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EXISTENCE OF PERIODIC SOLUTIONS OF NONAUTONOMOUS RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is concerned with non-linear functional differential equations of retarded type which are periodic in the independent variable t. The aim is to obtain explicit conditions which are sufficient for the existence of periodic solutions if there exists a bounded solution.

1. INTRODUCTION

In this work we give consider the following equation

$$\begin{cases} \dfrac{d}{dt}x(t) = F(t, x_t), \text{ for } t \geq 0, \\ x_0 = \varphi \in C\big([-r, 0], \mathbf{R}^n\big) = C, \end{cases} \tag{1}$$

where C is the space of continuous functions on $[-r, 0]$ with values in $\mathbf{R}^n$ endowed with the uniform norm topology, F is a continuous function on $\mathbf{R} \times C$ with values in $\mathbf{R}^n$, and for every $t \geq 0$ the function $x_t \in C$ is defined by

$$x_t(\theta) = x(t + \theta), \quad \text{for } \theta \in [-r, 0].$$

We will study the following problem:

Under what conditions, the existence of bounded solutions implies the existence of periodic solutions?

Note that the answer to this problem was given by Massera [3] in the scalar case

$$\frac{d}{dt}x(t) = f(t, x(t)), \tag{2}$$

which states that if Problem (2) has the uniqueness property with respect

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to the initial condition and if f is ω periodic in t, then the existence of a bounded solution which is defined in $[0,\infty]$ implies the existence of an ω periodic solution. If $n \geq 3$, the Massera theorem is not true, a counter example was given by Massera.

This paper extends the work of R. A. Smith [4] to the nonautonomous case

$$\frac{d}{dt}x(t) = F(x_t). \tag{3}$$

2. Main results

Throughout this paper, K^* denotes the transpose of a real $r \times s$ matrix K. In the sequel we consider the following hypotheses.

(H_1) $F : \mathbf{R} \times C \to \mathbf{R}^n$, is a continuous function, ω periodic in t and, there exists $k > 0$ such that

$$|F(t,\varphi) - F(t,\psi)| \leq k|\varphi - \psi|, \quad \text{for every } \varphi, \psi \in C \text{ and } t \in \mathbf{R}.$$

(H_2) There exists a continuous function U on C with values in $\mathbf{R}$ and an symmetric $(n \times n)$-matrix P such that for all bounded solutions x and y of (1) which are defined on $\mathbf{R}$ one has,

$$U(x_t - y_t) \leq 0, \quad \text{for all } t \in \mathbf{R}$$

$$U(\phi) - \phi(0)^* P\phi(0) \geq \beta\Big[\int_{-r}^{0} |\phi(s)|ds\Big]^2,$$

where β is a positive constant.

(H_3) P has j negative eigenvalues and $(n-j)$ positive eigenvalues.

Remark 1. In [4], R. A. Smith considered similar hypotheses with $j = 2$ in the autonomous case, and he obtains the following result:

Theorem 1 ([4]). *Under the hypotheses* (H_1)-(H_3) *with* $j = 2$, *if Equation* (3) *has a bounded positive semi-orbit* Γ *and the omega limit set* $\Omega(\Gamma)$ *of* Γ *does not contain a critical point, then* $\Omega(\Gamma)$ *contains at least one periodic orbit.*

Our work is a contribution to the nonautonomous case under the above hypotheses with $j = 1$. We obtain

Theorem 2. *Assume that the hypotheses* (H_1)-(H_2) *are satisfied with* $j = 1$*. If Equation* (1) *has a bounded solution* x *which is defined on* $\mathbf{R}$*, then there exists an* ω *periodic solution* u *such that* $x(t) - u(t)$ *tends to zero as* t *tends to* $+\infty$*.*

