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NONLINEAR MONOTONE ILL-POSED PROBLEMS IN BANACH SPACES

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Abstract. In this paper, convergence rates for an operator version of Tikhonov regularization constructing on the base of subdifferential of uniformly convex functional on real reflexive Banach space are obtained without both closeness conditions and parameter selection method. Then, the obtained results are considered in combination with finite-dimensional approximations of the space. An example is given for the illustration.

1. Introduction

Let X be a real reflexive Banach space and X^* be its dual space. For the sake of simplicity, norms of X and X^* will be denoted by one symbol $\|.\|$. We write $\langle x^*, x\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Let A be a nonlinear operator with domain of definition $D(A) = X$ and range $R(A) \subseteq X^*$. Let f_0 be an element of $R(A)$.

Consider the nonlinear ill-posed problem

$$A(x) = f_0. \tag{1.1}$$

By ill-posedness we mean that solutions of (1.1) do not depend continuously on the data f_0. Various aspects about regularization of (1.1) were studied in detail, when A is compact or continuous and weakly closed, and $X = H$ is a Hilbert space (see, for instance, [5], [11-15]). The variational method of Tikhonov regularization consists of minimizing the functional

$$F_\alpha^\delta(x) = \|A(x) - f_\delta\|^2 + \alpha\|x\|^2, \tag{1.2}$$

where $\alpha > 0$ is the parameter of regularization and f_δ are the approximations of f_0 with the well-known informations

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$$\|f_\delta - f_0\| \ < \ \delta, \quad \delta \to 0.$$

In order to study convergence rates of this method one needs to have, in general, the following conditions (see [5]): (i) A is Fréchet differentiable, (ii) there exists a constant $L > 0$ such that $\|A'(x) - A'(y)\| \leq L\|x - y\|$, $x,\ y \in D(A)$, (iii) there exists an element $\omega \in H$ such that $A'^*(x_0)\omega = x_0$, where $A'^*(x_0)$ denotes the adjoint of derivative of A at x_0 being a norm-minimal solution of (1.1), and (iv) $L\|\omega\| \ < \ e \ = \ 1$. Usually, the last requirement names the *closeness condition*.

In [13] A. Neubauer estimated $e \approx 0.9476$ for a modification of (1.2). We see that the equation in condition (iii) is not explicitly defined because the operator $A'^*(x_0)$ and the right-hand side x_0 are not known. Therefore, the verification of (iv) is in general too difficult to realize. To exceed this difficulty, in [11] A. Neubauer developed an approach of [9] in the linear case for nonlinear problems involving compact operators. A big advantage of this approach is that rates are obtained by merely requiring smoothness conditions for the exact solution as in the linear case and parameter selection method.

In [2], when A is a monotone operator in H (see terminologies in [16]), the author obtained $e = 2$ for the following operator version of Tikhonov regularization

$$A(x) + \alpha x \ = \ f_\delta.$$

In [4], when A is a monotone operator in Banach space X, the convergence rates are obtained for solution of the regularized equation

$$A(x) + \alpha Bx = f_\delta,$$

where B is a linear and strongly monotone operator, i.e.

$$\langle Bx, x\rangle \geq m_B\|x\|^2, \quad m_B > 0.$$

In this case $e = 2m_B$.

For a real number s denotes by $[s]$ its integer part. In [3], the convergence rates are obtained for the following regularized equation

$$A(x) + \alpha U^s(x) = f_\delta, \quad f_\delta \in X^*, \tag{1.3}$$

where U^s is the nonstandard dual mapping of X satisfying the condition

$$\langle U^s(x),\ x\rangle = \|x\|^s, \quad \|U^s(x)\| = \|x\|^{s-1}, \quad s \geq 2,$$

without closeness conditions when $s \neq [s]$, Note that (1.3) was studied in [1]. These results can be applied to investigate nonlinear monotone ill-posed problems in the spaces of type ℓ_p, L_p, W_p with $p > 2$ and $p \neq [p]$. They are still open if $1 < p < 2$ or $2 \leq p = [p] < +\infty$.

In this paper, we shall show that using the following regularized equation (cf. [14])

$$A(x) + \alpha\partial\varphi(x) \;=\; f_\delta, \tag{1.4}$$

where $\partial\varphi$ is the subdifferential of the uniformly convex functional φ on X, and replacing the smoothness condition (iii) by a more general condition we can exclude condition (iv). Main results about convergence rates are presented in Section 2. An example is given in Section 3. Note that, the results in this paper are more general than the ones in [3].

