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P-ADIC HYPERBOLIC SURFACES

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Dedicated to Hoang Tuy on the occasion of his seventieth birthday

1. Introduction

A holomorphic curve in a projective variety X is said to be degenerate if it is contained in a proper algebraic subset of X. In 1979 M. Green and Ph. Griffiths [GG] conjectured that every holomorphic curve in a complex projective variety of general type is degenerate. Up to now this conjecture seems still far from being proved, but some progress have been made. M. Green [G] proved the degeneracy of holomorphic curves in the Fermat variety of large degree. To obtain the mentioned results, M. Green used the Nevanlinna theory for holomorphic curves. In [N] A. M. Nadel gave a class of projective hypersurfaces for which the conjecture is valid. Using the results on degeneracy of holomorphic curves Nadel constructed some explicit examples of hyperbolic hypersurfaces in $\mathbf{P}^3$. Nadel's techniques are based on Siu's theory of meromorphic connections. We refer the reader to the survey [Z2] for related topics.

For the p-adic case, the degeneracy of holomorphic curves in the Fermat variety of large degree is established in [HM]. In this note we are going to show that if X is a pertubation of the Fermat variety in $\mathbf{P}^n(\mathbf{C}_p)$ of degree large enough with respect to n and to the number of non-zero coefficients in the defining equation, then every holomorphic curve in X is degenerate. The proof provides sufficiently precise information on the position of the curve in X, which will be useful for applications. As a consequence, we give some explicit examples of p-adic hyperbolic surfaces in $\mathbf{P}^3(\mathbf{C}_p)$ and of curves in $\mathbf{P}^2(\mathbf{C}_p)$ with hyperbolic complements, and also explicit examples of hyperbolic surfaces with hyperbolic complements. Recall that a variety X is said to be p-adic hyperbolic if every holomorphic map from $\mathbf{C}_p$ into X is constant. These examples are different to the ones of [HM], which were given by using the p-adic Nevanlinna-Cartan theorem. While the degree of surfaces in [HM] as well as in all known explicit examples of complex

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hyperbolic surfaces is divided by some integer > 1, the degree d of the examples of this note is required only to be not less than 24 for hyperbolic surfaces and curves with hyperbolic complements or not less than 50 for hyperbolic surfaces with hyperblic complements. As in [HM] the main tool of this note is the height function defined in [H1]-[H3], [HM]. This function plays a role similar to the one of the Nevanlinna characteristic function in Green's arguments. Moreover, the height of a p-adic holomorphic function $f(z)$ gives information on the distribution of zeros of f and describes the growth of $|f(z)|$. In many cases we can use the height in the study of p-adic holomorphic functions such as the degree in the study of complex polynomials. The proof of Lemma 3.2 is such an example.

The paper is planned as follows. In §2 we recall some facts on heights of p-adic holomorphic functions and of p-adic holomorphic curves. Section 3 is devoted to the proof of the degeneracy of holomorphic curves in perturbations of the Fermat variety. These results are used in the last section to give explicit examples of p-adic hyperbolic surfaces in $\mathbf{P}^3(\mathbf{C}_p)$, curves with hyperbolic complements in $\mathbf{P}^2(\mathbf{C}_p)$, and hyperbolic surfaces with hyperbolic complements in $\mathbf{P}^3(\mathbf{C}_p)$.

2. Height of p-adic holomorphic functions

We recall some facts on heights of p-adic holomorphic functions for later use in this note. More details can be found in [H1]-[H3], [HM].

Let p be a prime number, Q_p the field of p-adic numbers, and C_p the p-adic completion of the algebraic closure of Q_p. The absolute value in Q_p is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the additive valuation on C_p which extends ord_p.

Let $f(z)$ be a p-adic holomorphic function on $\mathbf{C}_p$ represented by a convergent series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Since we have

$$\lim_{n\to\infty} \{v(a_n) + nv(z)\} = \infty$$

for every $z \in \mathbf{C}_p$, it follows that for every $t \in \mathbf{R}$ there exists an n for which $v(a_n) + nt$ is minimal.