If P is a symmetric matrix with only one negative eigenvalue and all other eigenvalues are positive, then there exists an invertible matrix M such that

$$M^*PM = \mathrm{Diag}(-1, 1, 1, \ldots, 1).$$

Set $x = M_{\mathrm{col}}(X, Y)$, where $\mathrm{col}(X, Y)$ is the column vector with the components $X \in \mathbf{R}$ and $Y \in \mathbf{R}^{n-1}$. One has

$$x^*Px = |Y|^2 - |X|^2.$$

Define the following mapping

$$\begin{aligned} Q_1 : \mathbf{R}^n &\to \mathbf{R} \\ x &\to X. \end{aligned}$$

We have

$$2|Q_1x|^2 + x^*Px = |Y|^2 + |X|^2 = |M^{-1}x|^2 \geq |M|^{-2}|x|^2.$$

From this we deduce that

$$2|Q_1x|^2 + x^*Px \geq |M|^{-2}|x|^2, \quad \text{for all } x \in \mathbf{R}^n. \tag{4}$$

Define the projection operator Π on C by:

$$\begin{aligned} \Pi : C &\to \mathbf{R} \\ \varphi &\to \sqrt{2}Q_1\varphi(0). \end{aligned}$$

Π is a bounded linear operator and

$$|\Pi| \leq \sqrt{2}|Q_1|.$$

For the proof of Theorem 2 we need the following lemmas.

Lemma 3. *For each* $a > 0$ *there exists* $b > 0$ *such that for every pair of bounded solutions* x, y *of Equation* (1) *in* $\mathbf{R}$ *with the property that* $|x_t| \leq a$ *and* $|y_t| \leq a$*, for all* $t \in \mathbf{R}$*, one has*

$$|Q_1|\sqrt{2}|x_t - y_t| \geq |\Pi(x_t - y_t)| \geq b|x_t - y_t|^2 \quad \textit{for all } t \in \mathbf{R}.$$

Proof. Let x and y to be two bounded solutions of Equation (1) such that

$$|x_t| \leq a \quad \text{and} \quad |y_t| \leq a \quad \text{for all } t \in \mathbf{R}. \tag{5}$$

Using (4), we obtain

$$|\Pi(x_t - y_t)|^2 \geq -(x(t) - y(t))^* P(x(t) - y(t)) + |M|^{-2}|x(t) - y(t)|^2.$$

Condition (H_2) implies that $U(x_t - y_t) \leq 0$, and

$$|\Pi(x_t - y_t)|^2 \geq |M|^{-2}|x(t) - y(t)|^2 + \beta\Big(\int_{-r}^{0} |x_t(s) - y_t(s)|ds\Big)^2, \tag{6}$$

for all $t \in \mathbf{R}$. On the other hand, one has

$$\int_{t+s}^{t} 2(x(u) - y(u))(F(x_u) - F(y_u))du = |x(t) - y(t)|^2 - |x(t+s) - y(t+s)|^2.$$

From this we get

$$|x(t+s) - y(t+s)|^2 \leq |x(t) - y(t)|^2 + 2k \int_{t+s}^{t} |x(u) - y(u)|\ |x_u - y_u|du,$$

for all $s \in [-r, 0]$. This yields

$$|x(t+s) - y(t+s)|^2 \leq |x(t) - y(t)|^2 + 2ka \int_{t-r}^{t} |x(u) - y(u)|du,$$

for all $s \in [-r, 0]$, and

$$|x_t - y_t|^2 \leq |x(t) - y(t)|^2 + 2ka \int_{-r}^{0} |x_t(u) - y_t(u)|du. \tag{7}$$

Hence

$$|x_t - y_t|^4 \leq |x(t) - y(t)|^4 + (2ka)^2\Big(\int_{-r}^{0} |x_t(u) - y_t(u)|du\Big)^2$$
$$+ 4ka|x(t) - y(t)|^2 \int_{-r}^{0} |x_t(u) - y_t(u)|du.$$

Using (5) and (6) we conclude that there exists a positive constant b such that

$$|x_t - y_t|^4 \leq \Big(|x(t) - y(t)|^2 + \Big(\int_{-r}^{0} |x_t(s) - y_t(s)|ds\Big)^2\Big) \leq b|\Pi(x_t - y_t)|^2,$$

for all $t \in \mathbf{R}$. This completes the proof of the lemma.

