ACTA MATHEMATICA VIETNAMICA
Volume 22, Number 2, 1997, pp. 493–500

RECONSTRUCTION OF IMAGES IN MAGNETIC RESONANCE IMAGING

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Dedicated to Hoang Tuy on the occasion of his seventieth birthday

1. Introduction

One of the most delicate applications of coherent wavelets is spatially localized nuclear magnetic resonance imaging (MRI) based on harmonic analysis of the Heisenberg nilpotent Lie group (cf. [3]). A local assembly of magnetic dipoles can be interpreted as a gyromagnetic spin system that can be excited to upper states and returned to the ground state by a pulse of RF power. Then, the nuclear magnetic resonance spectrum of the undamped two-level system constitutes an one-dimensional profile of nuclear density. More precisely, what is being imaged by the MRI method is a spatial representation of the nuclear resonance signal. MRI admits ray tracing fan postprocessors like massively parallel quantum holographic computers which are useful for three-dimensional visulization of stacks of individual tomographic planar slices. Because of the capability to determine distinct chemical signatures of different tissue types and multifocal lesions, MRI scanners are a real progress in radiological diagnosis, far better than X-ray computer tomography scanners for many applications. Plannar imaging spatial encoding of time-domain signals by symplectic spinors in two-dimensional local coorodinate frames, whereas plannar visualization is performed by the spatial decoding of symplectic spinors in a local laboratory coordinate frame. Based on a phase coherent reference wave, phase coherent wavelets allow to create a link between time-domain local differential phase encoding and spatial encoding in such a way that local differential phase and spatial position in the selectively excited plannar tomographic slice form essentially synonymous concepts which can be spatially decoded by an application of the symplectic Fourier transform (see [4]). Thus, the problem of reconstruction of two-dimensional pictures

Received September 5, 1996; in revised form March 15, 1997.
1991 Mathematics Subject Classification. 41A25, 42B10
Key words. Magnetic resonance imaging, Sympletic Fourier transform, Reconstruction of image, Truncation sampling series.

from the hologram echo, their symplectic Fourier transform naturally arises.

A two-dimensional image can be considered as a function f in $\mathbf{L}_2(\mathbf{R}^2)$ which "almost" vanishes outside the square $[-T,T]^2$ for some $T>0$. Then, its symplectic function $\breve{f}$ is in $\mathbf{L}_2(\mathbf{R}^2)$, too. We are interested in the minimal number of information about $\breve{f}$ in the form of the values at linear functionals, needed for reconstructing f from these values with a preassigned precision. Let $\{\ell_m\}_{m=1}^M$ be a family of linear continuous functionals in $\mathbf{L}_2(\mathbf{R}^2)$ and $\{\psi_m\}_{m=1}^M$ a family of functions in $\mathbf{L}_2(\mathbf{R}^2)$. Then, the function f can be reconstructed in $[-T,T]^2$ by a linear combination

$$\sum_{m=1}^{M} \ell_m(\breve{f})\psi_m. \tag{1}$$

Real images are mostly not bandlimited and have small smoothness. Thus, we shall assume that f belongs to the unit ball S of the space $B_2^{\alpha,\beta}$ of the common mixed smoothness (α,β) with $0<\alpha\le 1$ and $0<\beta\le 1$ (see the definition in Section 3). Given a precision $\varepsilon>0$, we let $M(\varepsilon,T)$ denote the minimum of such number M that there exists $\{l_m\}_{m=1}^M$ and $\{\psi_m\}_{m=1}^M$ satisfying the inequality

$$\|f-\sum_{m=1}^{M} \ell_m(\breve{f})\psi_m\|_{\mathbf{L}_2([-T,T]^2)} \le \varepsilon$$

for all $f\in S$.

In this paper we shall give the exact degree of $M(\varepsilon,T)$ for T large enough and corresponding optimal reconstruction formulae for functions from S (Section 3). To construct such a formula, we apply an approximation by truncation sums of a modified sampling series (Section 2).

