ACTA MATHEMATICA VIETNAMICA
Volume 21, Number 1, 1997, pp. 71–105

OPTIMIZATION OF C-ORTHOGONAL POSYNOMIALS

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Dedicated to Hoang Tuy on the occasion of his seventieth birthday

ABSTRACT. We introduce a new class of posynomials, called c-orthogonal posynomials, and we consider the corresponding c-orthogonal programs. The treatment of such programs is motivated by the fact that c-orthogonal posynomial programs having a positive degree of difficulty can be solved under weak assumptions, while "normal" posynomial programs with such a positive degree reduce in general the spectacular power of geometric programming. The optimal value of an unconstrained or constrained c-orthogonal program is equal to the sum of (positive) coefficients of the objectives, respectively.

Especially, using the gained results several interesting inequalities can be proved in a simple way.

1. Introduction

In the more than 30-year history of geometric programming the treatment of posynomial programs played an important role not only from the theoretical point of view but also by their applicability for solving real-life problems. Besides fundamental theoretical results ([1], [3], [4], [7], [10], [12], [21], [23], [28], [29], [32]), numerous algorithms were developed and/or tested concerning this class of geometric programming problems ([2], [5], [8], [9], [13], [21], [27], [31], [36], [37], [41]). Moreover, it is impressive to see a large number of papers devoted to quite different applications (cf. [6]).

Generalizations with respect to other classes of functions, for instance quadratic functions or so-called composite functions were considered and

Received January 27, 1997
Key words. Geometric programming, posynomial, c-orthogonality condition, duality, optimality conditions.
1991 Mathematics Subject Classification. Primary: 90C28; Secondary: 90C15.
Financial support from the Direción General de Investigación Cientifica y Técnica (DG-ICYT) under the grant SAB95-0275 is gratefukky acknowledged.

the corresponding scalar geometric optimization problems were studied ([11], [18], [19], [20], [24], [25], [26], [30], [32], [33], [34], [35], [39], [40]). Another generalization tends to problems with more than one objective function, so-called geometric vector optimization problems ([14]-[17]).

Also of some interest is the way of specializing the class of posynomials. In the geometric programming literature one can find posynomial programs of the following type:

$$\min\{g_0(t) := t_1^{-1}t_2t_3 + t_1t_2^{-1}t_3^{-1} \mid t \in B_p\}$$

$$B_p := \Big\{t \in \mathbf{R}^3 \mid t > 0,\ g_1(t) := \frac{1}{2}t_1t_2^{-2}t_3 + \frac{1}{4}t_1^{-3}t_2^3t_3^{-2} + \frac{1}{4}t_1t_2 \le 1\Big\}.$$

For this optimization problem the "*degree of difficulty*" or "*number of degrees of freedom*" which is introduced in [10] as "*number of terms minus rank A minus 1*" is for the problem above equal to 1 (matrix A is defined in Section 3.1). In the case that the matrix A is of full rank which means rank $A = m$, where m is the number of the variables t_j, $j = 1, \dots, m$, the degree of difficulty is zero, if the number of terms is equal to $m + 1$. Otherwise, the degree is positive or negative and the spectacular power of geometric programming is reduced. In that case the dual program, described in Section 3.1, must be solved.

Even in the recent literature systematic solution methods for geometric programming problems with a large degree of difficulty are hardly developed (see [21], [27], [41]).

Such a positive degree of difficulty can also occur for posynomial programs with functions of the type g_0, g_1. These posynomials have a property which may be denoted as c-orthogonality and treated in Chapter 2. In Chapter 3, an investigation of posynomial programs including such c-orthogonal functions will be done.

It can be shown that for *unconstrained c-orthogonal posynomial programs* the optimal value is equal to the sum of the (positive) coefficients (Theorem 3.1). This assertion remains true for *constrained c-orthogonal posynomial programs* under certain conditions. Duality results for programs will be given, too (Theorem 3.4, Theorem 3.5).

Some examples will demonstrate the usefulness of these results, especially the power of c-orthogonality for proving interesting inequalities.

2. C-ORTHOGONAL POSYNOMIAL

2.1. c-orthogonality

The original geometric programming problem was expressed in terms of polynomials, i.e. functions of the type

$$g_k(t) = \sum_{i\in[k]} c_i \prod_{j=1}^{m} t_j^{a_{ij}} \ , \quad k \in J_p^0,\ m \in \mathbf{N} \tag{2.1}$$

where $J_p^0 := \{0, 1, \dots, p\}$, $\mathbf{N}$ is the set of natural numbers, $t \in \mathbf{R}^m$, $t > 0$, $[k] = \{m_k, m_k+1, \dots, n_k\}$, $m_0 := 1$, $m_k = n_{k-1}+1$, $k = 1, \dots, p$, $n_p := n$, a_{ij} and c_i are reals, $c_i > 0$ for all $i \in [k]$.

Let $A_{[k]} = (a_{ij})$ be the $(n_k - m_k + 1) \times m$ - matrix of exponents due to the variables t_j of the function g_k and $c_{[k]} = (c_{m_k}, \dots, c_{n_k})^T$ the vector of the coefficients. Such functions may have a special property which is defined as follows:

Definition 2.1. A posynomial g_k is said to be c-orthogonal, if

$$A_{[k]}^T c_{[k]} = 0, \quad k \in J_p^0. \tag{2.2}$$

Because of (2.2) it follows immediately

$$(a_{m_k j}, \dots, a_{n_k j}) c_{[k]} = 0 \ , \quad j = 1, \dots, m. \tag{2.3}$$

This means that each c-orthogonal posynomial can be partitioned into m *c-orthogonal sub-posynomials* depending on one unique variable t_j:

$$g_k^j(t_j) = \sum_{i\in[k]} c_i t_j^{a_{ij}} \ , \quad k = 0, 1, \dots, p,\ j = 1, \dots, m. \tag{2.4}$$

Moreover, if each sub-posynomial of a posynomial g_k is c-orthogonal then, of course, g_k is c-orthogonal, too.

Therefore, the proof of c-orthogonality for a posynomial will often be done by proving that property for all partitioned sub-posynomials. Furthermore, the set of c-orthogonal posynomials is "closed" under the common operations addition and multiplication.

Theorem 2.1. *If $\mathcal{G}$ is the set of c-orthogonal posynomials and g_h, $g_\ell \in \mathcal{G}$, $h, \ell = 0, 1, \dots, p$, then*

(i)

$$\alpha g_h \in \mathcal{G}, \quad \alpha \in \mathbf{R}, \quad \alpha > 0, \tag{2.5}$$

(ii)

$$g_h + g_\ell \in \mathcal{G}, \tag{2.6}$$

(iii)

$$g_h \cdot g_\ell \in \mathcal{G}. \tag{2.7}$$

Proof.

(i) For g_h we have $A_{[h]}^T c_{[h]} = 0$ and thus $A_{[h]}^T \alpha c_{[h]} = 0$ for $\alpha \in \mathbf{R}$, $\alpha > 0$ which means that

$$\alpha g_h(t) = \sum_{i \in [h]} \alpha c_i \prod_{j=1}^{m} t_j^{a_{ij}}$$

is c-orthogonal.

(ii) W.l.o.g. we choose $h = 0$, $\ell = 1$, i.e., the posynomials

$$g_0(t) = \sum_{i \in [0]} c_i \prod_{j=1}^{m} t_j^{a_{ij}} \ , \quad g_1(t) = \sum_{i \in [1]} c_i \prod_{j=1}^{m} t_j^{a_{ij}}, \tag{2.8}$$

where

$$A_{[0]}^T c_{[0]} = 0 \ , \quad A_{[1]}^T c_{[1]} = 0. \tag{2.9}$$

Since $[0] \cap [1] = \emptyset$ and $[0] \cup [1] = \{1, \ldots, n_0, n_0 + 1, \ldots, n_1\}$ we obtain the sum of g_0 and g_1 according to

$$g(t) = g_0(t) + g_1(t) = \sum_{i=1}^{n_1} c_i \prod_{j=1}^{m} t_j^{a_{ij}}.$$

For g we have, regarding (2.9),

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ \vdots & \vdots & & \vdots \\ a_{n_0 1} & a_{n_0 2} & \ldots & a_{n_0 m} \\ a_{n_0+1,1} & a_{n_0+1,2} & \ldots & a_{n_0+1,m} \\ \vdots & \vdots & & \vdots \\ a_{n_1 1} & a_{n_1 2} & \ldots & a_{n_1 m} \end{pmatrix}^T \begin{pmatrix} c_1 \\ \vdots \\ c_{n_0} \\ c_{n_0+1} \\ \vdots \\ c_{n_1} \end{pmatrix} =$$

$$= A_{[0]}^T c_{[0]} + A_{[1]}^T c_{[1]} = 0.$$

Thus (2.6) is fulfilled.