Lemma 4. *Suppose that x and y are bounded solutions of Equation* (1) *in $\mathbf{R}$. If there exists an α such that $\Pi x_\alpha = \Pi y_\alpha$, then $x(t) = y(t)$ for all t in $\mathbf{R}$.*

Proof. If there exists an α such that $\Pi x_\alpha = \Pi y_\alpha$, by Lemma 3, we deduce that $x_\alpha = y_\alpha$. The hypothesis (H_1) implies that Equation (1) has the uniqueness property with the initial data. From this we obtain that $x_t = y_t$ for all $t \in \mathbf{R}$.

Proof of Theorem 2.

Let x be a solution of Equation (1) defined on $\mathbf{R}$ such that $x_t \in S_0$ for all t, where S_0 is a bounded closed subspace of C. Put $y(t) = x(t + \omega)$. Then y is also a solution of Equation (1) and $y_t \in S_0$ for all t.

By hypothesis (H_2), one has

$$U(x_t - y_t) \leq 0, \quad \text{for all } t \in \mathbf{R}.$$

If we define a by

$$a = \sup\Big\{|\varphi|, \varphi \in S_0\Big\}, \tag{8}$$

then by Lemma 3, there exists b such that

$$|\Pi(x_t - y_t)| \geq b|x_t - y_t|^2, \quad \text{for all } t. \tag{9}$$

If there exists t_0 such that $x_{t_0} = y_{t_0}$, by Lemma 4, one has $x(t) = y(t)$ for all t. This follows that x is periodic. If for all t, $x_t \neq y_t$, then we have

$$|\Pi(x_t - y_t)| > 0, \quad \text{for all } t. \tag{10}$$

Let $t_0 \in \mathbf{R}$, then one has

$$|\Pi(x_{t_0+n\omega} - x_{t_0+n\omega+\omega})| > 0, \quad \text{for all } n. \tag{11}$$

Hence $\Pi(x_t - y_t)$ is a continuous scalar function, so it has a constant sign. Then the sequence $(\Pi(x_{t_0+n\omega}))_n$ is monotone, which implies that $(x_{t_0+n\omega})_n$ is a Cauchy sequence. Let $\varphi \in S_0$ such that

$$\lim_{n\to\infty} x_{t_0+n\omega} = \varphi.$$

Let u be a solution of

$$\begin{cases} \dfrac{d}{dt}x(t) & = F(t, x_t) \\ x_{t_0} & = \varphi. \end{cases} \tag{12}$$

Then u is defined on $[t_0, +\infty[$. On the other hand, one has

$$u_{t_0+\omega} = \lim_{n\to\infty} x_{t_0+n\omega+\omega} = \lim_{n\to\infty} x_{t_0+n\omega} = u_{t_0}.$$

Hence, u is ω periodic. By hypothesis (H_1) we conclude that

$$|x_t - u_t| \leq \exp k\omega |x_{t_0+n\omega} - u_{t_0+n\omega}|, \text{ for all } t \in [t_0 + n\omega, t_0 + n\omega + \omega].$$

So,

$$|x_t - u_t| \leq \exp k\omega |x_{t_0+n\omega} - u_{t_0}|, \quad \text{for all } t \in [t_0 + n\omega, t_0 + n\omega + \omega].$$

Hence,

$$\lim_{n\to\infty} x_{t_0+n\omega} = u_{t_0},$$

which implies

$$\lim_{t\to\infty} (x_t - u_t) = 0. \tag{13}$$

This completes the proof of Theorem 2.

In the sequel we give an analogous to a theorem of Massera.