Definition 2.1. The *height* of $f(z)$ is defined by

$$h(f,t) = \min_{0\le n<\infty} \{v(a_n) + nt\}.$$

Now let us give a geometric interpretation of height. For each n we draw the graph Γ_n which depicts $v(a_n z^n)$ as a function of $v(z)$. This graph is a straight line with slope n. Then $h(f,t)$ is the boundary of the intersection of all of the half-planes lying under the lines Γ_n. In any finite segment $[r,s]$, $0 < r, s < +\infty$, there are only finitely many Γ_n which appear in $h(f,t)$. Thus, $h(f,t)$ is a polygonal line. The point t at which $h(f,t)$ has vertices is called *the critical point* of $f(z)$. A finite segment $[r,s]$ contains only finitely many critical points. It is clear that if t is a critical point, then $v(a_n) + nt$ attains its minimum at least at two values of n.

If $v(z) = t$ is not a critical point, then $f(z) \neq 0$ and $|f(z)| = p^{-h(f,t)}$. The function $f(z)$ has zeros when $v(z) = t_i$, where $t_o > t_1 > \dots$ is the sequence of critical points; and the number of zeros (counting multiplicity) for which $v(z) = t_i$ is equal to the difference $n_{i+1} - n_i$ between the slope of $h(f,t)$ at $t_i - 0$ and its slope at $t_i + 0$. It is easy to see that n_i and n_{i+1}, respectively, are the smallest and the largest values of n at which $v(n) + nt$ attains minimum.

Lemma 2.2. *Let $f(z)$ be a non-constant holomorphic function on $\mathbf{C}_p$. Then for t sufficiently small we have*

$$h(f',t) - h(f,t) \geq -t.$$

Lemma 2.3. *For a non-constant holomorphic function $f(z)$ in $\mathbf{C}_p$, $h(f,t) \longrightarrow -\infty$ as $t \to -\infty$.*

Lemma 2.4. *For holomorphic functions $f(z)$, $g(z)$ in $\mathbf{C}_p$ we have*

(i) $h(f+g,t) \geq \min\{h(f,t), h(g,t)\}$.

(ii) $h(fg,t) = h(f,t) + h(g,t)$.

The proofs of Lemmas 2.2-2.4 follow immediately from Definition 2.1 and the geometric interpretation of height.

Now let f be a p-adic holomorphic curve in the projective space $\mathbf{P}^n(\mathbf{C}_p)$, i.e., a holomorphic map from $\mathbf{C}_p$ to $\mathbf{P}^n(\mathbf{C}_p)$. We identify f with its representation by a collection of holomorphic functions on $\mathbf{C}_p$:

$$f = (f_1, f_2, \dots, f_{n+1}),$$

where the functions f_i have no common zeros.

Definition 2.5. The *height* of the holomorphic curve f is defined by

$$h(f,t) = \min_{1 \leq i \leq n+1} h(f_i,t).$$

We need the following lemma.

Lemma 2.6. *Let* $(g_1, \dots, g_{n+1})$ *be a representation of the same projective map as* $(f_1, \dots, f_{n+1})$*, where* g_i *are holomorphic functions. Then*

$$h(f,t) = \min_{1 \le i \le n+1} h(g_i, t) + C,$$

where C *is a constant.*

Proof. By the hypothesis there is a meromorphic function $\lambda(z)$ such that for every $i = 1, \dots, n+1$ we have

$$g_i(z) = \lambda(z) f_i(z).$$

Since $g_i(z)$ are holomorphic functions, and $f_i(z)$ have no common zeros, λ is a holomorphic function. Then by Lemma 2.3, $h(\lambda, t) < 0$ for t sufficiently small, or $\lambda(z)$ is constant. Lemma 2.6 is proved.

From Lemma 2.6 we can see that the height of a holomorphic curve is well defined modulo a bounded value.

3. Degeneracy of holomorphic curves

Let

$$M_j = z_1^{\alpha_{j,1}} \dots z_{n+1}^{\alpha_{j,n+1}}, \qquad 1 \le j \le s,$$

be distinct monomials of degree d with non-negative exponents. Let X be a hypersurface of degree d of $\mathbf{P}^n(\mathbf{C}_p)$ defined by

$$X: \quad c_1 M_1 + \dots + c_s M_s = 0,$$

where $c_j \in \mathbf{C}_p^*$ are non-zero constants. We call X *a perturbation of the Fermat hypersurface of degree* d if $s \ge n+1$ and

$$M_j = z_j^d, \quad j = 1, \dots, n+1.$$

Theorem 3.1. *Let* X *be a perturbation of the Fermat hypersurface of degree* d *in* $\mathbf{P}^n(\mathbf{C}_p)$ *and let* f *be a holomorphic curve in* X*. Assume that*

$$d \ge \frac{(n+1)(s-1)(s-2)}{2}.$$

Then the image of f *lies in a proper algebraic subset of* X*.*

If there is $f_i \equiv 0$, then f is degenerate, and we can assume that any $f_i \not\equiv 0$. The proof of Theorem 3.1 uses the following lemmas.

