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DUALITY FOR RADIANT AND SHADY PROGRAMS

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Dedicated to Hoang Tuy on the occasion of his seventieth birthday

Abstract. We consider the two classes of closed convex sets which are stable under dilatations and shrinkings respectively (homotheties of rates greater than one and less than one respectively). We define conjugacies for the classes of functions whose sublevel sets belong to these classes. For such functions, the conjugate can be defined on the dual space and an extra parameter is not needed. We apply these notions to the maximization of a convex function on a convex set and to the minimization of a convex function on the set of points outside a convex subset. We introduce several dual problems related to each of these problems and we give conditions ensuring there is no duality gap.

1. Introduction

In a number of optimization problems, a particular point plays a special role. In some cases this point is known to be a global minimizer of the objective function $f : X \to \overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty, \infty\}$ and one looks for non trivial local minimizers. In some other cases, this point is a singular point of the objective function. This may be the case for the fractional programming problem

$$(\mathcal{F}) \quad \text{minimize} \quad q(x) := \frac{n(x)}{d(x)} : x \in X.$$

Even when this point is not singular, it may be irrelevant for the problem under consideration. This is the case for the problems

$$(\mathcal{M}) \quad \text{maximize} \quad f(x) : x \in F$$

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$$(\mathcal{R}) \quad \text{minimize} \quad f(x) : x \in X \setminus C,$$

when the feasible set F does not contain that point, with a special emphasis to the cases the objective function f is convex and C, F are convex. These two cases (called anticonvex programs in [31]) are difficult to deal with, either from a theoretical viewpoint or from a numerical viewpoint, although some convexity properties are present. In particular, local solutions are not necessarily global solutions. Therefore they have been widely studied during the last few years (see for instance [14], [17]-[19], [31], [40]-[43], [44], [47]-[49], [53]-[56]).

A key tool for the study of such problems is a conjugacy scheme. Most conjugacy schemes require an extra parameter for the dual problem or are not symmetric. However, among the proposed conjugacy schemes, the ones devised by Atteia-Elqortobi ([1], [33]), Thach ([47]-[49]) and their variants by Rubinov and Simsek [38], Rubinov and Glover [37] are particularly attractive because they do not require the introduction of such an extra parameter as in the classical conjugacy schemes for quasiconvex functions ([11], [20], [23], [29], [32]-[35],...). In the present paper, as in [31], we follow this line of thought, also restricting our attention to special classes of quasiconvex functions. The classes we consider are the class of quasiconvex radiant functions and the class of quasiconvex shady functions, a function f being called radiant (resp. shady) if its sublevel sets are radiant (resp. shady). Here, a subset S of a vector space is said to be radiant (resp. shady) if it is stable under homotheties of rate less (resp. greater) than 1. We also add topological assumptions such as lower semicontinuity. Radiant (resp. shady) functions obviously attain their minimum (resp. maximum) at 0; such functions have been studied in [47]-[50], [38], [37] and [31]. Radiant functions is a rather large subclass of the class of quasiconvex functions; but the class of shady functions seems to be more restricted, although it encompasses the class of nonincreasing functions on $\mathbf{R}$ with domain $\mathbf{R}_+$ and the class of quasiconvex positively homogeneous functions of degree $d \geq 0$ which are nonpositive on their domains. We observe that radiant functions and shady functions are stable under several operations, in particular under composition with a nondecreasing function from $\mathbf{R}$ to $\mathbf{R}$.

The conjugates we study are defined on the dual space of the given space and have the similar property of being radiant or shady. Thus our conjugacy scheme is perfectly symmetric. Moreover, lower semicontinuous (l.s.c.) quasiconvex radiant (resp. shady) functions coincide with their biconjugates. In [47]-49] the biconjugate of a function f does not coincide with f unless f is upper semicontinuous (u.s.c.), a restrictive assumption we wish to avoid, indicator functions of closed subsets being of great use in optimization theory. Furthermore, the conjugacy and the duality relationships we obtain do not require some extra assumptions needed in [47]-[49].

In order to get these properties, we modify the definition of the conjugate of f, using four sorts of half-spaces. Accordingly, we consider four kinds of polar sets of a given subset, one of which being the usual polar set, another one being the one introduced by Atteia-Elqortobi in [1] and the other two being variants using strict inequalities. It appears that such slight changes have appealing consequences in terms of sublevel sets and, clearly, sublevel sets are important for quasiconvex functions.

These conjugacies have the advantage of entering into the general framework of the Fenchel-Moreau conjugacy (see [27], [2] and others such as [6], [24], [30], [35],...; cf. the forthcoming book [28] and its references) for which the conjugate of f is given by

$$f^c(y) := -\inf_{x \in X} (f(x) - c(x, y)),$$

where $c : X \times Y \to \overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty, \infty\}$ is a coupling function. In such a way, known results or tools (such as perturbational duality, subdifferentials) can be used easily, as in [20]-[24], [32], [33], [41]. We also point out the links with polarity, as expounded in [10], [58] (see Section 3).

Section 2 is devoted to two classes of convex subsets of a locally convex topological vector space (in brief l.c. space). The functions whose sublevel sets are of one of these two types are studied in Section 4, simultaneously with their conjugacies. Section 5 is devoted to the maximization of a radiant function on a convex subset and to a reverse convex program. More applications are given in [31], [47]-[50], [53]-[56].

2. Radiant sets, shady sets and functions

Recall that a subset S of a vector space X is said to be *starshaped* if for any $x \in S$ and any $t \in]0, 1]$ one has $tx \in S$. We will say that a subset S of X is *radiant* if it is starshaped, convex and contains 0 or is empty. Thus a set is radiant iff it is convex and contains 0 or is empty. We will say that a subset S of a vector space X is *co-starshaped* if for any $x \in S$ and any $t \geq 1$ one has $tx \in S$. If moreover $S = X$ or does not contain 0 we say that S is *strictly co-starshaped*. If S is convex and strictly co-starshaped we say that S is *shady*. Thus, for a subset S of X, S is co-starshaped (resp. strictly co-starshaped) iff $S^c := X \setminus S$ is starshaped (resp. radiant). It is easy to see that a subset S of $X \setminus \{0\}$ is strictly co-starshaped iff its image by the inversion $x \mapsto \|x\|^{-2}x$ is starshaped.

Since any intersection of starshaped (resp. co-starshaped) subsets of a vector space X is starshaped (resp. co-starshaped), given a subset A of X there exists a smallest starshaped (resp. co-starshaped) set S containing A; we call it the starshaped (resp. co-starshaped) hull of A.