Corollary 5. *Assumme that the hypotheses* (H_1)-(H_3) *are satisfied with* $j = 1$*. If Equation* (1) *has a bounded solution which is defined on* $[0, +\infty]$*, then there exists an* ω *periodic solution of Equation* (1)*.*

Proof. It suffices to show that Equation (1) has a bounded solution which is defined on $\mathbf{R}$. Let x be a bounded solution of Equation (1) which is defined on $[0, +\infty]$. For a large n, the following sequence $(x_n(t))_n = (x(t + n\omega))_n$

is equicontinuous on $[-1,1]$. By Ascoli-Arzela Theorem, there exists a subsequence $\big(x_n^1\big)_n$ of $\big(x_n\big)_n$ and a continuous function $x^{(1)}$ such that

$$x_n^{(1)} \underset{x\to\infty}{\longrightarrow} x^{(1)}, \quad \text{uniformly in } [-1,1].$$

For each n, x_n is a solution of Equation (1). This implies that $x^{(1)}$ is also solution of Equation (1) which is defined on $[-1,1]$. Similarly, one can find $x^{(2)}$ and a subsequence $\big(x_n^{(2)}\big)_n$ of $\big(x_n^{(1)}\big)_n$ such that

$$x_n^{(2)} \underset{n\to\infty}{\longrightarrow} x^{(2)}, \quad \text{uniformly in } [-2,2].$$

$x^{(2)}$ is also solution of Equation (1) on $[-2,2]$. Following the same procedure, one can find a continuous function x which is defined on $\mathbf{R}$ and a subsequence $\big(x_{k_n}\big)_n = \big(x_n^{(n)}\big)_n$ of $\big(x_n\big)_n$ such that

$$x_{k_n} \underset{n\to\infty}{\longrightarrow} v, \quad \text{uniformly in every compact set of } \mathbf{R}.$$

For each n

$$x(t) = x^{(n)}(t), \quad \text{for all } t \in [-n,n],$$

which implies that x is a bounded solution of Equation (1), which is defined on $\mathbf{R}$. By Theorem 2, there exists an ω periodic solution. This completes the proof of Corollary 5.

3. Example

Consider the following differential equation

$$\frac{d}{dt}x(t) = Ax(t) + BF(t, g(x_t)). \tag{14}$$

The parameters and functions on the right-hand side are defined as follows

$$g(\varphi) = \int_{-r}^{0} G(\theta)\varphi(\theta)d\theta, \tag{15}$$

where for each θ in $[-r,0]$, B is an $(n\times r)$-matrix, $G(\theta)$ is an $(s\times n)$-matrix.

The function $F : \mathbf{R}\times\mathbf{R}^s \to \mathbf{R}^r$ satisfies the inequality

$$|F(t,y) - F(t,z)| \le \sigma|y-z|, \quad \text{for all } t \text{ in } \mathbf{R}, y, z \text{ in } \mathbf{R}^s,$$

and

$$F(t+\omega, y) = F(t,y) \quad \text{for all } t \text{ in } \mathbf{R} \text{ and } y \text{ in } \mathbf{R}^s.$$

$A = \operatorname{diag}(a_1, a_2, a_3)$, where a_1, a_2, a_3 are squares matrices such that there are positive constants λ, μ with the following properties:

$$-\lambda > \operatorname{Re} z \quad \text{for all eigenvalues } z \text{ of } a_1 \tag{16}$$

$$\mu > \operatorname{Re} z > -\lambda \quad \text{for all eigenvalues } z \text{ of } a_2 \tag{17}$$

$$\operatorname{Re} z > \mu \quad \text{for all eigenvalues } z \text{ of } a_3 \tag{18}$$

By [1, Ch. 6], these conditions ensure that there exist unique real symmetric matrices v_k, w_k, such that

$$(a_k + \lambda I_k)^* v_k + v_k^*(a_k + \lambda I_k) = -I_k,$$
$$(a_k - \mu I_k)^* w_k + w_k^*(a_k - \mu I_k) = -I_k,$$

for $k = 1, 2, 3$, where I_k denotes the unit matrix of the same size as of a_k. Define symmetric $(n \times n)$ matrices P_k, P_w by

$$P_v = \operatorname{Diag}(v_1, v_2, v_3), \quad P_w = \operatorname{Diag}(w_1, w_2, w_3)$$

These matrices depend only on A, λ, μ.