2. Reconstruction by truncation sampling series

We recall that if $f\in\mathbf{L}_1(\mathbf{R}^2)$, then the symplectic Fourier transform $\breve{f}$ of f is defined by

$$\breve{f}(u,v) := \int_{\mathbf{R}^2} f(x,y)e^{2\pi i \det \left(\begin{smallmatrix} x & u \\ y & v\end{smallmatrix}\right)}dxdy.$$

Obviously,

$$\breve{f}(u,v)=\hat{f}(-v,u)$$

where

$$\hat{f}(u,v) := \int\limits_{\mathbf{R}^2} f(x,y)e^{-2\pi i(xu+yv)}dxdy$$

is the Fourier transform of f. Therefore, one can easily verify that all facts from the theory of Fourier transforms are valid in the corresponding modified form for the symplectic Fourier transform. In particular, the symplectic Fourier transform $\breve{f}$ can be defined for f in $\mathbf{L}_2(\mathbf{R}^2)$. Moreover, for $f \in \mathbf{L}_2(\mathbf{R}^2)$ the Parseval equality

$$\|\breve{f}\|_{\mathbf{L}_2(\mathbf{R}^2)} = \|f\|_{\mathbf{L}_2(\mathbf{R}^2)} \tag{2}$$

holds. The symplectic Fourier transform is self-inverse, i.e.

$$(\breve{f})\breve{} = f.$$

A function $f \in \mathbf{L}_2(\mathbf{R}^2)$ is called symplectically bandlimited to

$$Q_{\sigma,\sigma'} := \{(x,y) \in \mathbf{R}^2 : |x| \leq \sigma, |y| \leq \sigma'\}$$

iff supp $\breve{f} \subset Q_{\sigma,\sigma'}$. We now formulate a modification of the Whittaker-Kotelnikov-Shannon sampling theorem for symplectically bandlimitted functions, which can be proved in a way similar to the proof of Theorem 1 in [1].

Let $\nu > 1$ be a fixed natural number. For $\sigma, \sigma' > 0$, the function Φ is defined by

$$\Phi(x,y) := \varphi(\sigma,x)\varphi(\sigma',y), \tag{3}$$

where

$$\varphi(\sigma,t) := \mathrm{sinc}(4\pi\sigma t)\{\mathrm{sinc}(2\pi\sigma t/\nu)\}^{\nu} \tag{4}$$

and sinc(.) is the sinc-functions defined by

$$\mathrm{sinc}(t) := \begin{cases} t^{-1}\sin t & \text{for} \quad t \neq 0 \\ 1 & \text{for} \quad t = 0. \end{cases}$$

Put $h = 1/4\sigma$, $h' = 1/4\sigma'$.

Theorem 1. *Every function $f \in \mathbf{L}_2(\mathbf{R}^2) \cap C(\mathbf{R}^2)$ symplectically bandlimited to $Q_{\sigma,\sigma'}$, can be represented by the series*

$$(5) \qquad f(x,y) = \sum_{(k,s)\in\mathbf{Z}^2} f(h's, hk)\Phi(y-hk, x-h's)$$

converging uniformly on $\mathbf{R}^2$.

The representation (5) is satisfactory for approximation in the square $[-T,T]^2$ of symplectically bandlimited functions by the following truncation sum

$$(6) \qquad (F_{M,M'}f)(x,y) := \sum_{|k|\le M, |s|\le M'} f(h's, hk)\Phi(y-hk, x-h's).$$

For the error of this approximation we have the following estimate.

Lemma 1. *Let* $f \in \mathbf{L}_2(\mathbf{R}^2) \cap C(\mathbf{R}^2)$ *be symplectically bandlimited to* $Q_{\sigma,\sigma'}$. *Then, for any* $M > T/h$ *and* $M' \ge T/h'$ *we have*

$$\begin{aligned}\|f - F_{M,M'}f\|_{\mathbf{L}_2([-T,T]^2)} \le c_\nu \|f\|_2 T^{1/2}\{&\sigma^{1-\nu}(Mh-T)^{-\nu-1/2} \\ &+ {\sigma'}^{1-\nu}(M'h'-T)^{-\nu-1/2}\}.\end{aligned}$$

This lemma can be proved analogously to the proof of Theorem 3 in [1].

3. Reconstruction of images with small smoothness

We first define a mixed small smoothness (α,β) with $0 < \alpha \le 1, 0 < \beta \le 1$ for functions f in $\mathbf{L}_2(\mathbf{R}^2)$, with the aid of the second moduli of smoothness. For $u, v \ge 0$, we let the mixed second modulus $\omega^2(f;u,v)$ be defined by

$$(7) \qquad \omega^2(f;u,v) := \sup_{|u'|\le u, |v'|\le v} \Big(\int_{\mathbf{R}^2} |\Delta^2_{u',v'} f(x,y)|^2 dxdy\Big)^{1/2},$$