(iii) Let us choose g_0, g_1 again according to (2.8), (2.9). Then

$$\begin{aligned}
g_0(t) \cdot g_1(t) &= \sum_{i=1}^{n_0} c_i \prod_{j=1}^{m} t_j^{a_{ij}} \cdot \sum_{i=n_0+1}^{n_1} c_i \prod_{j=1}^{m} t_j^{a_{ij}} \\
&= \sum_{i=1}^{n_0} c_i c_{n_0+1} \prod_{j=1}^{m} t_j^{a_{n_0+1,j}} \cdot \prod_{j=1}^{m} t_j^{a_{ij}} \\
&\;\;\vdots \\
&= \sum_{i=1}^{n_0} c_i c_{n_1} \prod_{j=1}^{m} t_j^{a_{n_1,j}} \cdot \prod_{j=1}^{m} t_j^{a_{ij}}.
\end{aligned}$$

Setting

$$\begin{aligned}
&d_1 := c_1 c_{n_0+1}, \dots, d_{n_0} := c_{n_0} c_{n_0+1}, \\
&d_{n_0+1} := c_1 c_{n_0+2}, \dots, d_{2n_0} := c_{n_0} c_{n_0+2}, \\
&\vdots \\
&d_{(n_1-1)n_0+1} := c_1 c_{n_1}, \dots, d_{n_1 n_0} := c_{n_0} c_{n_1},
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&b_{1j} := a_{1j} + a_{n_0+1,j}, \dots, b_{n_0,j} := a_{n_0,j} + a_{n_0+1,j}, \\
&b_{n_0+1,j} := a_{1j} + a_{n_0+2,j}, \dots, b_{2n_0,j} := a_{n_0,j} + a_{n_0+2,j}, \\
&\vdots \\
&b_{(n_1-1)n_0+1,j} := a_{1j} + a_{n_1,j}, \dots, b_{n_1 n_0,j} := a_{n_0,j} + a_{n_1,j},
\end{aligned} \tag{2.11}$$

we obtain the posynomial

$$g(t) := g_0(t) \cdot g_1(t) = \sum_{i=1}^{n_1} d_i \prod_{j=1}^{m} t_j^{b_{ij}}.$$

For g we have

$$B^T d = \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & & \vdots \\ b_{n_0 1} & \dots & b_{n_0 m} \\ \vdots & & \vdots \\ b_{(n_1-1)n_0+1,1} & \dots & b_{(n_1-1)n_0+1,m} \\ \vdots & & \vdots \\ b_{n_1 n_0,1} & \dots & b_{n_1 n_0,m} \end{pmatrix}^T \begin{pmatrix} d_1 \\ \vdots \\ d_{n_0} \\ \vdots \\ d_{(n_1-1)n_0+1} \\ \vdots \\ d_{n_1 n_0} \end{pmatrix} =$$

$$\begin{pmatrix} b_{11}d_1 + \cdots + b_{n_0 1}d_{n_0} + \cdots + b_{*,1}d_* + \cdots + b_{n_1 n_0,1}d_{n_1 n_0} \\ \vdots \\ b_{1m}d_1 + \cdots + b_{n_0 m}d_{n_0} + \cdots + b_{*,m}d_* + \cdots + b_{n_1 n_0,m}d_{n_1 n_0} \end{pmatrix}, \tag{2.12}$$

where

$$b_{*,1}d_* = b_{(n_1-1)n_0+1,1}d_{(n_1-1)n_0+1},$$
$$b_{*,m}d_* = b_{(n_1-1)n_0+1,m}d_{(n_1-1)n_0+1}.$$

Taking into consideration (2.9) - (2.11), it follows

$$B^T d = (c_{n_0+1} + \cdots + c_{n_1})A^T_{[0]}c_{[0]} + (c_1 + \cdots + c_{n_0})A^T_{[1]}c_{[1]} = 0.$$

Thus (2.7) is satisfied. □

From (2.3) and (2.7) the following assertions can be concluded immediately.

Corollary 2.1. *Each posynomial*

$$g(t) \equiv \alpha = \text{const} \quad \textit{for all } t \in \mathbf{R}^m,\ t > 0$$

is c-orthogonal.

Corollary 2.2. *If g is a c-orthogonal posynomial, then g^n, $n \in \mathbf{N}$, is c-orthogonal, too.*

2.2. Examples of c-orthogonal posynomials

Example 2.1. For $t \in \mathbf{R}^2$, $t > 0$, we consider the following two posynomials simultaneously:

$$g_0(t) = t_1 + t_1^{-1},$$
$$g_1(t) = t_1^{-1} + 2t_2^3 + 3t_1^2 t_2^{-2} + t_1^{-5}.$$

Then

$$[0] = \{1, 2\},$$
$$c_{[0]} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad A_{[0]} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix},$$

$$[1] = \{3, 4, 5, 6\},$$
$$c_{[1]} = \begin{pmatrix} c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \quad A_{[1]} = \begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \\ a_{51} & a_{52} \\ a_{61} & a_{62} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \\ 2 & -2 \\ -5 & 0 \end{pmatrix}.$$

Because of $A_{[k]}^T c_{[k]} = 0$, $k = 0, 1$, both of the posynomials g_0, g_1 are c-orthogonal according to Definition 2.1.

Using (2.2), we get for g_0 and g_1

$$(a_{11}, a_{21})c_{[0]} = 0, \quad (a_{12}, a_{22})c_{[0]} = 0, \tag{2.13}$$

and

$$(a_{31}, a_{41}, a_{51}, a_{61})c_{[1]} = 0, \quad (a_{32}, a_{42}, a_{52}, a_{62})c_{[1]} = 0, \tag{2.14}$$

respectively.

Since $a_{32} = a_{41} = a_{62} = 0$, relation (2.14) can be rewritten as

$$(a_{31}, a_{51}, a_{61}) \begin{pmatrix} c_3 \\ c_5 \\ c_6 \end{pmatrix} = 0, \quad (a_{42}, a_{52}) \begin{pmatrix} c_4 \\ c_5 \end{pmatrix} = 0.$$

Therefore, a partitioning of g_1 into two c-orthogonal sub-posynomials according to (2.4) is justified:

$$g_1^1(t_1) = t_1^{-1} + 3t_1^2 + t_1^{-5}, \quad g_1^2(t_2) = 2t_2^3 + 3t_2^{-2}. \tag{2.15}$$

From (2.13) we conclude that g_0 can be partitioned formally into the two c-orthogonal sub-posynomials (cf. Corollary 2.1)

$$g_0^1(t_1) = t_1 + t_1^{-1} = g_0(t), \quad g_0^2(t_2) = 1 \cdot t_2^0 + 1 \cdot t_2^0 = 2 = \text{const.} \tag{2.16}$$

Example 2.2. The c-orthogonal of the posynomial

$$g(t) = \left(\sum_{j=1}^{m} t_j^2\right)^2 \left(\sum_{j=1}^{m} t_j^{-2}\right), \quad m \in \mathbf{N}, \tag{2.17}$$

it not obvious at the first glance. Reformulation of (2.17) leads to

$$\begin{aligned} g(t) &= \left(\sum_{j=1}^{m} t_j^2 + 2 \sum_{\substack{i<\ell \\ i,\ell=1,\dots,m}} t_i t_\ell\right) \left(\sum_{j=1}^{m} t_j^{-2}\right) \\ &= \left(\sum_{j=1}^{m} t_j^2\right) \left(\sum_{j=1}^{m} t_j^{-2}\right) + 2 \left(\sum_{\substack{i<\ell \\ i,\ell=1,\dots,m}} t_i t_\ell\right) \left(\sum_{j=1}^{m} t_j^{-2}\right). \end{aligned} \tag{2.18}$$

The first term in (2.18) yields

$$\left(\sum_{j=1}^{m} t_j^2\right) \left(\sum_{j=1}^{m} t_j^{-2}\right) = m + \sum_{\substack{i<\ell \\ i,\ell=1,\dots,m}} (t_i^2 t_\ell^{-2} + t_i^{-2} t_\ell^2),$$

where the posynomial

$$g_0(t) := m = mt^0 = \text{const.}$$

is c-orthogonal according to Corollary 2.1. The c-orthogonality of the posynomial

$$g_1(t) = t_i^2 t_\ell^{-2} + t_i^{-2} t_\ell^2$$

is obvious by (2.3). Therefore, using Theorem 2.1, the first term in (2.18) is c-orthogonal.