Lemma 3.2. *Let $f = (f_1, \ldots, f_{n+1})$ be a holomorphic curve and let M be a monomial as above. Then for every $k \geq 0$ we have the following representation*

$$\frac{(M \circ f)^{(k)}}{M \circ f} = \frac{Q_k}{f_1^k \ldots f_{n+1}^k},$$

where Q_k is a holomorphic function and

$$h(Q_k, t) \geq k \sum_{i=1}^{n+1} h(f_i, t) - kt$$

for t sufficiently small.

Proof. We use induction on k. The case $k = 0$ is trivial. Assume that Lemma 3.2 holds for k. For simplicity we set

$$\varphi = f_1 \ldots f_{n+1}. \tag{1}$$

Then we have

$$h(\varphi, t) = \sum_{i=1}^{n+1} h(f_i, t).$$

The induction hypothesis gives

$$(M \circ f)^{(k)} = \frac{Q_k . M \circ f}{\varphi^k}.$$

Then we have

$$\frac{(M \circ f)^{(k+1)}}{M \circ f} = \frac{Q_{k+1}}{\varphi^{k+1}},$$

where

$$Q_{k+1} = \varphi . Q_k' + \varphi . Q_k . \frac{(M \circ f)'}{M \circ f} - k Q_k . \varphi'.$$

Note that the function $\dfrac{(M \circ f)'}{(M \circ f)}$ has only simple poles at the zeros of $f_1, \ldots, f_{n+1}$. Therefore, the function $\varphi . \dfrac{(M \circ f)'}{(M \circ f)}$ is holomorphic. Hence, Q_{k+1} is a holomorphic function.

On the other hand, by Lemmas 2.3 and 2.4,

$$\begin{aligned} h(Q_{k+1},t) \geq \min\big\{&h(\varphi,t)+h(Q'_k,t),\\ &h(\varphi,t)+h(Q_k,t)+h((M\circ f)',t)-h(M\circ f,t),\\ &v(k)+h(Q_k,t)+h(\varphi',t)\big\}. \end{aligned}$$

Then by Lemma 2.2 we obtain

$$\begin{aligned} h(Q_{k+1},t) \geq \min\big\{&h(\varphi,t)+h(Q_k,t)-t, h(\varphi,t)+h(Q_k,t)-t,\\ &v(k)+h(Q_k,t)+h(\varphi,t)-t\big\}\\ =\ &h(\varphi,t)+h(Q_k,t)-t. \end{aligned} \tag{2}$$

The conclusion for $k+1$ follows from (1), (2) and the induction hypothesis for k.

Notice that the representation in Lemma 3.2 does not depend on the degree d. This fact is important for applications.

Lemma 3.3. *Let X be a perturbation of the Fermat hypersurface of degree d in $\mathbf{P}^n(\mathbf{C}_p)$ and let f be a holomorphic curve in X. Assume that*

$$d \geq \frac{(n+1)(s-1)(s-2)}{2}.$$

If $\{M_j\circ f,\ j=1,\dots,s-1\}$ are linearly independent, then f is a constant map.

Proof. For simplicity we set

$$g_j(z)=c_jM_j\circ f(z)/c_sM_s\circ f, \quad j=1,\dots,s-1.$$

Then the meromorphic functions $\{g_1,\dots,g_{s-1}\}$ satisfy the following relation:

$$g_1+\dots+g_{s-1}\equiv -1.$$

We are going to show that $\{g_1,\dots,g_{s-1}\}$ are linearly dependent. For this purpose we apply the Wronskian techniques of Nevanlinna, Bloch, Cartan [C],(see also [L], Ch. VII).