The closure of a starshaped (resp. co-starshaped) subset of a topological vector space is starshaped (resp. co-starshaped), so that we can speak of the closed starshaped (resp. co-starshaped) hull of a set. Similarly, the convex hull of a co-starshaped subset is a co-starshaped subset, so that we can easily describe the co-starshaped convex hull of a subset. Let us note, however, that it may happen that the convex hull $\mathrm{co}(S)$ of a co-starshaped subset S of $X \setminus \{0\}$ contains 0 (take $X = \mathbf{R}$, $S := \{x \in X : |x| \geq 1\}$). Thus, the class of strictly co-starshaped subsets is not invariant by the operation of taking convex hulls.

A class of elementary radiant (resp. co-starshaped) subsets of a l.c. space X is formed by the open or closed half-spaces containing 0 (resp. not containing 0) to which one adjoins $\emptyset$ and X. Another class is formed by the open (or closed) half-spaces containing (resp. not containing) 0 in their interiors. These classes can serve to give dual characterizations of radiant (resp. co-starshaped) hulls which satisfy a further topological property. In this respect, let us recall that a subset of a topological vector space is said to be *evenly convex* if it is the whole space or an intersection of open half-spaces. We will say that a subset is evenly radiant, if it is radiant and evenly convex. Part (a) of the following characterizations is well known. The notation we use will be justified in the following section where it will be shown that the four hulls we consider are associated with the closure operations deduced from polarities.

Proposition 2.1. *Let C be a nonempty subset of a l.c. space X.*

(a) *C is closed and radiant iff C is the intersection of the family $\mathcal{H}_0(C)$ of closed half-spaces containing C and containing 0 in their interior.*

(b) *C is evenly radiant iff it is the intersection of the family $\mathcal{H}_\wedge(C)$ of open half-spaces of X containing C and 0.*

(c) *C is closed and shady iff it is the intersection of the family $\mathcal{H}_\nabla(C)$ of closed half-spaces of X containing C and not containing 0.*

Proof. Assertion (a) is a consequence of the bipolar theorem or of the Hahn-Banach theorem, observing that any closed half-space $[y_0 \leq r]$ separating C and an element $x_0 \in X \setminus C$ can be changed into the closed half-space $[y \leq 1]$ for some $s > 0$, $y = sy_0 \in Y \setminus \{0\}$ and $[y \leq 1]$ is an element of $\mathcal{H}_0(C)$. Assertion (b) is immediate.

Assertion (c) is proved in [49], Theorem 5.1. Let us give a short proof similar to the preceding justification. Given a nonempty closed convex costarshaped subset C of X and $x_0 \in X \setminus C$ we observe that the segment $S := co(0, x_0) := [0,1]x_0$ does not intersect C as C is costarshaped. Let $y_0 \in Y \setminus \{0\}$ and $r \in \mathbf{R}$ be such that $S \subset [y_0 < r]$ and $C \subset [y_0 \geq 1]$. Then we have $r > 0$ and for $y := r^{-1}y_0$ we have $[y \geq 1] \in \mathcal{H}_\nabla(\mathbf{C})$ and $x_0 \notin [y \geq 1]$. □

The preceding proposition suggests to introduce the following terminology; we will say that a subset C of X is *evenly shady* if it is the whole space or an intersection of open half-spaces whose closures do not contain 0. Observe that there exist evenly convex subsets which are co-starshaped and which closures do not contain 0 but are not evenly shady.

Example 2.1. Let $C := \{(r,s) \in \mathbf{R}^2 : r > 1,\ s > 0\}$. Then one can check that for $x := (t,0)$ with $t > 1$ one cannot find $y \in \mathbf{R}^2$ such that $\langle w, y\rangle > 1$ for each $w \in C$ and $\langle x, y\rangle \leq 1$.

Corollary 2.2. *Let S be a nonempty subset of a l.c. space X with dual space Y.*

(a) *The closed radiant hull S^∞ of S is the intersection of the family $\mathcal{H}_0(S)$ of closed half-spaces containing 0 in their interior.*

(b) *The evenly radiant hull $S^{\wedge\wedge}$ of S is the intersection of the family $\mathcal{H}_\wedge(S)$ of open half-spaces of X containing S and 0.*

(c) *The closed convex shady hull $S^{\nabla\nabla}$ of S is the intersection of the family $\mathcal{H}_\nabla(S)$ of closed half-spaces of X containing S and not containing 0 in their interiors.*

(d) *The evenly shady hull $S^{\vee\vee}$ of S is the intersection of the family $\mathcal{H}_\vee(S)$ of open half-spaces of X containing S and not containing 0 in their closure.*

Proof. Assertion (d) follows from our definition whereas the other ones are consequences of the fact that if C is the hull of S in one of the three cases one has $\mathcal{H}(C) = \mathcal{H}(S)$ when $\mathcal{H}$ denotes the corresponding family of half-spaces. □

It is possible to characterize the class of radiant (resp. shady) closed convex subsets by their support function

$$\sigma_C(y) := \sup_{x \in C} \langle x, y\rangle \qquad y \in Y.$$

However, since the same support function may correspond to several subsets C if these sets are not closed and convex, one cannot expect a characterization for evenly convex subsets.

Proposition 2.3. *Let C be a nonempty closed convex subset of X.*

(a) *C is radiant iff σ_C takes its values in $\mathbf{R}_+ \cup \{\infty\}$;*

(b) *C is co-starshaped iff σ_C takes its values in $\mathbf{R}_- \cup \{\infty\}$;*

(c) *C is shady iff σ_C takes its values in $\mathbf{R}_- \cup \{\infty\}$ and at least one of its values is negative.*

Proof. Both assertions use the fact that

$$C = \{x \in X : \langle x, y\rangle \leq \sigma_C(y) \ \forall y \in Y\}.$$

Then the first one immediate. The if part of the second one follows from this relation; the only if part stems from the fact that if C is co-starshaped and if $\sigma_C(y)$ is positive then one can find some $x \in C$ such that $\langle x, y\rangle > 0$, so that $\sigma_C(y) \geq \sup_{t \geq 1} \langle tx, y\rangle = \infty$. If C is shady, i.e. if $0 \notin C$ in our case, we can separate C and $\{0\}$ by using some $\overline{y} \in Y$ and some $r < 0$ such that $C \subset [\overline{y} \leq r]$; then $\sigma_C(\overline{y}) \leq r < 0$. The converse is obvious. □

3. Interpretation in terms of polarities

It is convenient to adopt a unified approach to the preceding characterizations using the notion of polarity. The one we follow here is not as general as the one in [58] which uses the general framework of complete lattices, but it is well-adapted to our purposes; see also [10], [22], [41]. Let us recall that a *polarity* (or duality) between two sets X, Y is a mapping $P := \mathcal{P}(X) \to \mathcal{P}(Y)$, where $\mathcal{P}(X)$ (resp. $\mathcal{P}(Y)$) is the power set of X (resp. Y), i.e. the set of all subsets of X (resp. Y), which satisfies the condition

$$P\Big(\bigcup_{i \in I} A_i\Big) = \bigcap_{i \in I} P(A_i)$$

for any family $(A_i)_{i \in I}$ of subsets of X.