To prove the c-orthogonality of the second term in (2.18), we simplify in the following manner:

$$
\begin{aligned}
&2\left(\sum_{\substack{i<\ell\\ i,\ell=1,\dots,m}} t_i t_\ell\right)\left(\sum_{j=1}^{m} t_j^{-2}\right) = \\
&2\left(\sum_{\substack{i<\ell\\ i,\ell=1,\dots,m}} (t_i t_\ell^{-1} + t_i^{-1} t_\ell) + \sum_{j=1}^{m}\Big(t_j^{-2} \sum_{\substack{i<\ell\\ i,\ell=1,\dots,m\\ i,\ell\neq j}} t_i t_\ell\Big)\right).
\end{aligned}
\tag{2.19}
$$

Since in (2.19)

$$g_2(t) = \sum_{\substack{i<\ell\\ i,\ell=1,\dots,m}} (t_i t_\ell^{-1} + t_i^{-1} t_\ell)$$

is a c-orthogonal posynomial by (2.3) and Theorem 2.1, it remains to show that the posynomial

$$g_3(t) = \sum_{j=1}^{m}\Big(t_j^{-2} \sum_{\substack{i<\ell\\ i,\ell=1,\dots,m\\ i,\ell\neq j}} t_i t_\ell\Big), \quad m \in \mathbf{N},$$

is c-orthogonal, too. For proving this property we use the idea of partitioning g_3 into sub-posynomials. First, we consider all terms of g_3 containing t_1:

$$
\begin{aligned}
g_{31}(t) = {} & t_1^{-2} t_2 (t_3 + \dots + t_m) + t_1^{-2} t_3 (t_4 + \dots + t_m) + \dots + t_1^{-2} t_{m-1} t_m \\
& + t_2^{-2} t_1 (t_3 + \dots + t_m) + t_3^{-2} t_1 (t_2 + t_4 + t_5 + \dots + t_m) + \dots \\
& + t_m^{-2} t_1 (t_2 + \dots + t_{m-1}).
\end{aligned}
$$

By partitioning of g_{31} into sub-posynomials depending only on exact one variable t_j, $j = 1, \dots, m$, we obtain with respect to t_1:

$$
\begin{aligned}
g_{31}^{1}(t_1) &= t_1^{-2} + t_1^{-2} + \dots + t_1^{-2} + t_1 + t_1 + \dots + t_1 \\
&= [(m-2) + (m-3) + \dots + 2 + 1] t_1^{-2} + (m-1)(m-2) t_1 \\
&= \binom{m-1}{2} t_1^{-2} + (m-1)(m-2) t_1,
\end{aligned}
$$

because t_1^{-2} occurs in g_{31} $\binom{m-1}{2}$-times and t_1 occurs $(m-1)(m-2)$-times in that sum. Taking into account (2.3), we have for g_{31}^1

$$\binom{m-1}{2}(-2)+(m-1)(m-2)\cdot 1=0.$$

This means that g_{31}^1 is c-orthogonal.

Analogously, one can prove the c-orthogonality of the other sub-posynomials $g_{31}^j(t_j)$, $j=2,\dots,m$, and moreover, of all the remaining sub-posynomials of g_3.

Since both terms in (2.18) are c-orthogonal it follows by Theorem 2.1 that the posynomial (2.17) has this property, too.

Example 2.3. The posynomial

$$g(t)=\Big(\sum_{j=1}^m t_j\Big)\Big(\sum_{j=1}^m t_j^{-1}\Big)=m+\sum_{i<\ell}^m(t_it_\ell^{-1}+t_i^{-1}t_\ell) \tag{2.20}$$

is c-orthogonal because of (2.3), Corollary 2.1, and Theorem 2.1.

Example 2.4. A generalization of the c-orthogonal posynomial (2.17) is the posynomial

$$h(t):=(t_1+\cdots+t_m)^p(t_1^{-p}+\cdots+t_m^{-p}),\quad t>0,\ m\in\mathbf{N},\ p\ge 1. \tag{2.21}$$

To prove that $h(t)$ is c-orthogonal we use the polynomial expression

$$(t_1+\cdots+t_m)^p=\sum_{k_1+\cdots+k_m=p}\binom{p}{k_1,\dots,k_m}t_1^{k_1}t_2^{k_2}\dots t_m^{k_m}, \tag{2.22}$$

$k_i\in\mathbf{N}$. Then (2.21) turns to

$$\begin{aligned}h(t)=&\sum_{k_1+\cdots+k_m=p}\binom{p}{k_1,\dots,k_m}t_1^{k_1-p}t_2^{k_2}\dots t_m^{k_m}\\&+\sum_{k_1+\cdots+k_m=p}\binom{p}{k_1,\dots,k_m}t_1^{k_1}t_2^{k_2-p}\dots t_m^{k_m}\\&\vdots\\&+\sum_{k_1+\cdots+k_m=p}\binom{p}{k_1,\dots,k_m}t_1^{k_1}t_2^{k_2}\dots t_m^{k_m-p}.\end{aligned} \tag{2.23}$$

Now we prove (2.3) for any sub-posynomial of $h(t)$.

Therefore, we consider w.l.o.g. the sub-posynomial

$$\begin{aligned} h^1(t_1) = &\sum_{k_1+\dots+k_m=p} \binom{p}{k_1,\dots,k_m} t_1^{k_1-p} \\ &+ (m-1) \sum_{k_1+\dots+k_m=p} \binom{p}{k_1,\dots,k_m} t_1^{k_1}. \end{aligned} \tag{2.24}$$

To verify (2.3) we get from (2.24) and the well-known relation

$$\sum_{k_1+\dots+k_m=p} \binom{p}{k_1,\dots,k_m} = m^p \tag{2.25}$$

the equality

$$\begin{aligned} &\sum_{k_1+\dots+k_m=p} \binom{p}{k_1,\dots,k_m}(k_1-p) + (m-1) \sum_{k_1+\dots+k_m=p} \binom{p}{k_1,\dots,k_m} k_1 \\ &= -p \sum_{k_1+\dots+k_m=p} \binom{p}{k_1,\dots,k_m} + m \sum_{k_1+\dots+k_m=p} \binom{p}{k_1,\dots,k_m} k_1 =: A. \end{aligned} \tag{2.26}$$

Setting $\alpha := k_1$, we conclude from (2.26)

$$A = -pm^p + m \sum_{\alpha=0}^{m} \sum_{\alpha+k_2+\dots+k_m=p} \binom{p}{\alpha, k_2,\dots,k_m} \alpha. \tag{2.27}$$

Because of

$$\begin{aligned} &\sum_{\alpha=0}^{p} \sum_{\alpha+k_2+\dots+k_m=p} \binom{p}{\alpha, k_2,\dots,k_m} \alpha \\ &= \sum_{\alpha=1}^{p} \alpha \sum_{k_2+\dots+k_m=p-\alpha} \frac{p!}{\alpha! k_2! \dots k_m!} \\ &= p \sum_{\alpha=1}^{p} \frac{(p-1)!}{(p-\alpha)!(\alpha-1)!} \sum_{k_2+\dots+k_m=p-\alpha} \frac{(p-\alpha)!}{k_2! \dots k_m!} \\ &= p \sum_{\alpha=1}^{p} \binom{p-1}{\alpha-1} (m-1)^{p-\alpha} = p \sum_{\alpha=1}^{p} \binom{p-1}{\alpha-1} (m-1)^{(p-1)-(\alpha-1)} \cdot 1^{\alpha-1} \\ &= p((m-1)+1)^{p-1} = pm^{p-1}, \end{aligned}$$

it follows in (2.27) immediately $A = 0$, whence (2.3) is satisfied. Thus, $h^1(t_1)$ is c-orthogonal.