Define the following logarithmic Wronskian:

$$L_s(g) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \frac{g_1'}{g_1} & \frac{g_2'}{g_2} & \dots & \frac{g_{s-1}'}{g_{s-1}} \\ \dots & \dots & \dots & \dots \\ \frac{g_1^{(s-2)}}{g_1} & \frac{g_2^{(s-2)}}{g_2} & \dots & \frac{g_{s-1}^{(s-2)}}{g_{s-1}} \end{vmatrix}$$

and the logarithmic Wronskians $L_i = L_i(g_1, \dots, g_{s-1})$ by

$$L_1(g) = L_1(g_1, \dots, g_{s-1}) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ 0 & \frac{g_2'}{g_2} & \dots & \frac{g_{s-1}'}{g_{s-1}} \\ \dots & \dots & \dots & \dots \\ 0 & \frac{g_2^{(s-2)}}{g_2} & \dots & \frac{g_{s-1}^{(s-2)}}{g_{s-1}} \end{vmatrix}$$

and similarly for all $i = 2, \dots, s-1$, where the column $\{1, 0, \dots, 0\}$ is the i-th column.

If $\{g_1, \dots, g_{s-1}\}$ are linearly independent, then the projective maps

$$(M_1 \circ f, \dots, M_s \circ f) \quad \text{and} \quad L = (L_1, L_2, \dots, L_s)$$

are equal (see [L]).

Now we can apply Lemma 3.2 to the determinants. Typically, the first term in the expansion of $L_1(g)$ can be written in the form

$$\frac{Q_1 \dots Q_{s-2}}{\varphi \dots \varphi^{s-2}} = \frac{R}{\varphi^{(s-1)(s-2)/2}}.$$

The denominator $\varphi^{(s-1)(s-2)/2}$ is a common denominator of all the terms in all the expansions of all the determinants $L_i(g)$. Hence, we have an equality of projective maps:

$$(M_1 \circ f, \dots, M_s \circ f) = (L_1 \dots, L_s) = (R_1, \dots, R_s),$$

where, by Lemma 3.2, the R_j are holomorphic functions and satisfy the following condition (for t sufficiently small).

$$\begin{aligned} h(R_j,t) &= \sum_{k=1}^{s-2} h(Q_k,t) \\ &\geq (h(\varphi,t)-t)\sum_{k=1}^{s-2} k \\ &= \frac{(s-1)(s-2)}{2}h(\varphi,t) - \frac{(s-1)(s-2)}{2}t \\ &\geq \frac{(n+1)(s-1)(s-2)}{2}h(f,t) - \frac{(s-1)(s-2)}{2}t. \end{aligned}$$

Since $M_1 \circ f, \ldots, M_s \circ f$ have no common zeros, by Lemma 2.6 we have

$$\begin{aligned} \min_{1\leq j\leq s} h(M_j \circ f,t) &\geq \min_j h(R_j,t) + 0(1) \\ &\geq \frac{(n+1)(s-1)(s-2)}{2}h(f,t) - \frac{(s-1)(s-2)}{2}t + 0(1). \end{aligned}$$

Because X is a perturbation of the Fermat hypersurface of degree d we get

(3) $$\min_{1\leq j\leq n+1} h(M_j \circ f.t) = d \min_{1\leq j\leq n+1} h(f_j,t) = dh(f,t).$$

For other monomials we have

$$h(M_j \circ f,t) = \sum_{k=1}^{n+1} \alpha_{jk}h(f_k,t) \geq dh(f,t).$$

Thus we obtain

(4) $$dh(f,t) \geq \frac{(n+1)(s-1)(s-2)}{2}h(f,t) - \frac{(s-1)(s-2)}{2}t + 0(1).$$

When $d = (n+1)(s-1)(s-2)/2$ we have a contradiction as $t \to -\infty$, and when $d > \dfrac{(n+1)(s-1)(s-2)}{2}$ the inequality (4) gives us

$$h(f,t) \geq -Nt + 0(1),$$

where N is a strictly positive number. So by Lemma 2.4, f is a constant map. Lemma 3.3 is proved.

Proof of Theorem 3.1. It suffices to notice that by Lemma 3.3 the image of f is contained in the proper algebraic subset of X defined by the equation

$$a_1 z_1^d + a_2 z_2^d + \cdots + a_{n+1} z_{n+1}^d + a_{n+1} M_{n+2} + \cdots + a_{s-1} M_{s-1} = 0,$$

where not all a_j are zeros.

Remark 3.4. There is no similar result in the complex case.