where

$$\Delta^2_{uv} f(x,y) := \Delta^2_u \circ \Delta^2_v f(x,y),$$

$$\Delta^2_u f(x,y) := f(x+2u,y) - 2f(x+u,y) + f(x,y),$$

$$\Delta^2_v f(x,y) := f(x,y+2v) - 2f(x,y+v) + f(x,y).$$

The partial second moduli of smoothness $\omega_x^2(f;u)$ and $\omega_y^2(f;v)$ are defined by replacing Δ_{uv}^2 in the right part of (7) by Δ_u^2 and Δ_v^2, respectively. We let $B_2^{\alpha,\beta}$ denote the space of all functions f in $\mathbf{L}_2(\mathbf{R}^2)$ for which the norm

$$\|f\|_{\mathbf{B}_2^{\alpha,\beta}} := |f|_{\alpha,\beta} + |f|_\alpha + |f|_\beta + \|f\|_{\mathbf{L}_2(\mathbf{R}^2)}$$

is finite where

$$|f|_{\alpha,\beta} := \Big(\int_0^\infty \int_0^\infty \Big\{u^{-\alpha}v^{-\beta}\omega^2(f;u,v)\Big\}^2 dudv/uv\Big)^{1/2},$$

$$|f|_\alpha := \Big(\int_0^\infty \Big\{u^{-\alpha}\omega_x^2(f;u)\Big\}^2 du/u\Big)^{1/2},$$

$$|f|_\beta := \Big(\int_0^\infty \Big\{u^{-\beta}\omega_y^2(f;v)\Big\}^2 dv/v\Big)^{1/2}.$$

The set $B_2^{\alpha,\beta}$ consists of all functions in $\mathbf{L}_2(\mathbf{R}^2)$ with the common small mixed smoothness (α,β). In what follows, without the loss of generality we shall assume that $\alpha \le \beta$. For $(k,s) \in \mathbf{Z}_+^2 := \{(m,n) \in \mathbf{Z}^2 : m \ge 0, n \ge 0\}$, define

$$(\gamma_{ks} f)(x,y) = \int_{\Delta_{ks}} f(u,v) e^{2\pi i \,\det\begin{pmatrix} u & x \\ v & y\end{pmatrix}} dudv,$$

where $\Delta_{ks} := \{(x,y) \in \mathbf{R}^2 : [2^{k-1}] \le |x| < 2^k, [2^{s-1}] \le |y| < 2^s\}$ (here $[t]$ denotes the integer part of $t \in \mathbf{R}$). The following norms equivalence

$$\|f\|_{B_2^{\alpha,\beta}} \approx \Big(\sum_{(k,s)\in\mathbf{Z}_+^2} \{2^{\alpha s+\beta k}\|\gamma_{ks}\breve{f}\|_{\mathbf{L}_2(\mathbf{R}^2)}\}^2\Big)^{1/2} \tag{8}$$

can be proved by the common method for establishing norms equivalence of Besov spaces (see [2]). We need this equivalence for estimating the error of reconstruction of functions of the unit ball

$$S := \{f \in B_2^{\alpha,\beta} : \|f\|_{B_2^{\alpha,\beta}} \le 1\}.$$

For any $\xi > 0$, we let

$$G_\xi := \{(k,s) \in \mathbf{Z}_+^2 : \beta k + \alpha s \le \xi\}. \tag{9}$$

Let f be an arbitrary function of S. We shall reconstruct f in the square $[-T,T]^2$ from linear continuous functionals of $\breve{f}$ by the following truncation sum

$$K_\xi \breve{f} = \sum_{(k,s)\in G_\xi} F_{M_k M'_s}\, \gamma_{ks}\breve{f}, \tag{10}$$

where $M_k = [2^k T]-2, M'_s = [2^s T]-2$ and the operators $F_{M_k M'_s}$ are defined by (3),(4),(6) with $\sigma = \rho = 2^k, \sigma' = \rho' = 2^s$ and $M = M_k, M' = M'_s$. We note that $K_\xi \breve{f}$ can be easily rewritten in the form (1). The number of functionals in $K_\xi \breve{f}$ is determined by the size T and parameter ξ. For any preassigned precission $\varepsilon > 0$, we shall search $\xi := \xi(\varepsilon)$ such that

$$\mathcal{J} := \|f - K_\xi \breve{f}\|_{\mathbf{L}_2([-T,T])^2} \le \varepsilon \tag{11}$$

for T large enough. We have

$$\mathcal{J} \le \mathcal{J}_1 + \mathcal{J}_2, \tag{12}$$

where

$$\mathcal{J}_1 := \|f - \sum_{(k,s)\in G_\xi} \gamma_{ks}\breve{f}\|_{\mathbf{L}_2(\mathbf{R}^2)},$$
$$\mathcal{J}_2 := \sum_{(k,s)\in G_\xi} \|\gamma_{ks}\breve{f} - F_{M_k M'_s}\gamma_{ks}\breve{f}\|_{\mathbf{L}_2([-T,T]^2)}.$$