Analogously, one can prove the c-orthogonality of the remaining sub-posynomials $h^j(t_j)$, $j = 2, \ldots, m$. Therefore, by Definition 2.1 the posynomial $h(t)$ id c-orthogonal.

Example 2.5. The posynomial

$$g(t) := \frac{t_1^m + t_2^m + \cdots + t_m^m}{t_1 t_2 \ldots t_m}\ , \quad m \in \mathbf{N},$$

is c-orthogonal, because for each sub-posynomial (see (2.4))

$$\begin{aligned} g^1(t_1) &:= t_1^{m-1} + (m-1)t_1^{-1}, \\ g^2(t_2) &:= t_2^{m-1} + (m-1)t_2^{-1}, \\ &\vdots \\ g^m(t_m) &:= t_m^{m-1} + (m-1)t_m^{-1}, \end{aligned}$$

relation (2.3) is easy to verify:

$$(m-1) \cdot 1 + (-1)(m-1) = 0, \quad m \in \mathbf{N}.$$

3. Optimization of c-orthogonal posynomials

3.1. c-orthogonal posynomial programs

For "classical" posynomial programs a duality theory is established in [10], and refined, for instance, in [31]. The duality approach described there is based on the inequality between the weighted arithmetic and geometric mean, related to the following optimization problems:

$$\begin{aligned} &P_p \ : \ \min\{g_0(t) \mid t \in B_p\}, \\ &B_p := \{t \in \mathbf{R}^m \mid t > 0;\ g_k(t) \le 1,\ k \in J_p\}, \end{aligned} \tag{3.1}$$

where g_o, g_k, $k \in J_p := \{1, \ldots, p\}$, are given according to (2.1).

$$\begin{aligned} &P_p^* \ : \ \max\Big\{v(y) := \prod_{k=0}^{p} \prod_{i \in [k]} \Big(\frac{c_i}{y_i}\Big)^{y_i} (\lambda_k(y))^{\lambda_k(y)} \mid y \in B_p^*\Big\}, \\ &B_p^* := \{y \in \mathbf{R}^m \mid y \ge 0;\ \lambda_0(y) = 1,\ A^T y = 0\}, \end{aligned} \tag{3.2}$$

where $\lambda_k(y) = \sum\limits_{i\in[k]} y_i$ and $A = \begin{pmatrix} A_{[0]} \\ \vdots \\ A_{[p]} \end{pmatrix}$, $A_{[k]}$, $k \in J_p^0$, described in Section 2.1.

In the context of geometric programming, P_p is called a *primal posynomial program* and P_p^* the corresponding *dual program.* For further investigations, P_p will be assumed to be c-orthogonal according to

Definition 3.1. Problem P_p is said to be a c-orthogonal posynomial program, if all functions g_k, $k = 0, 1, \ldots, p$, are c-orthogonal.

Moreover, in [10] it was shown that the programs P_p, P_p^* are equivalent to the following convex programs P, P^*.

$$\begin{aligned} P \quad : \quad & \min\{G_0(x) \mid x \in B\}, \\ & B := \{x \in \mathbf{R}^n \mid x \in \mathcal{P};\ G_k(x) \leq 1,\ k \in J_p\}, \end{aligned} \tag{3.3}$$

where

$$G_k(x) := \sum_{i\in[k]} c_i e^{x_i}, \quad k \in J_p^0, \tag{3.4}$$

$\mathcal{P}$ is the column space of the $n \times m$-matrix A defined by

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{pmatrix} = \begin{pmatrix} A_{[0]} \\ A_{[1]} \\ \vdots \\ A_{[p]} \end{pmatrix}. \tag{3.5}$$

The underlying transformation from program P_p to program P can be represented by

$$t_j = e^{r_j}, \quad j = 1, \ldots, m, \tag{3.6}$$

and

$$x = Ar, \tag{3.7}$$

or, coordinatewise

$$x_i = \sum_{j=1}^{m} a_{ij} r_j = \ln \prod_{j=1}^{m} t_j^{a_{ij}} \quad \forall i \in [k],\ k \in J_p^0. \tag{3.8}$$

The corresponding dual program is

$$
\begin{aligned}
P^* \quad &: \quad \max\{V(y) := \ln v(y) \mid y \in B^*\}, \\
&\quad\;\; B^* = B_p^*. \qquad (3.9)
\end{aligned}
$$

Remark 3.1. If P_p is a c-orthogonal program, then P is c-orthogonal, too. By use of the same matrix A and the same coefficient vector $c = \big(c_{[0]}^T, \ldots, c_{[p]}^T\big)^T$ for the programs P_p and P property (2.2) is preserved.

The purpose of the following theorem is twofold: It gives the minimal value of any c-orthogonal posynomial and will be used in Section 3.3 to prove some well-known inequalities or to create some new ones.

Theorem 3.1. *Let g_k, $k \in J_p^0$, be a c-orthogonal posynomial. Then*

$$\min_{t>0} g_k(t) = \sum_{i\in[k]} c_i = g_k(1,1,\ldots,1). \tag{3.10}$$

Proof. Since g_k is to consider on the positive orthant of $\mathbf{R}^m$ we have

$$g_k(1,1,\ldots,1) = \sum_{i\in[k]} c_i.$$

Therefore, the minimum value of any posynomial must be less or equal than the sum of the coefficients:

$$\min_{t>0} g_k(t) \le \sum_{i\in[k]} c_i. \tag{3.11}$$

For proving equality in (3.11) we use the c-orthogonality of g_k.

With the posynomial term (cf. [10])

$$u_i := c_i \prod_{j=1}^{m} t_j^{a_{ij}}, \quad i \in [k], \tag{3.12}$$

we form the expression

$$\prod_{i\in[k]} u_i^{c_i} = \prod_{i\in[k]} c_i^{c_i} \cdot \prod_{i\in[k]} t_1^{a_{i1}c_i} \cdots \prod_{i\in[k]} t_m^{a_{im}c_i},$$

and obtain because of the c-orthogonality property (2.3)

$$\prod_{i\in[k]} u_i^{c_i} = \prod_{i\in[k]} c_i^{c_i} t_1^{\sum\limits_{i\in[k]} a_{i1}c_i} \cdots t_m^{\sum\limits_{i\in[k]} a_{im}c_i} = \prod_{i\in[k]} c_i^{c_i}. \tag{3.13}$$

Introducing new variables w_i, $i \in [k]$, according to

$$u_i := \frac{c_i}{c} w_i, \quad \text{where} \quad c := \sum_{i\in[k]} c_i, \tag{3.14}$$

the left hand side of (3.13) can be written as

$$\prod_{i\in[k]} \Big(\frac{c_i}{c} w_i\Big)^{c_i} = \Big(\frac{1}{c}\Big)^c c_{m_k}^{c_{m_k}} \dots c_{n_k}^{c_{n_k}} w_{m_k}^{c_{m_k}} \dots w_{n_k}^{c_{n_k}}.$$

By (3.14) it follows

$$\prod_{i\in[k]} w_i^{c_i/c} = c \prod_{i\in[k]} c_i^{c_i/c} \cdot \prod_{i\in[k]} c_i^{-c_i/c} = c. \tag{3.15}$$

Since the positive numbers c_i/c, $i \in [k]$, are normalized weights, the well-known geometric mean-arithmetic mean inequality is valid for w_i, $i \in [k]$:

$$\sum_{i\in[k]} \frac{c_i}{c} w_i \geq \prod_{i\in[k]} w_i^{c_i/c}.$$

By (3.15), (3.14), (3.12) and (2.1) this inequality yields:

$$g_k(t) = \sum_{i\in[k]} u_i \geq \sum_{i\in[k]} c_i$$

and

$$\min_{t>0} g_k(t) \geq \sum_{i\in[k]} c_i.$$

Together with (3.11) we obtain (3.10). □

Remark 3.2. Choosing $k = 0$ (w.l.o.g.), relation (3.10) means that the point $t^0 := (1, 1, \dots, 1)^T$ is an optimal solution of each unconstrained c-orthogonal posynomial program P_p:

$$\min\{g_0(t) \mid t \in B_p\},$$

$$B_p := \{t \in \mathbf{R}^m \mid t > 0\}. \tag{3.16}$$

In Theorem 3.1 nothing is said about the uniqueness of the minimizer t_0 for an unconstrained c-orthogonal posynomial program. Other optimal solutions may exist; for instance, the problem

$$\min\{g_0(t) := t_1^2 t_2^{-2} + t_1^{-2} t_2^2 \mid t \in \mathbf{R}^2,\ t > 0\}$$

has the optimal solutions $t^0 = (a, a)$, $a \in \mathbf{R}$, $a > 0$, and its minimum value is $\min\limits_{t>0} g_0(t) = 2$.