It is easy to check that each of the four operations introduced in the preceding section are polarities. In fact, each of these polarities is associated with a family of half-spaces of X and therefore stems forms a general construction which can be described as follows. Given sets X, Y and a family $(E(y))_{y \in Y}$ of subsets of X indexed by Y, called the family of elementary subsets of X or the family of half-spaces of X, one defines a polarity P associated with these data by

$$P(A) := \{y \in Y : E(y) \supset A\}$$

for $A \in \mathcal{P}(X)$. It is easy to check that P is a polarity, whatever the family $(E(y))_{y \in Y}$ is. One can see E as a multimapping (or correspondence)

$E : Y \rightrightarrows X$, or, identifying it with its graph in $Y \times X$, as a subset of $Y \times X$. Then $P(\{a\}) = E^{-1}(a)$.

It has been observed in [41], Theorem 1 that in fact any polarity can be obtained in that way.

Now, with any polarity $P : \mathcal{P}(X) \to \mathcal{P}(Y)$ one can associate a dual polarity $P^* : \mathcal{P}(Y) \to \mathcal{P}(X)$ (also denoted by P when there is no rick of confusion) given by

$$P^*(B) := \{x \in X : B \subset P(x)\}.$$

In particular, denoting by P_X the multimapping (correspondence) $P_X : X \to \mathcal{P}(Y)$ obtained by restricting P to the singletons of X (i.e. $P_X(x) := P(\{x\})$) one has

$$P^*(\{y\}) = (P_X)^{-1}(y) \qquad \forall y \in Y.$$

When P is constructed with the help of a family $(E(y))_{y \in Y}$ of elementary subsets of X as above, one has

$$P^*(B) = \{x \in X : \forall b \in B x \in E(b)\} = \bigcap_{b \in B} E(b).$$

Thus $P^*(\{b\}) = E(b)$ and P^* is the unique polarity which extends E considered as a multimaping from Y into X. One easily checks that the composition $C := P^* \circ P$ of P and P^* is a closure operation in $\mathcal{P}(X)$ and similarly $P \circ P^*$ is a closure operation in $\mathcal{P}(Y)$ in the sense that C is extensive ($C(A) \supset A$ for each $A \in \mathcal{P}(X)$), idempotent ($C(C(A)) = C(A)$ for each $A \in \mathcal{P}(X)$) and homotone ($C(A) \subset C(A')$ whenever A, $A' \in \mathcal{P}(X)$ are such that $A \subset A'$). It will be convenient to say that a subset B of Y is P-*convex* (or P-closed) if it is the image by P of some subset of X; we adopt a similar terminology for P-convex subsets of X.

Considering a l.c. space X with (topological) dual Y and the four "half-spaces" of X associated with some $y \in Y$ given by

$$E_{\bowtie}(y) := \{x \in X : \langle x, y \rangle \bowtie 1\},$$

where the relation $\bowtie$ is $\leq$, $<$, $\geq$, $>$ respectively (so that $E_{\bowtie}(0)$ is either X or the empty set), we get four kinds of polar set. The first ones are the usual polar set and its strict analogue:

$$A^o := \{y \in Y : \forall x \in A \ \langle x, y \rangle \leq 1\},$$
$$A^\wedge := \{y \in Y : \forall x \in A \ \langle x, y \rangle < 1\};$$

the third one is the copolar set used in [1] and the fourth one is its strict analogue:

$$A^{\nabla} := \{y \in Y : \forall x \in A \ \langle x, y\rangle \geq 1\},$$
$$A^{\vee} := \{y \in Y : \forall x \in A \ \langle x, y\rangle > 1\}.$$

It is easy to check that the hulls of a subset S of X described in Proposition 2.2 are obtained by taking S^{PP}, where P is one of the four polarities P^0, $P^{\wedge}$, P^{∇}, $P^{\vee}$ given above. This observation justifies our notation.

Let us also remark that the preceding four polarities can be given whenever X and Y are sets linked by a coupling function c which has no linearity property. In particular, X and Y may be convex cones in some l.c. spaces and c be the usual evaluation coupling or some less familiar coupling.

Let us note the following obvious observation for latter use.

Lemma 3.1. *The polar A^0 (resp. $A^{\wedge}$, A^{∇}, $A^{\vee}$) of an arbitrary subset A of X is radiant (resp. evenly radiant, closed shady, evenly shady).*

4. Conjugates of shady and radiant functions

Let us first delineate the classes of functions we shall use.

Definition 1. An extended real-valued function f on a vector space X is said to be radiant (resp. evenly radiant, resp. shady, resp. evenly shady) if for each real number r its sublevel set

$$[f \leq r] := \{x \in X : f(x) \leq r\}$$

is radiant (resp. evenly radiant, resp. shady, resp. evenly shady).

Observe that the function f is radiant (resp. shady) iff for each number r its strict sublevel set

$$[f < r] := \{x \in X : f(x) < r\}$$

is radiant (resp. shady) since

$$[f < r] = \bigcup_{s<r} [f \leq s], \quad [f \leq r] = \bigcap_{s>r} [f < s].$$

Radiant (resp. shady) functions are the functions which are quasiconvex and nondecreasing (resp. nonincreasing) along rays emanating from 0. The class of radiant (resp. shady) functions enjoys interesting stability properties. Among the operations which leave invariant this class is the level sum defined by

$$(g \overset{\vee}{+} h)(x) := \inf_{w \in X} g(x-w) \vee h(w).$$

It is known that this operation is an important tool for the study of quasi-convex functions; it bears some analogy with the infimal convolution (see [60], [62] and their references).

