Thus, if $\overline{\varepsilon} = \varepsilon\dfrac{\Big(1+\dfrac{1}{U^0_{iL}}\Big)\big(U^0_{iM}\big)^2}{b_{iL}}$, then we can choose $\ell(\overline{\varepsilon}) = \ell(\varepsilon)$. Therefore, by the definition, $U^0_i(t)$ is almost periodic. The proposition is proved.

In the proof of the following theorem we use the idea in [1].

Theorem 3.2. *Suppose that all conditions in Theorem 2.3 hold. If, in addition, a_{ij}, b_i $(1 \le i, j \le n)$ are almost periodic, then the solution $u^0(t)$ in Theorem 2.3 is almost periodic.*

Proof. Let $\{\tau_m\}_{m=1}^{\infty}$ be an arbitrary sequence of numbers. Since $b_i(t)$, $a_{ij}(t)$ and $U^0_i(t)$ $(1 \le i, j \le n)$ are almost periodic, there exists a subsequence $\{\tau_{m_k}\}_{k=1}^{\infty}$ of $\{\tau_{m_k}\}_{m=1}^{\infty}$ such that $b_i(t+\tau_{m_k})$, $a_{ij}(t+\tau_{m_k})$, $U^0_i(t+\tau_{m_k})$ converge uniformly on $(-\infty, +\infty)$ to functions $b^*_i(t)$, $a^*_{ij}(t)$,

$U_i^0(t)$, respectively. It is not hard to see that $b_{iL}^* = b_{iL}$, $b_{iM}^* = b_{iM}$, $a_{ijL}^* = a_{ijL}$, $a_{ijM}^* = a_{ijM}$, $U_{iL}^{0*} = U_{iL}^0$ and $U_{iM}^{0*} = U_{iM}^0$ $(1 \le i, j \le n)$. Furthermore, by Proposition 3.1 it follows that for each $i = 1, \dots, n$ the logistic equation

$$U' = U\big[b_i^*(t) - a_{ii}^*(t)U\big], \tag{3.4i}$$

has a unique solution defined on $(-\infty, +\infty)$ which is bounded above and below by positive constants. It is easy to see that $U_i^{0*}(t)$ is that unique solution.

Since $b_i(t+\tau_{m_k}) - \sum_{j\in J_i} a_{ij}(t+\tau_{m_k})U_j^0(t+\tau_{m_k})$ $(1 \le i \le n)$ converges uniformly on $(-\infty, +\infty)$ to $b_i^*(t) - \sum_{j\in J_i} a_{ij}^*(t)U_j^{0*}(t)$ as $k \to \infty$, it follows that

$$b_i^*(t) \ge \sum_{j\in J_i} a_{ij}^*(t)U_j^{0*}(t) + \varepsilon_1, \quad 1 \le i \le n,\ t \in \mathbf{R}. \tag{3.5}$$

Similarly, we have

$$\alpha_i a_{ii}^*(t) \ge \sum_{j\in J_i} a_{ji}^*(t)\alpha_j + \varepsilon_2, \quad 1 \le i \le n,\ t \in \mathbf{R}. \tag{3.6}$$

By Theorem 1.4 and 2.3, it follows that the system

$$u_i' = u_i\Big[b_i^*(t) - \sum_{j=1}^n a_{ij}^*(t)u_j\Big]; \quad 1 \le i \le n, \tag{3.7}$$

has a unique solution u^{0*} defined on $(-\infty, +\infty)$ such that

$$\eta_i \le u_i^{0*}(t) \le \Delta_i, \quad 1 \le i \le n,$$

where η_i, Δ_i are positive numbers satisfying

$$\eta_i < \min\Big\{\varepsilon_1/a_{iiM}^*,\ \inf_{t\in\mathbf{R}} U_i^{0*}(t)\Big\} = \min\Big\{\varepsilon_1/a_{iiM},\ \inf_{t\in\mathbf{R}} U_i^0(t)\Big\},$$

$$\Delta_i = U_{iM}^{0*} = U_{iM}^0.$$

Let us denote $S = \big\{(u_1, \dots, u_n) \in \mathbf{R}^n : \eta_i \le u_i \le \Delta_i,\ 1 \le i \le n\big\}$. We claim that $u^0(t+\tau_{m_k})$ converges to $u^{0*}(t)$ uniformly on $(-\infty, +\infty)$ as $t \to$

∞, which will show that $u^0(t)$ is almost periodic. Suppose by contradiction that the claim is false. Then there exist a subsequence $\{\tau_{m_{k_\ell}}\}_{\ell=1}^{\infty}$ of $\{\tau_{m_k}\}_{k=1}^{\infty}$, a sequence of numbers $\{S_\ell\}$, and a fixed number $\alpha > 0$ such that $\|u^0(S_\ell + \tau_{m_{k_\ell}}) - u^{0*}(S_\ell)\| \geq \alpha$ for all ℓ.