For a constrained c-orthogonal posynomial program the following assertions can be shown.

Theorem 3.2. *Let the c-orthogonal program P_p be given. Then*

$$B_p \neq \emptyset \quad \text{implies} \quad \hat{t} := (1, 1, \ldots, 1)^T \in B_p. \tag{3.17}$$

Proof. Suppose $\hat{t} \notin B_p$. Then there exists at least an index $k_1 \in J_p$ so that $g_{k_1}(\hat{t}) > 1$. Since each posynomial is assumed to be c-orthogonal, we conclude according to Theorem 3.1

$$g_{k_1}(t) \geq \min_{t>0} g_{k_1}(t) = \sum_{i \in [k_1]} c_i = g_{k_1}(\hat{t}) > 1,$$

that means there is no t satisfying $g_{k_1}(t) \leq 1$ which implies $B_p = \emptyset$. □

Theorem 3.3 (Weak Duality Theorem). *Let P_p, P_p^* be given and let P_p be c-orthogonal. If $B_p \neq \emptyset$ and $B_p^* \neq \emptyset$, then*

$$\text{(i)} \qquad g_0(t) \geq v(y) \quad \forall t \in B_p,\ \forall y \in B_p^*, \tag{3.18}$$

$$\text{(ii)} \qquad \min_{t \in B_p} g_0(t) \geq \max_{y \in B_p^*} v(y). \tag{3.19}$$

The proofs of (i) and (ii) are given in [10] for non c-orthogonal programs and can be applied directly to the case of c-orthogonal programs.

A corresponding theorem is true for the programs P, P^*.

Theorem 3.4 (Direct Duality Theorem). *Let P_p, P_p^* be given and let P_p be c-orthogonal, $r \cdot \operatorname{int} B_p \neq \emptyset$.*

If t^0 is an optimal solution of P_p, then there exists a dual optimal solution $y^0 \in B_p^*$ such that

$$\text{(i)} \qquad y_i^0 = \begin{cases} \dfrac{c_i}{\sum\limits_{i\in[0]} c_i}, & i \in [0], \qquad (3.20) \\ \lambda_k(y^0)c_i, & i \in [k],\ k \in J_p, \qquad (3.21) \end{cases}$$

$$\text{(ii)} \qquad g_0(t^0) = \min_{t\in B_p} g_0(t) = \max_{y\in B_p^*} v(y) = v(y^0). \tag{3.22}$$

Proof. (i) Since t^0 is an optimal solution of P_p, we have

$$g_0(t^0) = \min_{t\in B_p} g_0(t) \le g_0(t) \quad \forall t \in B_p. \tag{3.23}$$

Therefore, using the problem P equivalent to P_p, relation (3.23) can be written as

$$G_0(x^0) = \min_{x\in B} G_0(x) \le G_0(x), \quad \forall x \in B, \tag{3.24}$$

which means that x^0 in an optimal solution of P. Hence the system

$$G_0(x) - G_0(x^0) < 0, \quad G_k(x) - 1 \le 0 \quad (k \in J_p),\ x \in B,$$

is not solvable.

Since we assumed $r \cdot \operatorname{int} B_p \neq \emptyset$, the equivalent set $r \cdot \operatorname{int} B$ is nonempty, too. It exists an $\overline{x} \in r \cdot \operatorname{int} B$ such that $G_k(\overline{x}) < 1$ for all $k \in J_p$. Now, applying a standard result of convex analysis (see [38], Theorem 2.1.1) it follows the existence of a vector $(u^0, w^0) \in \mathbf{R}_+ \times \mathbf{R}_+^p$, $u^0 \neq 0$, such that

$$u^0(G_0(x) - G_0(x^0)) + \sum_{k=1}^{p} w_k^0(G_k(x) - 1) \ge 0, \quad \forall x \in B.$$

W.l.o.g. we assume $u^0 = 1$ and obtain on the one hand

$$G_0(x) + \sum_{k=1}^{p} w_k^0(G_k(x) - 1) \ge G_0(x^0), \quad \forall x \in B. \tag{3.25}$$

Because of $G_k(x^0) - 1 \le 0$ it is on the other hand

$$G_0(x^0) + \sum_{k=1}^{p} w_k(G_k(x^0) - 1) \le G_0(x^0), \quad \forall w_k \in \mathbf{R}_+, \tag{3.26}$$

and thus

$$G_0(x^0) + \sum_{k=1}^{p} w_k^0 (G_k(x^0) - 1) = G_0(x^0). \tag{3.27}$$

Using the denotation

$$L(x, w) := G_0(x) + \sum_{k=1}^{p} w_k (G_k(x) - 1), \quad (x, w) \in B \times \mathbf{R}_+^p, \tag{3.28}$$

it follows along with (3.25)-(3.27):

$$L(x^0, w) \leq L(x^0, w^0) = G_0(x^0) \leq L(x, w^0), \quad \forall x \in B, \ \forall w \in \mathbf{R}_+^p, \tag{3.29}$$

and moreover, $x^0 \in B$ is a global minimizer of $L(x, w^0)$ on $B \times \mathbf{R}_+^p$. Consequently, the relation

$$\frac{\partial L(x, w)}{\partial x_i}\Big|_{\substack{x=x^0 \\ w=w^0}} = 0, \quad i \in [k], \ k \in J_p^0, \tag{3.30}$$

is satisfied. Since $x \in \mathcal{P}$ is chosen according to (3.7), we obtain by (3.30)

$$0 = \sum_{i \in [0]} \frac{\partial G_0(x)}{\partial x_i}\Big|_{x=x^0} \frac{\partial x_i}{\partial r_q}\Big|_{r=r^0} + \sum_{k=1}^{p} w_k^0 \sum_{i \in [k]} \frac{\partial G_k(x)}{\partial x_i}\Big|_{x=x^0} \frac{\partial x_i}{\partial r_q}\Big|_{r=r^0},$$

and therefore

$$0 = \sum_{i \in [0]} a_{iq} c_i e^{x_i^0} + \sum_{k=1}^{p} w_k^0 \sum_{i \in [k]} a_{iq} c_i x^{x_i^0}, \quad j \in J_m. \tag{3.31}$$

Dividing (3.31) by $g_0(t^0)$ we conclude, regarding (3.8), the relation

$$0 = \sum_{i \in [0]} a_{iq} \frac{c_i \prod\limits_{j=1}^{m} t_j^{0^{a_{ij}}}}{\sum\limits_{i \in [0]} c_i \prod\limits_{j=1}^{m} t_j^{0^{a_{ij}}}}$$

$$+ \sum_{k=1}^{p} w_k^0 \sum_{i \in [k]} a_{iq} \frac{c_i \prod\limits_{j=1}^{m} t_j^{0^{a_{ij}}}}{\sum\limits_{i \in [0]} c_i \prod\limits_{j=1}^{m} t_j^{0^{a_{ij}}}}, \quad j \in J_m. \tag{3.32}$$

Because of the assumptions $r \cdot \operatorname{int} B_p \neq \emptyset$ and t^0 an optimal solution of P_p it follows by Theorem 3.1, Theorem 3.2 and the fact that $B_r \subset \mathbf{R}^m_+$ contains the element $(1, 1, \ldots, 1)^T$:

$$\min_{t \in B_p} g_0(t) = g_0(t^0) = \sum_{i \in [0]} c_i = g_0(1, 1, \ldots., 1).$$

That means, the optimal solution t^0 can be chosen as $t^0 = (1, 1, \ldots, 1)^T$. Thus, equality (3.32) leads to

$$0 = \sum_{i \in [0]} a_{iq} \frac{c_i}{\sum\limits_{i \in [0]} c_i} + \sum_{k=1}^{p} w_k^0 \sum_{i \in [k]} a_{iq} \frac{c_i}{\sum\limits_{i \in [0]} c_i} \ , \quad j \in J_m. \tag{3.33}$$

Hence the vector y^0 with the coordinates

$$y_i^0 = \begin{cases} \dfrac{c_i}{\sum\limits_{i \in [0]} c_i} \ , & i \in [0], \quad (3.34) \\ \dfrac{w_k^0 c_i}{\sum\limits_{i \in [0]} c_i} \ , & i \in [k], \ k \in J_p. \quad (3.35) \end{cases}$$

satisfies the so-called orthogonality condition of P_p^*:

$$A^T y = 0. \tag{3.36}$$

Moreover, regarding in (3.2) the denotation for λ_k, $k \in J_p^0$, we obtain immediately from (3.34) and (3.35)

$$\lambda_0(y^0) = \sum_{i \in [0]} y_i^0 = 1, \tag{3.37}$$

$$\lambda_k(y^0) = \sum_{i \in [0]} y_i^0 = \frac{w_k^0 \sum\limits_{i \in [k]} c_i}{\sum\limits_{i \in [0]} c_i} \ , \quad k \in J_p, \tag{3.38}$$

respectively.