4. Hyperbolic surfaces in $\mathbf{P}^3(\mathbf{C}_p)$

In this section we apply Theorem 3.1 to give explicit examples of p-adic surfaces in $\mathbf{P}^3(\mathbf{C}_p)$ as well as examples of curves in $\mathbf{P}^2(\mathbf{C}_p)$ with hyperbolic complements, and of hyperbolic surfaces with hyperbolic complements.

Without loss of generality we may assume that in the defining equation of X, the first coefficients $c_i = 1,\ i = 1, \ldots, n+1$.

Theorem 4.1. *Let X be a surface in $\mathbf{P}^3(\mathbf{C}_p)$ defined by the equation*

$$X:\ z_1^d + z_2^d + z_3^d + z_4^d + c z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4} = 0, \tag{5}$$

where $c \neq 0$, $\sum_{i=1}^{4} \alpha_i = d$, and if there is an exponent $\alpha_i = 0$, the others must be $\neq 1$. Then X is hyperbolic if $d \geq 24$.

Proof. First of all let us recall a result from [HM] (Theorem 4.3).

Lemma 4.2. *Let X be the Fermat hypersurface of degree d in $\mathbf{P}^n(\mathbf{C}_p)$, and let $f = (f_1, \ldots, f_{n+1})$ be a holomorphic curve in X. Assume that any $f_j \not\equiv 0$. If $d \geq n^2 - 1$, then either f is a constant curve, or there is a decomposition of the set of indices $\{1, \ldots, n+1\} = \cup I_\xi$ such that every I_ξ contains at least two elements, and if $i, j \in I_\xi$, f_i is equal to f_j multiple a constant (if $n = 2$ there exists only one class).*

Now let X be a hypersurface satisfying the hypothesis of Theorem 4.1, and let $f = (f_1, f_2, f_3, f_4):\ \mathbf{C}_p \longrightarrow X$ be a holomorphic curve in X. Suppose that for some i, $f_i \equiv 0$, for example, $f_4 \equiv 0$. If $\alpha_4 = 0$, the map (f_1, f_2, f_3) from $\mathbf{C}_p$ into $\mathbf{P}^2(\mathbf{C}_p)$ has the image contained in a projective curve of positive genus . By Berkovich's theorem (see [Be], also [Ch]) (f_1, f_2, f_3) is a constant map. From it and (5) it follows that f is a constant map.

Hence, we can assume that any $f_i \not\equiv 0$. From the proof of Theorem 3.1 it follows that $\{f_1^d, \ldots, f_4^d\}$ are linearly dependent. Suppose that

$$a_1 f_1^d + \cdots + a_4 f_4^d \equiv 0,$$

where not all a_i are zeros. Consider the following possible cases:

i) $a_i \neq 0,\ i = 1, \dots, 4$. By Lemma 4.2, f is a constant map, or we can assume that $f_1 = c_1 f_2,\ f_3 = c_2 f_4$. Then we can substitute this relation to (5) and show that f is a constant map.

ii) Only one coefficient, say, $a_4 = 0$. Then (f_1, f_2, f_3) is a constant map by Lemma 4.2, so f is.

iii) Two coefficients, say, $a_1 = a_2 = 0$. Then we have $f_3 = c_3 f_4$. Substitute this relation into (5) we obtain

$$f_1^d + f_2^d + \varepsilon_1 f_3^d + \varepsilon_2 f_1^{\alpha_1} f_2^{\alpha_2} f_3^{\alpha_3+\alpha_4} \equiv 0,$$

where $\varepsilon_2 \neq 0$. If $\varepsilon_1 \neq 0$, then the image of the map (f_1, f_2, f_3) from $\mathbf{C}_p$ into $\mathbf{P}^2(\mathbf{C}_p)$ is contained in a projective curve of positive genus , and (f_1, f_2, f_3) is a constant map, so f is (again by Berkovich's theorem).

Now suppose that $\varepsilon_1 = 0$. Then the image of the map (f_1, f_2, f_3) is contained in the following curve in $\mathbf{P}^2(\mathbf{C}_p)$:

$$Y:\ z_1^d + z_2^d + \varepsilon_2 z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3+\alpha_4} = 0.$$

We are going to show that under the hypothesis of Theorem 4.1, the genus of Y is at least 1, then Theorem 4.1 follows from Berkovich's theorem.