Below, by $a \sim b$ we mean $a = O(b)$ and $b = O(a)$. The symbols $\rightharpoonup$ and $\to$ denote weak convergence and convergence in norm, respectively.

2. Main results

Assume that there exists a uniformly convex functional $\varphi(x)$ on X such that

$$\langle \partial\varphi(x) - \partial\varphi(y), x - y\rangle \geq m_\varphi \|x - y\|^\beta,\ \forall x,\ y \in X,\ m_\varphi > 0,\ 2 < \beta \neq [\beta].$$

The existence of such uniformly convex functionals is investigated in [8]. It is shown in [14] that for every fixed $\alpha > 0$, Eq. (1.4) has a unique solution $x_{\alpha\delta}$, $f_\delta \in X^*$ and the sequence $\{x_{\alpha\delta}\}$ converges in norm of X to x_0 if δ/α and α tend to zero, where x_0 satisfies the inequality

$$\langle \partial\varphi(x_0), x - x_0\rangle \geq 0,$$

for all x in the set S_0 of solutions of (1.1). We shall prove the following result.

Theorem 2.1. *Suppose the following conditions hold:*

(i) *A is $[\beta]$-times Fréchet differentiable in some neighbourhood of S_0,*

(ii) *There exists a constant $L > 0$ such that*

$$\|A^{([\beta])}(x) - A^{([\beta])}(z)\| \leq L\|x - z\|,$$

for $x,\ z \in S(x_0, r)$, where $S(x_0, r)$ is a ball with center x_0 and radius r, and

(iii) *The equation*

$$\Big(A'^{*}(x_0)+\frac{1}{2!}A^{(2)*}(x_0)y_1+\frac{1}{3!}A^{(3)*}(x_0)y_1y_2+...+\\ \frac{1}{[\beta]!}A^{([\beta])*}(x_0)y_1y_2...y_{[\beta]-1}\Big)\omega=\partial\varphi(x_0)$$

has a bounded solution $\omega(y_1,...,y_{[\beta]-1})$, $y_i\in S(x_0,r)$, $i=1,...,[\beta]-1$. *Then for a choice* $\alpha\sim\delta^{\mu}$, $0<\mu<1$, *we obtain*

$$\|x_{\alpha\delta}-x_0\|=O(\delta^{\theta}),\quad \theta=\ \min\,\{(1-\mu)/\beta,\mu/\beta\}.$$

Proof. By virtue of Eqs (1.1), (1.4) and the monotone property of A and $\partial\varphi$ we have

$$\alpha m_{\varphi}\|x_{\alpha\delta}-x_0\|^{\beta}\le\delta\|x_{\alpha\delta}-x_0\|-\alpha\langle\partial\varphi(x_0),x_{\alpha\delta}-x_0\rangle.$$

From this inequality and condition (iii) of the theorem it follows that

$$\begin{aligned}\alpha m_{\varphi}\|x_{\alpha\delta}-x_0\|^{\beta}\le\delta\|x_{\alpha\delta}-x_0\|+\alpha\big\langle\omega(y_1,...,y_{[\beta]-1}),\big(A'(x_0)\\ +\frac{1}{2!}A^{(2)}(x_0)y_1+\frac{1}{3!}A^{(3)}(x_0)y_1y_2+\cdots\\ +\frac{1}{[\beta]!}A^{([\beta])}(x_0)y_1,...,y_{[\beta]-1}\big)(x_0-x_{\alpha\delta})\big\rangle,\end{aligned}$$

where $y_i\in S(x_0,r)$.

Using Taylor expansion (see [16]) and taking $y_i=x_{\alpha\delta}-x_0$ we can write

$$\begin{aligned}A'(x_0)(x_0-x_{\alpha\delta})-\frac{1}{2!}A^{(2)}(x_0)(x_0-x_{\alpha\delta})^2+\frac{1}{3!}A^{(3)}(x_0)(x_0-x_{\alpha\delta})^3-\\ \cdots+\frac{(-1)^{[\beta]-1)}}{[\beta]!}A^{([\beta])}(x_0)(x_0-x_{\alpha\delta})^{[\beta]}=A(x_0)-A(x_{\alpha\delta})+r_{\alpha\delta},\\ \|r_{\alpha\delta}\|\le L\|x_{\alpha\delta}-x_0\|^{[\beta]+1}/([\beta]+1)!\ .\end{aligned}$$