From (3.34) it follows

$$y_i^0 > 0, \quad \forall i \in [0]. \tag{3.39}$$

Since $w_k^0 \in \mathbf{R}_+$, from (3.35) we infer

$$y_i^0 \geq 0, \quad \forall i \in [k],\ k \in J_p. \tag{3.40}$$

Thus, the vector y^0 satisfying (3.36), (3.37), (3.39), (3.40) is an element of B_p^*. Furthermore, from (3.27) we conclude

$$w_k^0(G_k(x^0) - 1) = 0, \quad \forall k \in J_p,$$

and by (3.8) we have

$$w_k^0(g_k(t^0) - 1) = 0, \quad \forall k \in J_p. \tag{3.41}$$

Thus, for $t^0 = (1, 1, \ldots, 1)^T$ it follows

$$w_k^0 g_k(t^0) = w_k^0 \sum_{i \in [k]} c_i = w_k^0, \quad k \in J_p, \tag{3.42}$$

and (3.38) yields

$$\lambda_k(y^0) = \frac{w_k^0}{\sum\limits_{i \in [0]} c_i}\ , \quad \forall k \in J_p, \tag{3.43}$$

Therefore, the vector y^0 according to (3.34) and (3.35) has the presentation (3.20) and (3.21), respectively:

$$y_i^0 = \begin{cases} \dfrac{c_i}{\sum\limits_{i \in [0]} c_i}\ , & i \in [0], \\ \lambda_k(y^0) c_i, & i \in [k],\ k \in J_p. \end{cases}$$

(ii) Since

$$\begin{aligned} v(y^0) &= \prod_{i \in [0]} \Big(\frac{c_i}{c_i} \sum_{i \in [0]} c_i \Big)^{\frac{c_i}{\sum_{i \in [0]} c_i}} \prod_{k=1}^{p} \prod_{i \in [k]} \Big(\frac{c_i}{c_i \lambda_k(y^0)} \Big)^{c_i \lambda_k(y^0)} \lambda_k(y^0)^{\lambda_k(y^0)} \\ &= \sum_{i \in [0]} c_i \prod_{k=1}^{p} \prod_{i \in [k]} \frac{1}{\lambda_k(y^0)^{\lambda_k(y^0)}} \lambda_k(y^0)^{\lambda_k(y^0)} \\ &= \sum_{i \in [0]} c_i, \end{aligned}$$

we have $g_0(t^0) = v(y^0)$. Using (3.18), we conclude $v(y^0) \geq v(y)\ \forall y \in B_p^*$, therefore (3.22) is satisfied. $\square$

Theorem 3.5 (Inverse Duality Theorem). *Let P_p, P_p^* be given and let P_p be c-orthogonal, and let $r \cdot \operatorname{int} B_p^* \neq \emptyset$. If y^0 is an optimal solution of P_p^*, then there exists a primal optimal solution $t^0 \in B_p$ such that*

$$\text{(i)}\quad c_i \prod_{j=1}^{m} t_j^{0\,a_{ij}} = \begin{cases} y_i^0 \sum\limits_{i\in[0]} c_i, & i \in [0], \\ \dfrac{y_i^0}{\lambda_k(y^0)}\,, & i \in [k],\ k \in J_p,\ \text{where } \lambda_k(y^0) > 0, \end{cases}$$

$$\text{(ii)}\qquad v(y^0) = \max_{y\in B_p^*} v(y) = \min_{t\in B_p} g_0(t) = g_0(t^0).$$

Proof. (i) Since y^0 is an optimal solution of P_p^*, we have

$$v(y^0) \geq v(y), \quad \forall y \in B_p^*,$$

and therefore

$$V(y^0) \geq V(y), \quad \forall y \in B^*.$$

Hence, the system

$$\begin{cases} V(y^0) - V(y) < 0, & (3.44)\\ A^T y = 0, & (3.45)\\ \lambda_0(y) - 1 = 0, & (3.46)\\ -\lambda_k(y) \leq 0, \quad \forall k \in J_p, & (3.47)\\ y \in B^* \end{cases}$$

is not solvable.

Since the function in (3.44) and (3.47) are convex (see [10]) and the functions in (3.45), (3.46) are affine, there exists a vector (using a standard result of convex analysis, see [38], Theorem 2.1.1) $(\eta^0, \rho^0, \mu^0, \tau^0) \in \mathbf{R}_+ \times \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}_+^p$, $\eta^0 \neq 0$, such that

$$\eta^0(V(y^0) - V(y)) + \sum_{j=1}^{m} \rho_j^0 \Big(\sum_{i\in[0]} a_{ij} y_i + \sum_{k=1}^{p} \sum_{i\in[k]} a_{ij} y_i \Big)$$

(3.48)

$$+\, \mu^0(\lambda_0(y) - 1) - \sum_{k=1}^{p} \tau_k^0 \lambda_k(y) \geq 0, \quad \forall y \in B^*.$$

W.l.o.g. we assume $\eta^0 = 1$ and use the denotation

$$L^*(y, \rho^0, \mu^0, \tau^0) = V(y) + \mathcal{A}_1(y) + \mathcal{A}_2(y) + \mathcal{A}_3(y),$$

where

$$\begin{aligned}
\mathcal{A}_1(y) &:= -\sum_{j=1}^{m} \rho_j^0 \Big(\sum_{i\in[0]} a_{ij} y_i + \sum_{k=1}^{p} \sum_{i\in[k]} a_{ij} y_i \Big), \\
\mathcal{A}_2(y) &:= -\mu^0(\lambda_0(y) - 1) = -\mu^0 \Big(\sum_{i\in[0]} y_i - 1 \Big), \\
\mathcal{A}_3(y) &:= \sum_{k=1}^{p} \tau_k^0 \lambda_k(y) = \sum_{k=1}^{p} \tau_k^0 \sum_{i\in[k]} y_i.
\end{aligned}$$

Then (3.48) yields

$$V(y^0) \geq L^*(y, \rho^0, \mu^0, \tau^0), \quad \forall y \in B^*.$$

Because of (3.45)-(3.47) it follows

$$L^*(y^0, \rho, \mu, \tau) \geq V(y^0), \quad \forall (\rho, \mu, \tau) \in \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}_+^p.$$

Therefore, we obtain

$$V(y^0) = L^*(y^0, \rho^0, \mu^0, \tau^0), \tag{3.49}$$

and moreover

$$\begin{aligned}
&L^*(y^0, \rho, \mu, \tau) \geq L^*(y^0, \rho^0, \mu^0, \tau^0) = V(y^0) \geq L^*(y, \rho^0, \mu^0, \tau^0), \\
&\qquad \forall (y, \rho, \mu, \tau) \in B^* \times \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}_+^p,
\end{aligned}$$

i.e. $y^0 \in B^* = B_p^*$ is a global minimizer of $L^*(y, \rho^0, \mu^0, \tau^0)$ on $B_p^* \times \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}_+^p$. Consequently, the relation

$$\frac{\partial L^*(y, \rho^0, \mu^0, \tau^0)}{\partial y_i}\Big|_{y_i = y_i^0} = 0, \quad i \in [k], \; k \in J_p^0$$

must be satisfied.