The genus of Y is equal to the number of integer points in the triangle with the vertices $(d, 0), (0, d)$ and (α_1, α_2) (see, for example, [Ho]). It suffices to consider the case $\alpha_1 + \alpha_2 < d$. Then it is easy to see that this triangle contains at least one integer point, unless the case $\alpha_1 + \alpha_2 = d - 1$. This case is excluded by the hypothesis of Theorem 4.1. The proof is completed.

Remark 4.1. In [HM] by using the method of K. Masuda and J. Noguchi [MN], we give the following examples of hyperbolic hypersurfaces in $\mathbf{P}^3(\mathbf{C}_p)$:

$$z_1^{4d} + \cdots + z_4^{4d} + t(z_1 z_2 z_3 z_4)^d = 0,\ d \geq 6 \quad (\deg\ X = 4d \geq 24),\ t \in \mathbf{C}_p^*.$$

Here we have examples with arbitrary degree ≥ 24 (not necessarily divided by 4). Notice that all known explicit examples of hyperbolic hypersurfaces in the complex case are of degree d divided by some number > 1 (2 in the case of Brody-Green's example, 3 in Nadel's example, and $3, 4$ in Masuda-Noguchi's examples). Indeed, in [MN] an algorithm is given to construct hyperbolic hypersurfaces of arbitrary "large enough" degree d. Here we obtain hyperbolic hypersurfaces with $d \geq 24$.

Remark 4.2. The following example shows that if among the exponents α_i two of them are 0, 1, then X may not be hyperbolic. The surface

$$X:\ z_1^{25} + z_2^{25} + z_3^{25} + z_4^{25} + z_1 z_2^{24} = 0$$

contains the holomorphic curve $(-1-z^{25}, 1, 1+z^{25}, z)$.

Now we use Theorem 3.1 to give explicit examples of curves in $\mathbf{P}^2(\mathbf{C}_p)$ with hyperbolic complements.

Theorem 4.3. *Let X be a curve in $\mathbf{P}^2(\mathbf{C}_p)$ defined by the following equation:*

$$X:\ z_1^d + z_2^d + z_3^d + cz_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0,$$

where $d \geq 24$, $d > \alpha_i \geq 0$, $\sum \alpha_i = d$. Then the complement of X is p-adic hyperbolic in $\mathbf{P}^2(\mathbf{C}_p)$.

Proof. Let $f = (f_1, f_2, f_3):\ \mathbf{C}_p \longrightarrow \mathbf{P}^2$ be a holomorphic curve with the image contained in the complement of X. Then the function

$$f_1^d + f_2^d + f_3^d + cf_1^{\alpha_1} f_2^{\alpha_2} f_3^{\alpha_3} \neq 0$$

for $z \in \mathbf{C}_p$, and is identically equal to a non-zero constant a. Hence, the image of the following holomorphic curve

$$(f_1, f_2, f_3, 1):\ \mathbf{C}_p \longrightarrow \mathbf{P}^3$$

is contained in the surface Y of $\mathbf{P}^3$ defined by the equation

$$Y:\ z_1^d + z_2^d + z_3^d - az_4^d + cz_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0.$$

By the proof of Theorem 3.1, $\{f_1^d, f_2^d, f_3^d, 1\}$ are linearly dependent:

$$c_1 f_1^d + c_2 f_2^d + c_3 f_3^d + c_4 \equiv 0,$$

where not all $c_i = 0$. There are the following cases:

i) $c_i \neq 0$ for all i. By Lemma 4.2 at least one of f_1, f_2, f_3 is constant. From this it follows that f is a constant map.

ii) $c_i = 0$ only for one i. Then (f_1, f_2, f_3) is a constant map, again by Lemma 4.2.

iii) If two $c_i = 0$, then either one of f_i is constant, or the ratio of two functions f_i, f_j is constant. In both cases, f is a constant map. Theorem 4.3 is proved.

Remark 4.3. We prove that the map $(f_1, f_2, f_3, 1) : \mathbf{C}_p \to Y$ is a constant map, although Y may not be hyperbolic.

Now we use the proof of Theorem 4.1 and Theorem 4.3 to give explicit examples of hyperbolic surfaces in $\mathbf{P}^3(\mathbf{C}_p)$ with hyperbolic complements.