Theorem 6. *Assume that* $A = \operatorname{Diag}(a_1, a_2, a_3)$ *satisfies* (16), (17), (18). *Furthermore assume that*

$$\sigma \max(|P_v B|, |P_w B|) < 2\Big[\int_{-r}^{0} |G(\theta)| \exp(-\lambda\theta) d\theta\Big]^{-1}.$$

Then (14) *satisfies* (H_1), (H_2) *and* (H_3).

Proof. The proof is similar to the one given in [4].

Appendix

For the existence of a bounded solution of Equation (1) one has the following result:

Theorem 7. *Assume that:*

(i) *There exists a linear bounded operator L from C into $\mathbf{R}^n$ such that*

$\left|\dfrac{F(t,\varphi)-L(\varphi)}{|\varphi|}\right| \to 0$ *as* $|\varphi| \to +\infty$ *uniformly with respect to* $t \in \mathbf{R}$.

(ii) *All roots of the following characteristic equation*

$$\det(zI - L(ez)) = 0 \tag{19}$$

have $\operatorname{Re} z < 0$.

Then every solution of Equation (1)*, which is defined on* $[0, +\infty]$*, is bounded.*

Proof. For $t \geq 0$, let $T(t)$ be the mapping of C into itself defined by $T(t)\phi = y_t$, where $y(t)$ is the solution of the following linear equation

$$\begin{cases} \dfrac{d}{dt}y(t) = L(y_k), \\ y_0 = \phi. \end{cases}$$

Since $\operatorname{Re} z < 0$ for every characteristic root of Equation (19), by [2], it follows that there exist some positive constants K, β such that

$$|T(t)\phi| \leq K \exp(-\beta t)|\phi|, \quad \text{for all } t \geq 0,\ \phi \in C. \tag{20}$$

Choose a constant ε such that $0 < \varepsilon < K^{-1}\beta$. Set $g(t,\phi) = F(t,\phi) - L(\phi)$. Then (i) means that there exists a constant $m > 0$ such that

$$|g(t,\phi)| \leq m + \varepsilon|\phi|, \quad \text{for all } t \in \mathbf{R},\ \phi \in C. \tag{21}$$

Also Equation (1) can be rewritten as

$$\frac{d}{dt}x(t) = L(x_t) + g(t, x_t). \tag{22}$$

By [2, p. 120], the solution of (22) with the initial data $x_0 = \phi$ satisfies

$$x_t = T(t)\phi + \int_0^t T(t-s)X_0 g(s, x_s)ds, \quad \text{for all } t \geq 0,$$

where $X_0(\theta)$ is an $(n \times n)$-matrix defined by the formula

$$X_0(\theta) = \begin{cases} 0 & \text{for} \quad -r \leq \theta < 0, \\ I & \text{for} \quad \theta = 0. \end{cases}$$

Here I denotes the unit matrix. This gives

$$|x_t| \leq K\ \exp(-\beta t)|\phi| + \int_0^t K\ \exp(-\beta(t-s))|g(s,x_s)|ds, \text{ for all } t.$$

If we put

$$u(t) = \exp \beta t|x|,$$

then one has,

$$u(t) \leq K(\phi) + \beta^{-1}Km\exp(\beta t) + \int_0^t K\varepsilon u(s)ds, \text{ for all } t. \tag{23}$$

A generalized Gronwall's Lemma gives

$$|u(t)| \leq (\beta - \varepsilon K)^{-1}Km\exp(\beta t) + K|\phi|\exp(K\varepsilon t), \text{ for all } t \geq 0. \tag{24}$$

Since $\varepsilon < K^{-1}\beta$, this shows that $|x_t|$ is bounded in $[0,+\infty]$. This completes the proof of Theorem 7.

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