Hence we obtain

$$\begin{aligned}\alpha m_{\varphi}\|x_{\alpha\delta}-x_0\|^{\beta}\le\delta\|x_{\alpha\delta}-x_0\|+\alpha\|\omega_{\alpha\delta}\|\big(\delta+\alpha\|\partial\varphi(x_{\alpha\delta})\|\\ +L\|x_{\alpha\delta}-x_0\|^{[\beta+1]}/([\beta]+1)!\big),\end{aligned}$$

where $\omega_{\alpha\delta} = \omega_{\alpha\delta}(x_{\alpha\delta} - x_0, ..., x_{\alpha\delta} - x_0)$. The rest of the proof is proceeded similarly as in [4]. □

Remark 2.1. Note that for the proof of this theorem we only need the differentiability of A in some neighbourhood of x_0. If $A^{(2)}(x_0) = \cdots = A^{([\beta])}(x_0) = 0$ (for instance, A is linear on S_0) condition (iii) of Theorem 2.1 has the common form as in condition (iii) in the introduction with the right-hand side $\partial\varphi(x_0)$. We shall see this in an example in Section 3.

For numerical approximations one has to approximate the infinite dimensional Banach space X by a sequence of finite-dimensional subspaces X_n:

$$X_1 \subset X_2 \subset \; ... \; \subset X_n \; ... \subset X, \quad P_n x \to x, \; n \to +\infty, \; \forall x \in X,$$

where P_n denotes the bounded projection from X onto X_n. Now, instead of (1.4) consider the finite-dimensional problems

$$A_n(x) + \alpha\partial\varphi_n(x) \;=\; f_{\delta n}, \quad x \in X_n, \tag{2.1}$$

where $A_n = P_n^* A P_n$, $\partial\varphi_n = P_n^* \partial\varphi P_n$ and $f_{\delta n} = P_n^* f_\delta$. It is easy to verify that A_n and $\partial\varphi_n$ are monotone and continuous in X_n. Therefore, Eq. (2.1) has a unique solution $x_{\alpha\delta}^n$ for $\alpha > 0$, $f_\delta \in X^*$, and the sequence $\{x_{\alpha\delta}^n\}$ converges to $x_{\alpha\delta}$ as $n \to \infty$ (see [14]).

Theorem 2.2. *Suppose the following conditions hold:*

(i) *A is $[\beta]$-times Fréchet differentiable at some neighbourhood $\mathcal{U}_0$ of S_0,*

(ii) *There exists a constant $L > 0$ such that*

$$\|A^{([\beta])}(x) - A^{([\beta])}(y) \le L\|x - y\|, \quad x \;\in\; S_0, \; y \;\in\; \mathcal{U}_0,$$

and

(iii) *$\alpha = \alpha(n, \delta)$ is chosen such that α, $\delta/\alpha \to 0$ and*

$$\Big(\gamma_n(x)\|(I - P_n)x\| + L\|(I - P_n)x\|^{[\beta]+1}/([\beta] + 1)!\Big)/\alpha \;\to 0, \; \forall x \in S_0,$$

as $n \to \infty$, where I is the identity operator in X and $\gamma_n(x)$ is defined by

$$\gamma_n(x) = \max\Big\{\|A'(x)(I - P_n)\|, \frac{1}{2!}\|A^{(2)}(x)(I - P_n)\|\|(I - P_n)x\|, \\ \ldots, \frac{1}{[\beta]!}\|A^{([\beta])}(x)(I - P_n)\|\|(I - P_n)x\|^{[\beta]-1}\Big\}.$$

Then the sequence $\{x^n_{\alpha\delta}\}$ *converges to* x_0.