Theorem 4.4. *Let X be a surface in $\mathbf{P}^3(\mathbf{C}_p)$ of degree $d \geq 50$ defined by the following equation*

$$X: \; z_1^d + \cdots + z_4^d + c z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4} = 0, \tag{6}$$

where $c \neq 0$, and if there is an exponent $\alpha_i = 0$, the others must be at least 2. Then X is hyperbolic and the complement of X in $\mathbf{P}^3(\mathbf{C}_p)$ is also hyperbolic.

Proof. By Theorem 4.1 it remains to prove that the complement of X is hyperbolic. Let $f = (f_1, \ldots, f_4)$ be a curve with the image contained in the complement of X. As in the proof of Theorem 4.3 there is a constant $a \neq 0$ such that the map $(f_1, f_2, f_3, f_4, 1)$ has the image contained in the hypersurface Y of degree d in $\mathbf{P}^4(\mathbf{C}_p)$ defined by the following equation:

$$Y: \; z_1^d + z_2^d + z_3^d + z_4^d + a z_5^d + c z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4} = 0. \tag{7}$$

From the proof of Theorem 3.1 it follows that when $d \geq \dfrac{(4+1)(6-1)(6-2)}{2} = 50$, $\{f_1^d, f_2^d, f_3^d, f_4^d, 1\}$ are linearly dependent. We have

$$\sum_{i=1}^{4} \varepsilon_i f_i^d + \varepsilon_5 \equiv 0,$$

where not all $\varepsilon_i = 0$.

If $\varepsilon_5 = 0$ then we can repeat the proof of Theorem 4.1 for showing that f is a constant map (notice that the hypothesis of Theorem 4.1 is fulfilled).

Assume that $\varepsilon_5 \neq 0$. From Lemma 4.2 it follows that either f is a constant map, and we are done, or at least one of f_i, say, f_4 is constant. Substitute $f_4 = \textit{const}$ into (7), we can see that the image of the map $(f_1, f_2, f_3, 1)$ is contained in the surface defined by the following equation

$$Z: \; z_1^d + z_2^d + z_3^d + a' z_4^d + c' z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\beta_4} = 0,$$

where $a', c' \neq 0$, $\beta_4 = d - (\alpha_1 + \alpha_2 + \alpha_3)$.

Again, by the proof of Theorem 3.1, $\{f_1^d, f_2^d, f_3^d, 1\}$ are linearly dependent. We have

$$\delta_1 f_1^d + \delta_2 f_2^d + \delta_3 f_3^d + \delta_4 \equiv 0,$$

If $\delta_4 = 0$, then f is a constant map by using similar arguments to the ones in the proof of Theorem 4.1. To show why we need the hypothesis that if one $\alpha_1 = 0$, the others must be at least two, we consider the case

$\delta_4 \neq 0$. By Lemma 3.2, either f is a constant map, or at least one of f_i, say f_3 is constant. Substitute $f_3 = \text{const}$, $f_4 = \text{const}$ into the equation (6) we obtain $f_1 = \varepsilon f_2$ with some constant ε. Finally, since the map $(f_1, f_2, f_3, f_4, 1)$ has the image contained in Y we have

$$Af_2^d + Bf_2^{\alpha_1+\alpha_2} + C \equiv 0,$$

where A, B, C are constants, and $B \neq 0$. By the hypothesis of Theorem 4.4, $\alpha_1 + \alpha_2 \neq 0, d$, and then $f_2 = \mathit{const}$. Theorem 4.4 is proved.

Remark 4.4. Theorems 4.3 and 4.4 give the first examples of hyperbolic hypersurfaces with hyperbolic complements in the p-adic case. In the complex case, the existence of such hypersurfaces is proved by M. Zaidenberg [Z2]. A. Nadel [N] gives first examples of such curves in $\mathbf{P}^2$, and explicit examples of such surfaces in $\mathbf{P}^3$ are given recently by K. Masuda and J. Noguchi [MN].

Remark 4.5. The following example shows that when the sum of two exponents from $\{\alpha_i\}$ is 0 or d, the complement of the surface X may not be hyperbolic. Consider the surface X defined by the equation:

$$X: \; z_1^{51} + z_2^{51} + z_3^{51} + z_4^{51} + z_3^{25} z_4^{26} = 0.$$

Then X is hyperbolic by Theorem 4.1, but the complement of X in $\mathbf{P}^3(\mathbf{C}_p)$ contains the following holomorphic curve:

$$f = (z, -z, 1, 1).$$

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