Lemma 4.1. *The class of radiant (resp. shady) functions is stable under suprema and level sums. Moreover, if $f : W \times X \to \overline{\mathbf{R}}$ is radiant (resp. shady), then the performance function $p : W \to \overline{\mathbf{R}}$ given by $p(w) := \inf_{x \in X} f(w,x)$ is radiant (resp. shady).*

Proof. Stability under suprema sterms from the relation

$$[f \leq r] = \bigcap_{i \in I} [f_i \leq r]$$

for $f := \sup_{i \in I} f_i$. Since the level sum $g \overset{\vee}{+} h$ of the functions g, h satisfies the relation

$$[g \overset{\vee}{+} h < r] = [g < r] + [h < r], \quad \forall r,$$

stability for the level sum stems from stability of the class of radiant (resp. shady) sets under addition. It can also be deduced from the fact that the performance function p associated to a radiant (resp. shady) function f is radiant (resp. shady). This last fact is easy to check. $\square$

Now let us define appropriate conjugacies. One can associate a conjugacy to any polarity by setting for an extended real-valued function f on X

$$f^P(y) := \sup_{x \in X \setminus P^*(y)} -f(x) \quad \text{for } y \in Y, \tag{4.1}$$

where P^* (also denoted by P if there is no rick of confusion, in particular when P is symmetric) is the dual polarity associated with P. When P is defined by a family $(E(y))_{y \in Y}$ of elementary subsets of X one has

$$f^P(y) = \sup_{x \in X \setminus E(y)} -f(x) \quad \text{for } y \in Y.$$

Thus f^P is a supremum of cliff functions, a function being a cliff function associated with the family of half-spaces $(E(x))_{x \in X}$ if it takes some finite value r on some half-space $Y \setminus E(x)$ and the value $-\infty$ on $E(x)$. Taking the four polarities introduced previously, we get the following conjugates:

$$\begin{aligned} f^{o}(y) &:= -\inf\{f(x) : \langle x, y\rangle > 1\}, \\ f^{\wedge}(y) &:= -\inf\{f(x) : \langle x, y\rangle \geq 1\}, \\ f^{\nabla}(y) &:= -\inf\{f(x) : \langle x, y\rangle < 1\}, \\ f^{\vee}(y) &:= -\inf\{f(x) : \langle x, y\rangle \leq 1\}. \end{aligned}$$

The conjugates f^0 and f^{∇} have been defined in [1]; $f^{\wedge}$ is a variant of the conjugate f^H extensively studied in [47]-[49]. While the study of the conjugate f^{∇} has been limited to [1], we have learned after the writting of a first version of the present paper that a conjugate directly related to $f^{\vee}$ has been introduced in [50]. In fact the R-conjugate of f given in [50] is related to $f^{\vee}$ via the formula $f^{\vee}(y) = f^R(-y)$.

As observed in [32], [33], this conjugacy enters the framework of the generalized Fenchel-Moreau conjugacy: introducing the coupling function

$$c^P(x, y) := -\iota_{X\setminus P^*(y)}(x) = -\iota_{Y\setminus P(x)})(y), \tag{4.2}$$

where, for a subset A of X, ι_A stands for the indicator function of A given by $\iota_A(x) = 0$ if $x \in A$, $\iota_A(x) = \infty$ otherwise, one has

$$f^P(y) = -\inf_{x\in X} \left(f(x) - c^P(x, y)\right)$$

with the convention $(+\infty) - (+\infty) = +\infty$. This observation enables one to use the general properties of such conjugacies. In particular, one has the following properties which can also be proved directly.

Proposition 4.2. (a) *If $f \leq g$ then $f^P \geq g^P$;*

(b) *for any family $(f_i)_{i\in I}$ of functions on X one has $\left(\inf_{i\in I} f_i\right)^P = \sup_{i\in I} f_i^P$.*

(c) *for any function f and any real number r one has $(f+r)^P = f^P - r$.*

More specific properties are stated in the next propositions.

Proposition 4.3. (a) *For any function f and any $r \in \mathbf{R}_+$ one has $(rf)^P = rf^P$;*

(b) *For any function g and any nondecreasing upper semicontinuous (u.s.c.) function $h : \mathbf{R} \to \mathbf{R}$ one has for each $y \in Y$*

$$(h \circ g)^P(y) = -h(-g^P(y)).$$

(c) *For any function f one has* $\sup_{y \in Y} f^P(y) = -\inf_{x \in X \setminus P^*(Y)} f(x)$.

(d) *In particular, one has*

$$\sup_{y \in Y} f^0(y) = -\inf_{x \in X \setminus \{0\}} f(x) = \sup_{y \in Y} f^{\wedge}(y),$$
$$\sup_{y \in Y} f^{\vee}(y) = -\inf_{x \in X} f(x) = \sup_{y \in Y} f^{\nabla}(y).$$

Proof. The first assertion follows from the fact that for any subset A of X and any positive number r one has $r\iota_A = \iota_A$. The second one is a consequence of the equality

$$\inf h(S) = h(\inf S)$$

for any subset S of $\mathbf{R}$, in particular for $S = g(X \setminus P^*(y))$. The third one follows from the equality

$$\bigcup_{y \in Y} (X \setminus P^*(y)) = X \setminus P^*(Y).$$

The last one follows from the relations $P^*(Y) = \{0\}$ for $P = P^0$, $P = P^{\wedge}$, $P^*(Y) = \emptyset$ for $P = P^{\vee}$, $P = P^{\nabla}$. □

Example 4.1. The relation of assertion (b) is not necessarily valid if h is not supposed to be u.s.c., even when h is increasing: taking $X = \mathbf{R}$, $g(x) = x$, $h(r) = r$ for $r \leq 1$, $h(r) = r+1$ for $r > 1$ one gets $(h \circ g)(1) = -2$, $-h(-g^0(1)) = -h(1) = -1$.

Proposition 4.4. *For any extended real-valued function f on X, the conjugate f^P is P-quasiconvex in the sense that for each $r \in \mathbf{R}$ the sublevel set $[f^P \leq r]$ is P-convex. More precisely, one has*

$$[f^P \leq r] = P([f < -r]).$$

Proof. It suffices to prove the preceding relation. It follows from the equivalences

$$\begin{aligned}
(y \in [f^P \leq r]) &\Leftrightarrow (x \in X \setminus E(y) \Rightarrow f(x) \geq -r) \\
&\Leftrightarrow (x \in [f < -r] \Rightarrow x \in E(y)) \\
&\Leftrightarrow ([f < -r] \subset E(y)) \\
&\Leftrightarrow (y \in P([f < -r])).
\end{aligned}$$

□

The preceding formula shows that the conjugacy we defined coincides with the one introduced in [58]. It contains Proposition 2.1 of [31] which deals with the cases of the polarities P^o and $P^\wedge$ for which the conjugate f^0 (resp. $f^\wedge$), is l.s.c. radiant (resp. evenly radiant). Let us state the two other cases.

Corollary 4.5. *The conjugate $f^\vee$ (resp. f^∇) of any function f is evenly shady (resp. shady and l.s.c).*

Corollary 4.6. *The biconjugate $f^{PP} := (f^P)^P$ of any function f is such that*

$$[f^{PP} \leq r] = \bigcap_{s>r} [f < s]^{PP}.$$

This formula characterizes the biconjugate.

Proof. One has

$$\begin{aligned} [f^{PP} \leq r] &= [f^P < -r]^P \\ &= \Big(\bigcup_{s>r} [f^P \leq -s] \Big)^P \\ &= \bigcap_{s>r} [f^P \leq -s]^P \\ &= \bigcap_{s>r} [f < s]^{PP}. \end{aligned}$$

□

Obviously one has $f^\wedge \geq f^0$ (resp. $f^\nabla \geq f^\vee$). It is convenient to introduce a terminology for the cases in which equality holds.