Since the partial derivatives of $V(y)$, $\mathcal{A}_1(y)$, $\mathcal{A}_2(y)$, $\mathcal{A}_3(y)$ are given according to

$$\begin{aligned}
\frac{\partial V(y)}{\partial y_i}\Big|_{y_i=y_i^0} &= \ln\frac{c_i}{y_i^0}\,\lambda_k(y^0), && i\in[k],\quad k\in J_p^0,\\
\frac{\partial \mathcal{A}_1(y)}{\partial y_i}\Big|_{y_i=y_i^0} &= -\sum_{j=1}^m \rho_j^0 a_{ij}, && i\in[k],\quad k\in J_p^0,\\
\frac{\partial \mathcal{A}_2(y)}{\partial y_i}\Big|_{y_i=y_i^0} &= -\mu^0, && i\in[0],\\
\frac{\partial \mathcal{A}_3(y)}{\partial y_i}\Big|_{y_i=y_i^0} &= \tau_k^0, && i\in[k],\quad \forall k\in J_p,
\end{aligned}$$

we obtain the formulas (taking into consideration $\lambda_0(y^0)=1$):

$$\frac{\partial L^*(y,\rho^0,\mu^0,\tau^0)}{\partial y_i}\Big|_{y_i=y_i^0} =$$

$$= \begin{cases} \ln\dfrac{c_i}{y_i^0} - \displaystyle\sum_{j=1}^m \rho_j^0 a_{ij} - \mu^0 = 0, & i\in[0], \qquad (3.50a)\\[2ex] \ln\dfrac{c_i}{y_i^0}\,\lambda_k(y^0) - \displaystyle\sum_{j=1}^m \rho_j^0 a_{ij} + \tau_k^0 = 0, & i\in[k],\ k\in J_p. \qquad (3.50b)\end{cases}$$

Setting $\rho_j^0 := -\ln t_j^0$, $j=1,\dots,m$, it follows from (3.50a) and (3.50b)

$$c_i\prod_{j=1}^m {t_j^0}^{a_{ij}} = e^{\ln y_i^0+\mu^0} = y_i^0 e^{\mu_0},\quad i\in[0], \tag{3.51a}$$

$$c_i\prod_{j=1}^m {t_j^0}^{a_{ij}} = e^{\ln\frac{y_i^0}{\lambda_k(y^0)}-\tau_k^0} = \frac{y_i^0}{\lambda_k(y^0)}e^{-\tau_k^0},\quad i\in[k],\ k\in J_p, \tag{3.51b}$$

respectively.

Summing (3.51a) and (3.51b) over all i, we get

$$g_0(t^0) = \sum_{i\in[0]} c_i\prod_{j=1}^m {t_j^0}^{a_{ij}} = e^{\mu^0} \tag{3.52a}$$

and

$$g_k(t^0) = \sum_{i\in[k]} c_i \prod_{j=1}^{m} {t_j^0}^{a_{ij}} = e^{-\tau_k^0}, \quad k \in J_p, \tag{3.52b}$$

respectively.

From (3.52a) it is easy to see that

$$g_0(t^0) > 0$$

and by the assumption $\tau_k^0 \in \mathbf{R}_+$ for all $k \in J_p$, from (3.52b) it follows

$$g_k(t^0) \leq 1, \quad \forall k \in J_p.$$

So the existence of as feasible $t^0 \in B_p$ of the assumed c-orthogonal program P_p is shown.

In this case, by Theorem 3.2 we conclude that $(1, 1, \ldots, 1) \in B_p$.

Applying Theorem 3.1, we have (because of $B_p \subset \mathbf{R}_+^m$)

$$\min_{t\in B_p} g_0(t) = \sum_{i\in[0]} c_i = g_0(t^0). \tag{3.53}$$

Therefore, identifying t^0 as minimizer, from (3.53) we conclude

$$\prod_{j=1}^{m} {t_j^0}^{a_{ij}} = 1, \quad \forall i \in [0]. \tag{3.54}$$

Thus, (3.52a) becomes

$$\sum_{i\in[0]} c_i = e^{\mu^0}. \tag{3.55a}$$

Furthermore, taking into account the feasibility of y^0, from (3.49) we get

$$\sum_{k=1}^{p} \tau_k^0 \lambda_k(y^0) = 0.$$

Since $\tau_k^0 \geq 0$, $\lambda_k(y^0) \geq 0$ for each $k \in J_p$, we obtain $\tau_k^0 \lambda_k(y^0) = 0\ \forall k \in J_p$. Hence $\tau_k^0 = 0$ if $\lambda_k(y^0) > 0$, $k \in J_p$, and in that case (3.52b) leads to

$$g_k(t^0) = \sum_{i\in[k]} c_i \sum_{j=1}^{m} {t_j^0}^{a_{ij}} = e^{-\tau_k^0} = 1, \quad k \in J_p. \tag{3.55b}$$

Substituting (3.55a) and (3.55b) into (3.51a) and (3.51b), respectively, it follows assertion (i):

$$c_i \prod_{j=1}^{m} {t_j^0}^{a_{ij}} = \begin{cases} y_i^0 \sum\limits_{i\in[0]} c_i, & i \in [0], \qquad (3.56a) \\ \dfrac{y_i^0}{\lambda_k(y^0)}\,, & i \in [k],\ k \in J_p,\ \text{where } \lambda_k(y^0) > 0. \qquad (3.56b) \end{cases}$$

(ii) By (3.56a) and (3.56b) we get for dual function v of P_p^*:

(3.57)

$$v(y^0) = \prod_{i\in[0]} \Bigg(\frac{c_i \sum\limits_{i\in[0]} c_i}{c_i \prod\limits_{j=1}^{m} {t_j^0}^{a_{ij}}}\Bigg)^{y_i^0} \prod_{k=1}^{p} \prod_{i\in[k]} \Bigg(\frac{c_i}{c_i \prod\limits_{j=1}^{m} {t_j^0}^{a_{ij}} \lambda_k(y^0)}\Bigg)^{y_i^0} \lambda_k(y^0)^{\lambda_k(y^0)}.$$

Taking into consideration that $\lambda_0(y^0) = 1$ and $A^T y^0 = 0$ for $y^0 \in B_p^*$, (3.57) becomes

$$v(y^0) = \sum_{i\in[0]} c_i\ . \qquad (3.58)$$

Together with (3.53) it follows

$$v(y^0) := \max_{y\in B_p^*}\ v(y) = \min_{t\in B_p}\ g_0(t) = g_0(t^0),$$

and by (3.18) we have

$$g_0(t) \geq v(y^0) = g_0(t^0) \quad \forall t \in B_p,$$

which means that t^0 is an optimal solution of the c-orthogonal program P_p. □

Corollary 3.1. *Let the assumptions of Theorem 3.5 be satisfied. Then the assertions*

$$1 = \begin{cases} y_i^0\ \dfrac{\sum\limits_{i\in[0]} c_i}{c_i}\,, & i \in [0], \qquad (3.59a) \\ \dfrac{y_i^0}{c_i \lambda_k(y^0)}\,, & i \in [k], \quad k \in J_p \qquad (3.59b) \end{cases}$$

are true if and only if

$$\prod_{j=1}^{m} t_j^{0\,a_{ij}} = 1 \quad \forall i \in [k], \quad k \in J_p. \tag{3.60}$$

Proof. Let (3.59a), (3.59b) be satisfied. Then by (3.56a), (3.56b) it follows immediately (3.60).