Since b_i, a_{ij} and U_i^0 $(1 \leq i, j \leq n)$ are almost periodic, we may assume, without loss of generality, that $b_i(t+\tau_{m_{k_\ell}}+S_\ell)$, $a_{ij}(t+\tau_{m_{k_\ell}}+S_\ell)$, $U_i^0(t+\tau_{m_{k_\ell}}+S_\ell)$ converge uniformly on $(-\infty,+\infty)$ to $\hat{b}_i(t)$, $\hat{a}_{ij}(t)$, $\hat{U}_i^0(t)$, respectively, as $\ell \to \infty$. Hence $b_i^*(t+S_\ell) \to \hat{b}_i(t)$, $a_{ij}^*(t+S_\ell) \to \hat{a}_{ij}(t)$, $U_i^0(t+S_\ell) \to \hat{U}_i^0(t)$ $(1 \leq i, j \leq n)$ uniformly with respect to t on $(-\infty,+\infty)$ as $\ell \to +\infty$ and $\hat{b}_{iL} = b_{iL}$, $\hat{b}_{iM} = b_{iM}$, $\hat{a}_{ijL} = a_{ijL}$, $\hat{a}_{ijM} = a_{ijM}$, $\hat{U}_{iL}^0 = U_{iL}^0$ and $\hat{U}_{iM}^0 = U_{iM}^0$. Since $u^0(t) \in S$ for all t in $(-\infty,+\infty)$, we can assume without loss of generality that $u^0(S_\ell+\tau_{m_{k_\ell}}) \to \xi$ as $\ell \to \infty$, where $\xi \in S$. Similarly we may assume that $u^{0*}(S_\ell) \to \xi^* \in S$ as $\ell \to \infty$. Therefore $\|\xi - \xi^*\| \geq \alpha$. For each $\ell = 1, 2, \ldots$, $u^0(t + \tau_{m_{k_\ell}} + S_\ell)$ is a solution of the system

$$(3.8l)\quad u_i' = u_i\Big[b_i(t+\tau_{m_{k_\ell}}+S_\ell) - \sum_{j=1}^{n} a_{ij}(t+\tau_{m_{k_\ell}}+S_\ell)u_j\Big],\ 1 \leq i \leq n.$$

Consider the solution $\hat{u}^0(t)$ of

$$(3.9)\qquad u_i' = u_i\Big[\hat{b}_i(t) - \sum_{j=1}^{n} \hat{a}_{ij}(t)u_j\Big], \quad 1 \leq i \leq n,$$

having the initial value $\hat{u}^0(0) = \xi$. We have two systems (3.8l) and (3.9), where the right-hand side of (3.8l) converges uniformly to the right-hand side of (3.9) on any compact subset of $\mathbf{R}^{n+1} = \{(t, u_1, \ldots, u_n) : t \in \mathbf{R},\ u_i \in \mathbf{R},\ 1 \leq i \leq n\}$ as $\ell \to \infty$. Also the initial values satisfy the property that $u^0(\tau_{m_{k_\ell}} + S_\ell) \to \xi$ as $\ell \to \infty$. Hence it follows that $u^0(t + \tau_{m_{k_\ell}} + S_\ell)$ converges to $\hat{u}^0(t)$ uniformly on compact subintervals of the domain of $\hat{u}^0(t)$. This implies that $\hat{u}^0(t) \in S$ for all $t \in \mathbf{R}$.

Now recall that $u^{0*}(t)$ is the unique solution of (3.7) with $u^{0*}(t) \in S$ for all $t \in \mathbf{R}$. For each integer ℓ, $u^{0*}(t + S_\ell)$ is a solution of

$$(3.10l)\qquad u_i' = u_i\Big[b_i^*(t+S_\ell) - \sum_{j=1}^{n} a_{ij}^*(t+S_\ell)u_j\Big], \quad 1 \leq i \leq n,$$

with $u^{0*}(S_\ell) \to \xi^*$ as $\ell \to \infty$.

Since $b_i^*(t+S_\ell) \to \hat{b}_i(t)$, $\quad a_{ij}^*(t+S_\ell) \to \hat{a}_{ij}(t)$ $(1 \le i,j \le n)$ as $\ell \to \infty$ uniformly with respect to t on $(-\infty,+\infty)$, it follows that if $\hat{u}^{0*}(t)$ is the solution of (3.9) with $\hat{u}^{0*}(0) = \xi^*$, then $u^{0*}(t+S_\ell) \to \hat{u}^{0*}(t)$ as $t \to \infty$ uniformly on any compact subintervals of the domain of $\hat{u}^{0*}$. By the same argument given before, we have $\hat{u}^{0*}(t) \in S$ for any $t \in \mathbf{R}$. We also have $\hat{u}^0(t) \in S$ for any $t \in \mathbf{R}$. Using the same argument as in the proof of the fact that (3.7) has a unique solution $u^{0*}(t) \in S$ for $t \in \mathbf{R}$, we get that (3.9) has a unique solution defined on $(-\infty,+\infty)$ which is in S for any $t \in (-\infty,+\infty)$. Hence $\hat{u}^0 \equiv \hat{u}^{0*}$. But $\hat{u}^0(0) = \xi$, $\hat{u}^{0*}(0) = \xi^*$ and $\|\xi - \xi^*\| \ge \alpha > 0$, which is a contradiction. The theorem is proved.

One can show that conditions (0.3) imply conditions (0.5) and (0.6) by using completely the same argument in [7]. Thus, from Theorems 1.4, 2.2, 2.3 and 3.2 we get the following corollary.

Corollary 2.3. *Suppose that b_i, a_{ij} $(1 \le i,j \le n)$ are continuous and bounded above and below by positive constants. If conditions* (0.2) *hold, then the system* (0.1) *has a unique solution u^0 defined on* $(-\infty,+\infty)$, *whose components are bounded above and below by positive constants and $u_i(t) - u_i^0(t) \to 0$ as $t \to +\infty$, $1 \le i \le n$, for any solution $u(t)$ of* (0.1) *with $u(t_0) > 0$ for some $t_0 \in \mathbf{R}$.*

If, in addition, a_{ij}, b_i $(1 \le i,j \le n)$ are almost periodic then $u^0(t)$ is also almost periodic.

Acknowledgment

I am grateful to Professor Tran Van Nhung for his valuable comments.

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