Definition 2. A function f is said to be a Thach (resp. co-Thach) function if $f^\wedge = f^0$ (resp. $f^\nabla = f^\vee$).

A criteria for such a property is as follows.

Lemma 4.7. *Suppose that f is almost shady (resp. almost-starshaped) in the sense that for each $x \in X \setminus \{0\}$ and each $s > f(x)$ there exists $t > 1$ (resp. $t < 1$) such that $f(tx) < s$. Then f is a Thach function (resp. a co-Thach function). In particular, any function such that for each $x \in X \setminus \{0\}$ the radial function $f_x : r \mapsto f(rx)$ is nonincreasing (resp. nondecreasing) or u.s.c. (resp. l.s.c.) is a Thach function (resp. a co-Thach function).*

Proof. Given $y \in Y \setminus \{0\}$ and $s < f^\wedge(y)$ we can find x such that $\langle x, y \rangle \geq 1$ and $-f(x) > s$. As f is quasi-shady, there exists $t > 1$ such that $-f(tx) >$

s. As $\langle tx, y\rangle > 1$ we get $f^0(y) > s$. Thus $f^0(y) = f^\wedge(y)$; as this relation obviously holds for $y = 0$, the result is proved in the Thach case; the other case is similar. □

Let us note that, in contrast to the convex case, our conjugacies are of interest for improper functions, in particular for functions taking the value $-\infty$, as the following example shows.

Example 4.2. Let us associate to a subset S of X its indicator function ι_S given by $\iota_S(x) = 0$ if $x \in S$, $\iota_S(x) = \infty$ otherwise. Let us also define the canyon function γ_S of S by $\gamma_S(x) = -\infty$ if $x \in S$, $\gamma_S(x) = 0$ otherwise and the valley function v_S (of which M. Volle made a great use under the name "characteristic function" of S) given by $v_S(x) = -\infty$ for $x \in S$, $v_S(x) = +\infty$ for $x \in X \setminus S$. Then one can check (or refer to [59], Prop. 4 for the last relation) that

$$\begin{aligned}(\iota_S)^P &= \gamma_{P(S)},\\ (\gamma_S)^P &= \iota_{P(S)},\\ (v_S)^P &= v_{P(S)}.\end{aligned}$$

In particular, these relations hold for each of the four conjugacies we defined above. It follows that

$$\begin{aligned}(\iota_S)^{PP} &= \iota_{PP(S)},\\ (\gamma_S)^{PP} &= \gamma_{PP(S)},\\ (v_S)^{PP} &= v_{PP(S)}.\end{aligned}$$

It is known that for closed convex functions continuity at 0 corresponds to an inf-compactness property of the conjugate. Let us give similar properties for the present conjugacies; see also [47], [48]. Here X is endowed with its weak topology associated with the coupling $\langle ., .\rangle$, Y is endowed with the Mackey topology and f is said to be inf-compact if for each $r \in \mathbf{R}$ the sublevel set $[f \leq r]$ is compact.

Proposition 4.8. (a) *If f inf-compact, then $f^\wedge$ is u.s.c. on Y.*

(b) *If g is u.s.c. at 0 and $g(0) = -\infty$, then g^0 is inf-compact.*

Proof. (a) Suppose on the contrary that $g := f^\wedge$ is not u.s.c. at some $y \in Y$: there exists a net $(y_i)_{i\in I}$ with limit y and a real number r such that

$$\lim\sup_i g(y_i) > r > g(y).$$

Without loss of generality we may suppose $g(y_i) > r$ for each $i \in I$, so that there exists some $x_i \in [y_i \geq 1]$ such that $f(x_i) < -r$. Since f is inf-compact, we may assume that the net $(x_i)_{i \in I}$ (or a subset) converges to some $x \in [f \leq -r]$. Since $\{x_i : i \in I\}$ is contained in a compact set, we have $\langle x, y \rangle \geq 1$. Therefore $g(y) = f^{\wedge}(y) \geq -f(x) \geq r > g(y)$, a contradiction.

(b) For each real number r we have $[g^0 \leq r] = [g < -r]^0$. By assumption $[g < -r]$ is a neighborhood of 0, so that its polar set $[g < -r]^0$ is weakly compact. □

Corollary 4.9. *Suppose X is the dual space of a Banach space Y, f (resp. g) is an extended real-valued function on X (resp. Y). If f is coercive and l.s.c. for the weak topology then $f^{\wedge}$ is u.s.c. for the norm topology. If g is u.s.c. at 0 and $g(0) = -\infty$, then g^0 is coercive.*

Proof. We use the fact that the Mackey topology on Y is the norm topology and that g^0 is coercive iff it is inf-compact. □

Corollary 4.10. *Suppose X is the dual space of a Banach space Y, f is a Thach function on X and $f = f^{\infty}$. Then f is coercive iff f^0 is u.s.c.. If g is a Thach function on Y and $g = g^{\wedge\wedge}$, then g is u.s.c. iff $g^{\wedge}$ is coercive.*

Let us observe that one can replace the assumption $g(0) = -\infty$ in the preceding statements by the assumption $g(0) = \inf_{y \in Y} g(y)$ by suitably modifying the coercivity of inf-compactness property (see [47], [48]). In fact we may study functions taking their values in some sub-interval $[\alpha, w]$ of $\overline{\mathbf{R}}$, taking an increasing homeomorphism from this interval onto $\overline{\mathbf{R}}$ or setting $\sup \emptyset = \alpha$ in $[\alpha, w]$, a consistent convention. In particular, we can restrict our attention to nonnegative functions as in [38].

Now let us compare our conjugacies with known ones. Part of the following comparison with the usual Fenchel-Legendre conjugacy is given in [49], Theorem 2.2.

Proposition 4.11. *Denoting by f^* the Fenchel-Legendre conjugate of f, for any function f one has*

$$f^{\wedge}(y) \leq \inf_{r \geq 0} (f^*(ry) - r).$$
$$f^{\vee}(y) \leq \inf_{r \geq 0} (f^*(-ry) + r)$$

for each $y \in Y \setminus \{0\}$, with equality and attainment in the infimum when f is closed proper convex.