Proof. From (2.1) we have

$$\begin{aligned}&\big\langle A_n(x^n_{\alpha\delta}) - A_n(x_n) + \alpha(\partial\varphi_n(x^n_{\alpha\delta}) - \partial\varphi_n(x_n)), x^n_{\alpha\delta} - x_n\big\rangle \\ &= \big\langle f_{\delta n} - A_n(x_n), x^n_{\alpha\delta} - x_n\big\rangle + \alpha\big\rangle\partial\varphi_n(x_n), x_n - x^n_{\alpha\delta}\big\rangle,\end{aligned} \tag{2.2}$$

where $x_n = P_n x$, $x \in S_0$. As

$$\begin{aligned}A(P_n x) &= A(x) + A'(x)(P_n x - x) + \frac{1}{2}A^{(2)}(x)(P_n x - x)^2 \\ &\quad + \cdots + \frac{1}{[\beta]!}A^{([\beta])}(P_n x - x)^{[\beta]} + r^n, \\ \|r^n\| &\le L\|(I - P_n)x)\|^{[\beta]+1}/([\beta]+1)!, \quad x \in S_0,\end{aligned}$$

from (2.2) together with $\langle f_{\delta n} - A_n(x_n), x^n_{\alpha\delta} - x_n\rangle = \langle f_\delta - A(x_n), x^n_{\alpha\delta} - x_n\rangle$ and $I - P_n = (I - P_n)^2$ it follows that

$$\begin{aligned}m_\varphi\|x^n_{\alpha\delta} - x_n\|^\beta &\le \Big(\delta + [\beta]\gamma_n(x)\|(I - P_n)x\| \\ &\quad + L\|(I - P_n)x\|^{[\beta]+1}/([\beta]+1)!\Big)\|x^n_{\alpha\delta} - x_n\|/\alpha \\ &\quad + \langle\partial\varphi(x_n), x_n - x^n_{\alpha\delta}\rangle, \quad x_n = P_n x,\ x \in S_0.\end{aligned} \tag{2.3}$$

Applying the technique in the proof of Theorem 2.2 in [4], we get the conclusion of our theorem. □

Remark 2.2. From the above proof we can see that this theorem is still valid if condition (iii) is replaced by

(iii') $\quad \gamma^1_n(x)/\alpha \to 0, \quad \gamma^1_n(x) = \|(I - P_n)x)\|, \quad \forall x \in S_0.$

Theorem 2.3. *Assume that conditions* (i) - (iii) *of Theorem 2.1 and* (iii') *hold, and:*

(*) There exist two constants $L' > 0$, $\gamma' > 0$ *such that*

$$\langle\partial\varphi(y) - \partial\varphi(x_0), z\rangle \le L'\|y - x_0\|^{\gamma'}\|z\|, \quad \forall y,\ z \in S(x_0, r).$$

Choose $\alpha \sim (\delta + \tilde\gamma_n)^\mu$, $0 < \mu < 1$, *where*

$$\tilde\gamma_n = \max\Big\{\|(I - P_n)x_0\|, \sup_{y_i \in S(x_0,r)}\|(I - P_n)\omega\|, \|(I^* - P^*_n)f_0\|\Big\}.$$

Then

$$\|x^n_{\alpha\delta} - x_0\| = O\Big(\delta^{\theta_1} + \tilde{\gamma}_n^{\theta_2}\Big),$$

where

$$\theta_1 = \min\{(1-\mu)/(\beta-1), \mu/\beta\}, \quad \theta_2 = \min\{\theta_1, \gamma'/(\beta-1)\}.$$

Proof. Since

$$\begin{aligned}\|A(P_n x_0) - f_\delta\| &\le \delta + [\beta]\gamma_n(x_0)\|(I-P_n)x_0\| \\ &\qquad + L\|(I-P_n)x_0\|^{[\beta]+1}/([\beta]+1)!\ ,\end{aligned}$$

from (2.3) (with $x = x_0$) and condition (*) it follows that

$$\begin{aligned}&\alpha m_\varphi \|x^n_{\alpha\delta} - x_{0n}\|^\beta \\ &\le \Big(\delta + [\beta]\gamma_n(x_0)\tilde{\gamma}_n + L\tilde{\gamma}_n^{[\beta]+1}/([\beta]+1)!\Big)\|x^n_{\alpha\delta} - x_{0n}\| \\ &\quad + \alpha\langle\partial\varphi(x_0), x_{0n} - x^n_{\alpha\delta}\rangle + \alpha\langle\partial\varphi(x_{0n}) - \partial\varphi(x_0), x_{0n} - x^n_{\alpha\delta}\rangle \\ &\le \Big(\delta + [\beta]\gamma_n(x_0)\tilde{\gamma}_n + L\tilde{\gamma}_n^{[\beta]+1}/([\beta]+1)! + L'\alpha\tilde{\gamma}_n^{\gamma'}\Big)\|x^n_{\alpha\delta} - x_{0n}\| \\ &\quad + \alpha\langle\omega, A(x_0) - A(x^n_{\alpha\delta})\rangle + \alpha\|\omega\|\|r^n_{\alpha\delta}\|,\end{aligned}$$