By (8)-(9) and the Parseval equality (2), we obtain the following estimate for $\mathcal{J}_1$: f

$$\mathcal{J}_1 \le C_1 2^{-\xi} \tag{13}$$

with some constant C_1 not depending on $\breve{f}, \xi$ and T. Since the functions $\gamma_{ks}\breve{f} \in \mathbf{L}_2(\mathbf{R}^2)\cap C(\mathbf{R}^2)$ are symplectically bandlimited to $Q_{2^k,2^s}$, applying Lemma 1 to $\gamma_{ks}\breve{f}, (k,s) \in G_\xi$, and the Parseval equality (2), we have

$$\begin{aligned}\mathcal{J}_2 &\le C_2 \sum_{(k,s)\in G_\xi} \|\gamma_{ks}\breve{f}\|_{\mathbf{L}_2(\mathbf{R}^2)} T^{-\nu} \sum_{(k,s)\in G_\xi} \{2^{(1-\nu)k} + 2^{(1-\nu)s}\} \\ &\le C_3 \|\breve{f}\|_{\mathbf{L}_2(\mathbf{R}^2)} T^{-\nu} \sum_{(k,s)\in G_\xi} \{2^{(1-\nu)k} + 2^{(1-\nu)s}\} \le C_4 \xi T^{-\nu},\end{aligned} \tag{14}$$

with some constants C_2, C_3, C_4 not depending on $\breve{f}, \xi$ and T. We now use the preliminary estimates (13)-(14) for determining ξ. Taking

$$\xi = \xi(\varepsilon) := \log(2C_5/\varepsilon), \tag{15}$$

we get $\mathcal{J}_1 \leq \varepsilon/2$. Moreover, for all T satisfying the inequality

$$T \geq T_0(\varepsilon) := \Big(2C_5/\log(2C_4/\varepsilon)\Big)^{1/\nu},$$

we have $\mathcal{J}_2 \leq \varepsilon/2$. Thus, from (12)-(14) we can conclude that if ξ is defined by (15), then for all $f \in S$ and all $T \geq T_0(\varepsilon)$ the inequality (11) holds.

We note that $M(\varepsilon, T)$ does not exceed the number of functionals of $\breve{f}$ in $K_\xi \breve{f}$. Thus,

$$M(\varepsilon, T) \leq \sum_{(k,s) \in G_\xi} (2M_k + 1)(2M'_s + 1) \leq 4T \sum_{\beta k + \alpha s \leq \xi} 2^{k+s}.$$

The sum in the right part of the last inequality can be estimated by

$$\sum_{\beta k + \alpha s \leq \xi} 2^{k+s} \leq C_7 2^{\xi/\alpha} \xi^r$$

with some constant C_7 not depending on ξ, where $r = 1$ for $\alpha = \beta$ and $r = 0$ for $\alpha < \beta$. Hence, we obtain for all $T \geq T_0(\varepsilon)$

$$M(\varepsilon, T) \leq C_7 T \varepsilon^{1/\varepsilon} \log^r 1/\varepsilon.$$

The inverse inequality is also true. Namely, we proved that for all $T \geq T_0(\varepsilon)$

$$M(\varepsilon, T) \geq C_8 T \varepsilon^{1/\varepsilon} \log^r 1/\varepsilon.$$

Thus, we have the following

Theorem 2. *Let $\varepsilon > 0$ be a preassigned precision, the following estimates*

$$C \leq M(\varepsilon, T)/T\varepsilon^{1/\alpha} \log^r 1/\varepsilon \leq C'$$

hold for any T satisfying the inequality

$$T \geq C''/\log^{1/\nu} 1/\varepsilon$$

with some constants C, C', C'' *not depending on* ε *and* T *,where* $r = 1$ *for* $\alpha = \beta$ *and* $r = 0$ *for* $\alpha < \beta$*. Moreover, the reconstruction formula* (10) *is asymptically optimal for* $M(\varepsilon, T)$.

Remark. It is not difficult to see that one can study the same reconstruction problem for odinary Fourier transforms and that completely similar results can be obtained.

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