Proof. Let us prove the second assertion, the first one being proved in [49]. As

$$\begin{aligned}-f^{\vee}(y) &= \inf\{f(x) : x \in X,\ \langle x, y\rangle - 1 \leq 0\}\\ &\geq \inf_x \Big(\sup_{r\geq 0}(f(x) + r(\langle x, y\rangle - 1))\Big)\\ &\geq \sup_{r\geq 0} \Big(\inf_x (f(x) - \langle x, -ry\rangle - r)\Big),\end{aligned}$$

the inequality holds. Since the qualification condition $y(X) - \mathbf{R}_+ = \mathbf{R}$ is satisfied, the Lagrange multiplier rule yields the existence of a multiplier r for which equality holds when f is closed convex proper. □

Now let us relate the preceding conjugacies to the conjugacies adapted to general quasiconvexity, in which the conjugates of $f : X \to \overline{\mathbf{R}}$ are defined by

$$\begin{aligned}f^{\#}(y, r) &:= \sup\{\langle x, y\rangle : x \in [f \leq r]\},\\ f^{\flat}(y, r) &:= \sup\{\langle x, y\rangle : x \in [f < r]\};\end{aligned}$$

see [3], [20], [21], [32], [33] for instance. Since these functions are the support functions of the sublevel sets, they reflect in an accurate way the behavior of quasiconvex functions, but they are defined on a space larger than Y. In the reverse direction, the conjugates of a function $G : Y \times \mathbf{R} \to \overline{\mathbf{R}}$ are given by

$$\begin{aligned}G^{\flat}(x) &:= \sup_{y\in Y}(G_y)^e(\langle x, y\rangle),\\ G^{\#}(x) &:= \sup_{y\in Y}(G_y)^h(\langle x, y\rangle),\end{aligned}$$

where G_y is the partial function $s \mapsto G(y, s)$ and g^e (resp. g^h) denotes the epi or lower (resp. hypo or upper) quasi-inverse of $g : \mathbf{R} \to \overline{\mathbf{R}}$ defined by

$$\begin{aligned}g^e(r) &:= \inf\{s \in \mathbf{R} : g(s) \geq r\} = \sup\{t \in \mathbf{R} : g(t) < r\},\\ g^h(r) &:= \inf\{s \in \mathbf{R} : g(s) > r\} = \sup\{t \in \mathbf{R} : g(t) \leq r\}.\end{aligned}$$

Settting as in [33]

$$F(y, s) := \inf\{f(x) : x \in [y > s]\}$$

so that

$$f^0(y) = -F(y,1), \quad f^\nabla(y) = -F(-y,-1),$$

and noting that by [33], Proposition 3.6 one has

$$F_y := F(y,.) = \left(f_y^\flat\right)^h, \quad f_y^\flat := f^\flat(y,.) = (F_y)^e$$

for each $y \in Y \setminus \{0\}$ we get the first part of the following proposition.

Proposition 4.12. *For any extended real-valued function f, the conjugate $f^\flat$ determines the conjugates f^0, f^∇ of f by $f^0(y) = -\left(f_y^\flat\right)^h(1)$, $f^\nabla(y) = -\left(f_{-y}^\flat\right)^h(-1)$.*

Conversely, when f is radiant, $f_y^\flat := f^\flat(y,.) = (F_y)^e$ takes its values in $\mathbf{R}_+ \cup \{\infty\}$ and is determined by F_y, with $F_y(s) = -f^0(s^{-1}y)$ for $s > 0$, $F_y(s) = -\infty$ for $s \leq 0$.

Proof. Taking into account the relations $[y > s] = [s^{-1}y > 1]$ for $s > 0$, $[y > s] = [-s^{-1}y > -1] = [s^{-1}y < 1]$ for $s < 0$ we get

$$F_y(s) = -f^0(s^{-1}y) \quad \text{for } s > 0, \quad F_y(s) = -f^\nabla(s^{-1}y) \quad \text{for } s < 0.$$

Now, when f is radiant, for each real number r the sets $[f \leq r]$ and $[f < r]$ contain 0, so that $f^\flat(y,r)$ is a nonnegative number or $+\infty$. Then, since

$$f_y^\flat(r) \leq s \Leftrightarrow r \leq F_y(s) \tag{4.3}$$

by [33], Proposition 3.6 one has

$$\begin{aligned} f_y^\flat(r) &= \inf\{s > 0 : f_y^\flat(r) \leq s\} \\ &= \inf\{s > 0 : r \leq F_y(s)\}. \end{aligned}$$

Thus, it suffices to know F_y on $(0,\infty)$. $\square$

Similarly, when f is shady and l.s.c., Proposition 2.3 ensures that $f^\flat$ takes its values in $\mathbf{R}_- \cup \{\infty\}$. Then, whenever $f_y^\flat(r) < 0$ one has

$$\begin{aligned} f_y^\flat(r) &= \inf\{s < 0 : f_y^\flat(r) \leq s\} \\ &= \inf\{s < 0 : r \leq F_y(s)\} \end{aligned}$$

and the knowledge of F_y on $(-\infty,0)$ determines $f_y^\flat(r)$; thus if suffices to know $f^\nabla(y)$ in such a case.

In order to compare our conjugacy with the one of Passy and Prisman [29], we need the following lemma (see [61], Lemma 19).

Lemma 4.13. *If $g : Y \to \overline{\mathbf{R}}$ is quasiconvex, then the function $G : Y \times \mathbf{R} \to \overline{\mathbf{R}}$ given by $G(y, s) = g(s^{-1}y)$ for $y \in Y$, $s > 0$, $G(y, s) = +\infty$ for $y \in Y$, $s < 0$ is quasiconvex. If g is radiant, then for each $y \in Y$ the function $G_y : s \mapsto G(y, s)$ is nonincreasing.*

Clearly, G is positively homogeneous of degree 0 (in short p.h.) in the sense that $G(ry, rs) = G(y, s)$ for any $r > 0$, $(y, s) \in Y \times \mathbf{R}$. For such functions, Passy and Prisman introduced a conjugacy of the type

$$G^{\varpi}(x, r) := \sup\{-G(y, s) : \langle x, y\rangle + rs \geq 0\}.$$

When G is deduced from g as in the preceding lemma we get

$$G^{\varpi}(x, -1) := \sup\{-g(s^{-1}y) : \langle x, s^{-1}y\rangle \leq 1,\ s > 0, y \in Y\}.$$

This is nothing else than $g^{\wedge}(x)$. Clearly, one can introduce variants of the Passy-Prisman conjugacy which would yield the other conjugates we dealt with.

5. Dualities for anticonvex programs

In this section we consider the nonconvex programs $(\mathcal{F})$, $(\mathcal{M})$, $(\mathcal{R})$ of the introduction and

$$(\mathcal{P}) \text{ minimize } g(x) - h(x) : x \in X,$$

where g, h are extended real-valued functions on X, and $r - s$ means $r \dot{+} (-s)$, where the addition $\dot{+}$ (henceforth denoted by $+$ for simplicity) in $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty, -\infty\}$ satisfies the convention $\infty + (-\infty) = \infty$. We will show that such a formulation encompasses the problems considered in [31], [47]-[49] (but not the one treated in [50] which is of a different nature). For this purpose, we will use the canyon functions and the valley functions of Example 4.2 which are improper functions. Thus, we need an extension of the Topland-Singer duality to improper functions. Such an extension is given in [59]. We provide a proof for the reader's convenience. It shows that it only depends on the extensions to $\overline{\mathbf{R}}$ of usual rules for dealing with infima which are made explicit in [27].