where

$$\begin{aligned}r^n_{\alpha\delta} &= A(x^n_{\alpha\delta}) - A(x_0) + A'(x_0)(x_0 - x^n_{\alpha\delta}) - \frac{1}{2!}A^{(2)}(x_0)(x_0 - x^n_{\alpha\delta})^2 \\ &\quad + \frac{1}{3!}A^{(3)}(x_0)(x_0 - x^n_{\alpha\delta})^3 - \cdots + \frac{(-1)^{[\beta]-1}}{([\beta])!}A^{([\beta])}(x_0)(x_0 - x^n_{\alpha\delta})^{[\beta]}, \\ \omega &= \omega(x^n_{\alpha\delta} - x_0, ..., x^n_{\alpha\delta} - x_0), \text{ and} \\ \|r^n_{\alpha\delta}\| &\le L\|x^n_{\alpha\delta} - x_0\|^{[\beta]+1}/([\beta]+1)! \\ &\le L\|x^n_{\alpha\delta} - x_{0n}\|^{[\beta]+1}/([\beta]+1)! + O(\tilde{\gamma}_n).\end{aligned}$$

Let C_1 be a constant such that $\|P_n\| \le C_1$. Then

$$\begin{aligned}&\langle\omega, A(x_0) - A(x^n_{\alpha\delta})\rangle \\ &= \langle\omega, f_0 - f_{\delta n} + f_{\delta n} - A_n(x^n_{\alpha\delta})\rangle + \langle\omega, A_n(x^n_{\alpha\delta}) - A(x^n_{\alpha\delta})\rangle \\ &\le \|\omega\|\Big(\tilde{\gamma}_n + C_1(\delta + \alpha\|\partial\varphi(x^n_{\alpha\delta})\|)\Big) + \langle(P_n - I)\omega, A(x^n_{\alpha\delta})\rangle.\end{aligned}$$

Because of locally bounded property of a semicontinuous and monotone operator (see [6]), there exists a positive constant C_2 such that

$$\langle \omega, A(x_0) - A(x^n_{\alpha\delta})\rangle \leq \|\omega\|(\tilde{\gamma}_n + C_1\delta + C_2\alpha) + C_2\|(I - P_n)\omega\|.$$

Consequently, for a constant $C_3 > 0$ we get

$$\begin{aligned}
&\alpha\Big(m_\varphi - \frac{L\|\omega\|}{([\beta]+1)!}\|x^n_{\alpha\delta} - x_{0n}\|^{[\beta]+1-\beta}\Big)\|x^n_{\alpha\delta} - x_{0n}\|^\beta \\
&\leq \Big(\delta + [\beta]\gamma_n(x_0)\tilde{\gamma}_n + L\tilde{\gamma}_n^{[\beta]+1}/([\beta]+1)! + L'\alpha\tilde{\gamma}_n^{\gamma'}\Big)\|x^n_{\alpha\delta} - x_{0n}\| \\
&\quad + \|\omega\|\Big(C_1\delta + C_2\alpha + C_3\tilde{\gamma}_n\Big).
\end{aligned}$$

By [10], we have

$$\|x^n_{\alpha\delta} - x_{0n}\| = O(\delta^{\theta_1} + \tilde{\gamma}_n^{\theta_2}).$$

Thus

$$\|x^n_{\alpha\delta} - x_0\| = O(\delta^{\theta_1} + \tilde{\gamma}_n^{\theta_2}). \qquad \square$$

Remark 2.3. If X is a Hilbert space, we can take the functional $\varphi(x) = \|x\|^\beta$, $2 < \beta < 3$. Then $m_\varphi = 2^{2-\beta}$ and condition (iii) of Theorem 2.1 has the form

$$\Big(A'^*(x_0) + \frac{1}{2}A^{(2)*}(x_0)y\Big)\omega = \partial\varphi(x_0).$$

For the Banach spaces of Lebesgue's type ℓ_p, L_p, W^p_m, $1 < p < 2$ if $s = 2$, we can construct $\varphi(x)$ as in the Hilbert space case. For the case $2 < p$, $s = p$ and $p \neq [p]$, our results are still true if we use $\partial\varphi(x) = U^\beta(x)$ (see [3]). And we have $m_\varphi = 2^{2-p}/p$. If $p = [p]$, the number β is chosen so that $p < \beta < p + 1$. We can also use other forms of $\varphi(x)$ (see [8]).