Proposition 5.1 ([59]). *Let c be an arbitrary coupling between two sets X and Y and let g, h be extended real-valued functions on X. If $h^{cc} = h$ then*

$$\inf_{x \in X} \big(g(x) - h(x)\big) = \inf_{y \in Y} \big(h^c(y) - g^c(y)\big).$$

Moreover, if $h = h^{cc}$, if $\overline{x}$ is a solution to $(\mathcal{P})$ and if g and h are finite at $\overline{x}$, then, for any $\overline{y} \in \partial^c g(\overline{x}) \cap \partial^c h(\overline{x})$, $\overline{y}$ is a minimizer of $h^c - g^c$.

Here, for an extended real-valued function f, finite at $\overline{x}$, $\partial^c f(\overline{x})$ denotes the set of $\overline{y} \in Y$ such that $c(\overline{x}, \overline{y})$ is finite and

$$f(x) \geq f(\overline{x}) + c(x, \overline{y}) - c(\overline{x}, \overline{y}) \quad \forall x \in X.$$

Proof. As $h = h^{cc}$ we have

$$\begin{aligned}
\inf_{x \in X} \big(g(x) - h(x)\big) &= \inf_{x \in X} \big(g(x) + \inf_{y \in Y} (h^c(y) - c(x, y))\big) \\
&= \inf_{x \in X} \inf_{y \in Y} \big(g(x) + h^c(y) - c(x, y)\big) \\
&= \inf_{i \in Y} \big(h^c(y) + \inf_{x \in X} (g(x) - c(x, y))\big) \\
&= \inf_{y \in Y} (h^c(y) - g^c(y)).
\end{aligned}$$

The last assertion follows from the fact that when $\overline{y} \in \partial^c g(\overline{x}) \cap \partial^c h(\overline{x})$ the functions g, h (resp. h^c, g^c) are finite at $\overline{x}$ (resp. $\overline{y}$) and

$$\begin{aligned}
g(\overline{x}) + g^c(\overline{y}) &= c(\overline{x}, \overline{y}), \\
h(\overline{x}) + h^c(\overline{y}) &= c(\overline{x}, \overline{y}),
\end{aligned}$$

and the result follows by subtraction. □

We will need another result from [59] (Theorem 3); as shown by an example there, it is specific to the level conjugacy we use, in contrast to the preceding statement.

Proposition 5.2 ([59]). *Let X and Y be two sets and let $c = c^P$ be the coupling associated with a polarity $P : \mathcal{P}(X) \to \mathcal{P}(Y)$ as in (4.2). Then, for any extended real-valued functions g, h of X, one has the following inequality, which is an equality when $h = h^{PP}$:*

$$\inf_{x \in X} \big(g(x) \vee (-h)(x)\big) \leq \inf_{y \in Y} \big(h^P(y) \vee (-g^P)(y)\big).$$

Moreover, if $h = h^{PP}$, if $\overline{x}$ is a solution to $(\mathcal{P})$ and if g and h are finite at $\overline{x}$, then, for any $\overline{y} \in \partial^P g(\overline{x}) \cap \partial^P h(\overline{x})$ is a minimizer of $h^P \vee (-g^P)$.

Here $\partial^P f(\overline{x})$ denotes the subdifferential $\partial^c f(\overline{x})$ of f at $\overline{x}$ for $c = c^P$ and $r \vee s := \max(r, s)$.

Proof. It is a consequence of the relations $-h \leq -h^{PP}$, $(P^*)^* = P$ and of the equivalence

$$y \in Y \setminus P(x) \Leftrightarrow x \in X \setminus P^*(y),$$

which yields

$$\begin{aligned}\inf_{x\in X}\big(g(x)\vee(-h^{PP}(x))\big) &= \inf_{x\in X}\inf_{y\in Y\setminus P(x)} g(x)\vee h^P(y)\\ &= \inf_{y\in Y}\inf_{x\in X\setminus P^*(y)} g(x)\vee h^P(y)\\ &= \inf_{y\in Y}\big(h^P(y)\vee(-g^P(y))\big).\end{aligned}$$

The last assertion follows from the relations $-g^P(\overline{y}) = g(\overline{x})$, $h^P(\overline{y}) = -h(\overline{x})$. $\square$

Let us apply what precedes to the fractional programming problem

$$(\mathcal{F})\quad \text{minimize } q(x) := \frac{n(x)}{d(x)} : x\in X,$$

where n, $d : X\to\mathbf{R}$, with $d(x) > 0$, $n(x)\ge 0$ for each $x\in X$. Let e be the extension to $\overline{\mathbf{R}} := \mathbf{R}\cup\{-\infty,\infty\}$ of the exponential mapping given by $e(-\infty) = 0$, $e(\infty) = \infty$ and let ℓ be the extension of the logarithm given by $\ell(0) = -\infty$, $\ell(\infty) = \infty$.

Proposition 5.3. *Suppose* $(\ell\circ d)^{PP} = \ell\circ d$. *Then*

$$\inf\left\{\frac{n(x)}{d(x)} : x\in X\right\} = \inf\left\{\frac{n^P(x)}{d^P(y)} : y\in Y\right\}.$$

Proof. Since e is an isomorphism of complete lattices from $\overline{\mathbf{R}}$ onto $[0,1]$ with inverse denoted by ℓ, using Proposition 5.1 with $g=\ell\circ n$, $h=\ell\circ d$, we get

$$\begin{aligned}\ell\Big(\inf_X\frac{n(x)}{d(x)}\Big) &= \inf_X \ell\Big(\frac{n(x)}{d(x)}\Big) = \inf_X[\ell(n(x)) - \ell(d(x))]\\ &= \inf_Y\big[(\ell\circ d)^P(y) - (\ell\circ n)^P(y)\big]\\ &= \inf_Y\Big[\sup_{x\in X\setminus P^*(y)}(-\ell(d(x))) - \sup_{x\in X\setminus P^*(y)}(-\ell(n(x)))\Big]\\ &= \inf_Y\Big[-\ell\Big(\inf_{x\in X\setminus P^*(y)} d(x)\Big) + \ell\Big(\inf_{x\in X\setminus P^*(y)} n(x)\Big)\Big]\\ &= \inf_Y\big[\ell(-n^P(y)) - \ell(-d^P(y))\big]\\ &= \ell\Big(\inf_Y\frac{-n^P(y)}{-d^P(y)}\Big).\end{aligned}$$

Applying e to the extreme sides of this string of equalities we get the result. □

Now let us show that [48] Theorem 4.1 and [31], Prop. 4.1 are particular cases of Proposition 5.1; for another proof using Proposition 5.2 see [59], Corolary 5. These results deal with the maximization problem

$$(\mathcal{M}) \quad \text{maximize } f(x) : x \in F,$$

where $f : X \to \overline{\mathbf{R}}$ is an arbitrary function and the feasible set F is an arbitrary subset of X.