3. Application

Consider the nonlinear singular integral equation in the form (see [7])

$$\int_0^t (t-s)^{-\lambda}x(s) + F(x(t)) = f_0(t), \quad 0 < \lambda < 1, \tag{3.1}$$

where $f_0 \in L_q([0,1])$, $1 < q < +\infty$ and the nonlinear function $F(t)$ satisfies the following conditions:

- $|F(t)| \leq a_1 + a_2|t|^{p-1}, \ a_1, \ a_2 > 0, \ p^{-1} + q^{-1} = 1,$
  $F(t_1) \leq F(t_2)$ iff $t_1 \leq t_2$, and
- F is differentiable.

Thus, F is a monotone operator from $X = L_p([0,1])$ into $X^* = L_q([0,1])$. In addition, assume that F is a compact operator. Then (3.1) is an ill-posed problem, because the operator K defined by

$$Kx(t) = \int_0^t (t-s)^{-\lambda} x(s)ds,$$

also is compact. The above theoretical results can be applied to this problem. Note that in the cases $1 < p \leq 2$ the smooth condition (iii) in the introduction now has the form

$$\Big(K^* + F'^*(x_0) + \frac{1}{2}F^{(2)*}(x_0)y\Big)\omega = \partial\varphi(x_0), \tag{3.2}$$

where $x_0(t)$ is a norm-minimal solution of (3.1) and φ is as in Section 2 with $2 < p < 3$. If $F(x(t)) =$ const. for all solutions $x(t)$ of (3.1), then equation (3.2) has a simpler form $K^*\omega = \partial\varphi(x_0)$.

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References

1. Ya. I. Alber, *On solution of nonlinear problems involving monotone operators in Banach spaces*, Sibirskii Math. J. **16** (1975), 3-11 (in Russian).
2. Ng. Buong, *On nonlinear ill-posed problems,* JJIAM (to appear).
3. ________ , *Nonlinear ill-posed problems involving monotone operators in Banach spaces*, Math. Nachr. (to appear).
4. ________ , *Regularization by linear operators*, Acta Math. Vietnam **21** (1996), 135-145.
5. H. W. Engl, K. Kunisch and A. Neubauer, *Convergence rates for Tikhonov regularization of non-linear ill-posed problems*, Inverse Problems **5** (1989), 523-540.
6. H. Gajewski, K. Groger und K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferetialgleichungen*, Akademie Verlag, Berlin 1974.
7. A. I. Guseinov and X. Sh. Mukhtarov, *Introduction to the theory of nonlinear singular integral equations*, Nauka, Moskva, 1980 (in Russian).
8. R. Kluge, *Gleichmaßig monotone Potentialoperatoren auf einigen Funktionalraumen*, Math. Nachr. **126** (1986), 301-309.

9. F. Natterer, *Error bounds for Tikhonov regularization in Hilbert scales*, Appl. Analysis, **18** (1984), 29-37.

10. A. Neubauer, *An a-posteriori parameter choice for Tikhonovregularization in Hilbert scales leading to optimal convergence rates*, SIAM J. Numer. Math. **25** (1988), 1313-1326.

11. ________ , *Tikhonov regularization for nonlinear ill-posed problems: Optimal convergence rates and finite-dimensional approximation.* Inverse Problems **55** (1989), 541-577.

12. ________ , *Tikhonov regularization of Nonlinear ill-posed Problems in Hilbert Scales*, Johannes Kepler Univ. **417** (1990).

13. ________ and O. Scherzer, *Finite-dimensional approximation of Tikhonov regularization of nonlinear ill-posed problems*, Numer. Funct. Analysis Optim. **11** (1990), 85-89.

14. M. Sibony, *Sur l'approximation d'équation et inéquations aux derivées partielles nonlinéaires de type monotone*, J. Math. Anal. Appl. **34** (1972), 502-564.

15. A. N. Tikhonov and V. I. Arsenin, *Solutions of ill-posed problems*, Wiley, New York, 1977.

16. M. M. Vainberg, *Variational method and method of monotone operators*, Nauka, Moskva, 1972 (in Russian).

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