Corollary 5.4. *For any feasible set F and for any extended real-valued function f such that $f = f^{PP}$ one has*

$$\sup(\mathcal{M}) = \sup_{y \in Y \setminus P(F)} -f^P(y).$$

Proof. We can convert problem $(\mathcal{M})$ into

$$(\mathcal{M}') \quad \text{minimize } \iota_F(x) - f(x) : x \in X$$

so that Proposition 5.1 can be applied with $g = \iota_F$, $h = f$. Since $(\iota_F)^P = -\iota_{Y \setminus P(F)}$ we get the result. □

Proposition 4.1 of [31] corresponds to the cases $P = P^0$, $P = P^\wedge$. Let us state explicitely two other cases.

Corollary 5.5. *For any feasible set F and for any extended real-valued function f such that $f = f^{\nabla\nabla}$ (resp. $f = f^{\vee\vee}$) one has*

$$\sup_{x \in F} f(x) = \sup_{y \in Y \setminus P(F)} -f^{\nabla\nabla}(y)$$

$$(\text{resp. } \sup_{x \in F} f(x) = \sup_{y \in Y \setminus P(F)} -f^{\vee\vee}(y)).$$

Now let us turn to the reverse convex program

$$(\mathcal{R}) \quad \text{minimize } f(x) : x \in X \setminus C$$

which is studied in [48] in the case C is the interior of some convex subset D of X (so that $C^0 = D$) and let us relate it to dual problem

$$(\mathcal{R}^P) \quad \text{minimize } -f^P(y) \quad y \in P(C).$$

Corollary 5.6. *For any subset C of X and for any extended real-valued function f one has the following inequality which is an equality when $C = P^*(P(C))$:*

$$\inf_{x\in X\setminus C} f(x) \leq \inf_{y\in P(C)} -f^p(y).$$

Proof. It suffices to apply Proposition 5.2 and to observe as in [59] that since $(v_C)^P = v_{P(C)}$ one has

$$\begin{aligned}\inf_{x\in X\setminus C} f(x) &= \inf_{x\in X}(f(x) \vee (-v_C(x))) \\ &\leq \inf_{y\in Y} v_{P(C)}(y) \vee (-f^P(y)) = \inf_{y\in P(C)} -f^P(y).\end{aligned}$$

with equality when $C = P^*(P(C))$. One can also use Proposition 5.1 and observe that

$$\inf_{x\in X\setminus C} f(x) = \inf_{x\in X}(f(x) - \gamma_C(x)), \quad (\gamma_C)^P = \iota_{P(C)}.$$

□

This corollary encompasses Proposition 3.1 of [31] which corresponds to the cases $P = P^0$, $P = P^\wedge$ and two other cases we state now.

Corollary 5.7. *For any subset C of X and for any extended real-valued function f such that $f = f^{PP}$ one has the following inequality which is an equality when C is closed and shady (resp. evenly shady)*

$$\inf_{x\in X\setminus C} f(x) \leq \inf_{y\in C^\nabla} -f^\nabla(y)$$

$$(\text{resp. } \inf_{x\in X\setminus C} f(x) \leq \inf_{y\in C^\vee} -f^\vee(y).$$

Let us now present a unified characterization of solutions of $(\mathcal{R})$ which contains [49], Th. 7.1 and [41], Prop. 3.2 with a slight supplementary information.

Proposition 5.8. *Suppose C is P-convex, i.e. $C = P^*(P(C))$. Then the following assertions on $\overline{x} \in X \setminus C$ are equivalent:*

(a) *$\overline{x} \in X \setminus C$ is a solution to $(\mathcal{R})$;*

(b) *$P(C) \setminus P(\overline{x})$ is nonempty and, for each $\overline{y} \in P(C) \setminus P(\overline{x})$, $\overline{y}$ is a solution to $(\mathcal{R}^P)$ and one has $f(\overline{x}) + f^P(\overline{y}) = 0$.*

(c) *there exists a solution $\overline{y}$ to $(\mathcal{R}^P)$ such that $f(\overline{x}) + f^P(\overline{y}) = 0$.*

Proof. Let $\overline{x} \in X \setminus C$ be a solution to $(\mathcal{R})$. Since $C = P^*(P(C))$ there exists some $\overline{y} \in P(C)$ such that $\overline{x} \notin P^*(\overline{y})$, hence $\overline{y} \in P(C) \setminus P(\overline{x})$. Then one has

$$\inf_{y\in P(C)} -f^P(y) = \inf_{x\in X\setminus C} f(x) = f(\overline{x}) \geq -f^P(\overline{y}) \geq \inf_{y\in P(C)} -f^P(y),$$

so that $-f^P(\overline{y}) = f(\overline{x})$ and $\overline{y}$ is a solution to $(\mathcal{R}^P)$. Since (b) obviously implies (c) it remains to show that (c) implies (a). Now, if $\overline{y} \in P(C)$ satisfies $-f^P(\overline{y}) = f(\overline{x})$ then one has

$$\inf_{x \in X \setminus C} f(x) = \inf_{y \in P(C)} -f^P(y) = -f^P(\overline{y}) = f(\overline{x}),$$

so that $\overline{x}$ is a solution to $(\mathcal{R})$. □

The preceding result can be interpreted in terms of subdifferentials. Taking for c the coupling c^P and denoting the associated subdifferential by ∂^p (or ∂^0, $\partial^\wedge$, ∂^∇, $\partial^\vee$ in each of the specific cases we considered) we get the following criteria.

Corollary 5.9. *Suppose C is P-convex. If $\overline{x} \in X \setminus C$ is a solution to $(\mathcal{R})$ and if $f(\overline{x})$ is finite, then there exists $\overline{y} \in \partial^P f(\overline{x}) \cap (P(C) \setminus P(\overline{x}))$.*

Proof. The result is a consequence of the preceding corollary and of the following characterization of $\partial^P f(\overline{x})$: $\overline{y} \in \partial^P f(\overline{x})$ iff $f(\overline{x})$ is finite, $\overline{y} \notin P(\overline{x})$ and $f(\overline{x}) + f^P(\overline{y}) = 0$. □

Let us note that this condition is a necessary condition, not a necessary and sufficient condition. It becomes sufficient when one can check that $\overline{y}$ is a solution to the dual problem $(\mathcal{R}^P)$, a task which may be easier than solving the original problem $(\mathcal{R})$ when f^P and $P(C)$ are simple enough.

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