Since (3.60) is fulfilled by (3.53), from (3.56a) and (3.56b) we infer (3.59a) and (3.59b), respectively. □

Remark 3.3. For "classical" posynomial programs assertion (i) of Theorem 3.4 and Theorem 3.5 has the following presentations, respectively (see [10]):

$$\text{(i')} \qquad y_i^0 = \begin{cases} \dfrac{c_i \prod\limits_{j=1}^{m} t_j^{0\,a_{ij}}}{g_0(t^0)}, & i \in [0], \\ \lambda_k(y^0) c_i \prod\limits_{j=1}^{m} t_j^{0\,a_{ij}}, & i \in [k],\ k \in J_p, \end{cases}$$

$$\text{(ii')} \quad c_i \prod_{j=1}^{m} t_j^{0\,a_{ij}} = \begin{cases} y_i^0 v(y^0), & i \in [0], \\ \dfrac{y_i^0}{\lambda_k(y^0)}, & i \in [k],\ k \in J_p,\ \text{where } \lambda_k(y^0) > 0. \end{cases}$$

3.2. Examples

Example 3.1. Let be given

$$P_p : \min \left\{ g_0(t) := 2t_1^2 t_2 t_3^3 + t_1^{-4} t_2^{-2} t_3^{-6} \mid t \in B_p \right\}$$

$$B_p := \left\{ t \in \mathbf{R}^3 \mid t > 0,\ g_1(t) := \frac{1}{2} t_1 t_2^2 t_3^{-1} + \frac{1}{2} t_1^{-1} t_2^{-2} t_3 \leq 1 \right\}.$$

Since g_0, g_1 are c-orthogonal posynomials, the program P_p is c-orthogonal. Moreover, because of $\bar{t} = (1,1,1)^T \in B_p$ by Theorem 3.1 we get

$$g_0(t^0) = \sum_{i \in [0]} c_i = 3.$$

Therefore, $t^0 = \bar{t}$ is one optimal solution of P_p. To obtain all primal optimal solutions, we use Theorem 3.5. Solving the system $A^T y = 0$, $\lambda_0(y) = y_1 + y_2 = 1$, $y \geq 0$, where

$$A^T = \begin{pmatrix} 2 & -4 & 1 & -1 \\ 1 & -2 & 2 & -2 \\ 3 & -6 & -1 & 1 \end{pmatrix},$$

we get $y_1 = \dfrac{2}{3}$, $y_2 = \dfrac{1}{3}$, $y_3 = \alpha$, $y_4 = \alpha$.

Thus, the dual feasible set is

$$B_p^* := \Big\{ y \in \mathbf{R}^4 \mid y = \Big(\frac{2}{3}, \frac{1}{3}, 0, 0\Big)^T + \alpha(0,0,1,1)^T, \ \alpha \geq 0 \Big\},$$

and for each $y \in B_p^*$ we have $v(y) = 3$ which means that each $y \in B_p^*$ is a dual optimal solution.

Therefore, by Theorem 3.5, (i) we get

$$\begin{aligned} 2t_1^2 t_2 t_3^3 &= \frac{2}{3} \cdot 3 = 2, \\ t_1^{-4} t_2^{-2} t_3^{-6} &= \frac{1}{3} \cdot 3 = 1, \\ \frac{1}{2} t_1 t_2^2 t_3^{-1} &= \frac{\alpha}{2\alpha} \ = \frac{1}{2}, \\ \frac{1}{2} t_1^{-1} t_2^{-2} t_3 &= \frac{\alpha}{2\alpha} \ = \frac{1}{2}. \end{aligned} \tag{3.61}$$

Solving (3.61), we obtain the set of primal optimal solutions:

$$B_p^0 := \{ t^0 \in \mathbf{R}^3 \mid t_1^0 = \beta, \ t_2^0 = \beta^{-\frac{5}{7}}, \ t_3^0 = \beta^{-\frac{3}{7}}, \ \beta > 0 \}.$$

One can see that for $\beta = 1$ the point $\bar{t}$ is an element of B_p^0. From (3.61) it is obvious that

$$\prod_{j=1}^{m} t_j^{0\,a_{ij}} = 1, \quad \forall i \in [k], \ k \in J_p.$$

Therefore, by Corollary 3.1 we have

$$1 = y_i^0 \ \frac{\sum\limits_{i\in[0]} c_i}{c_i} = \begin{cases} \dfrac{2 \cdot 3}{3 \cdot 2}, & i = 1, \\[2mm] \dfrac{1 \cdot 3}{3 \cdot 1}, & i = 2, \end{cases}$$

$$1 = \frac{y_i^0}{c_i \lambda_k(y^0)} = \frac{y_i^0}{c_i(y_3^0 + y_4^0)} = \begin{cases} \dfrac{\alpha}{\frac{1}{2} \cdot 2\alpha}, & i = 3, \\[2mm] \dfrac{\alpha}{\frac{1}{2} \cdot 2\alpha}, & i = 4. \end{cases}$$

Example 3.2. Let be given

$$\begin{aligned} P_p \ : \quad & \min\{ g_0(t) := 3t_1^2 t_2^{-2} + t_1^6 t_2^{-6} \mid t \in B_p \}, \\ & B_p := \{ t \in \mathbf{R}^2 \mid t > 0, \ g_1(t) := t_1^{-1} + t_1 \leq 1 \}. \end{aligned}$$

Since g_0, g_1 are c-orthogonal posynomials, the program P_p is c-orthogonal. But $B_p = \emptyset$ because $(1,1)^T \notin B_p$ (Theorem 3.2).

Example 3.3

$$\min\{g_0(t) := t_1^{-1}t_2t_3 + t_1t_2^{-1}t_3^{-1} \mid t \in B_p\},$$

$$B_p := \Big\{t \in \mathbf{R}^3 \mid t > 0,\ g_1(t) := \frac{1}{2}t_1t_2^{-2}t_3 + \frac{1}{4}t_1^{-3}t_2^3t_3^{-2} + \frac{1}{4}t_1t_2 \le 1\Big\}.$$

Since g_0, g_1 are c-orthogonal posynomials, the program P_p is c-orthogonal. It is easy to see that $\bar{t} = (1,1,1)^T \in B_p$. Therefore by Theorem 3.1 we have

$$\min_{t \in B_p} g_0(t) = \sum_{i \in [0]} c_i = 2 = g_0(t^0).$$

To prove whether t^0 is unique or whether a set of primal optimal solutions exists we use Theorem 3.5. Solving the system

$$A^T y = \begin{pmatrix} -1 & 1 & 1 & -3 & 1 \\ 1 & -1 & -2 & 3 & 1 \\ 1 & -1 & 1 & -2 & 0 \end{pmatrix} (y_1, y_2, y_3, y_4, y_5)^T = 0,$$

$\lambda_0(y) = y_1 + y_2 = 1$, we get the following dual feasible set:

$$B_p^* := \Big\{y \in \mathbf{R}^5 \mid y = \Big(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\Big)^T + \alpha(0,0,2,1,1)^T,\ \alpha \ge 0\Big\}.$$

For each $y \in B_p^*$ we have $v(y) = 2$ which means that $y \in B_p^*$ is a dual optimal solution. Since

$$\begin{aligned} &y_i^0 \sum_{i \in [0]} c_i = \frac{1}{2} \cdot 2 = 1, \quad i = 1, 2, \\ &\frac{y_0^3}{c_3\lambda_1(y^0)} = \frac{2\alpha}{\frac{1}{2} \cdot 4\alpha} = 1, \\ &\frac{y_4^0}{c_4\lambda_1(y^0)} = \frac{y_5^0}{c_5\lambda_1(y^0)} = \frac{\alpha}{\frac{1}{4} \cdot 4\alpha} = 1. \end{aligned}$$

we obtain by Corollary 3.1 relation (3.60). Solving (3.60) it follows that $t^0 = (1,1,1)^T$ is the unique primal optimal solution.

3.3. Inequalities

Example 3.4. Let ABC be a triangle with the vertices A, B, C, being centres of three outside touching balls with the radii t_1, t_2, t_3, respectively (Fig. 1).

Fig. 1

Introducing the angles $\alpha_1 = \angle CAB$, $\alpha_2 = \angle ABC$ and $\alpha_3 = \angle BCA$, the inequality

$$\cot^2\frac{\alpha_1}{2}\cot^2\frac{\alpha_2}{2} + \cot^2\frac{\alpha_1}{2}\cot^2\frac{\alpha_3}{2} + \cot^2\frac{\alpha_2}{2}\cot^2\frac{\alpha_3}{3} \geq 27 = 3^3 \tag{3.62}$$

can be proved by solving the following unconstrained c-orthogonal posynomial program:

$$\min\Big\{g(t) := (t_1 + t_2 + t_3)^2(t_1^{-2} + t_2^{-2} + t_3^{-2}) \mid t \in \mathbf{R}^3,\ t > 0\Big\}.$$

By Example 2.2 it was shown that g is c-orthogonal, and applying Theorem 3.1 we get immediately

$$\min_{t>0} g(t) = g(t^0) = 27, \quad \text{where} \quad t_1^0 = t_2^0 = t_3^0. \tag{3.63}$$

Therefore, inequality (3.62) is proved if its left hand side can be identified with $g(t)$. For that purpose we write (3.62) as

$$\begin{aligned}
&\cot^2\frac{\alpha_1}{2}\cot^2\frac{\alpha_2}{2} + \cot^2\frac{\alpha_1}{2}\cot^2\frac{\alpha_3}{2} + \cot^2\frac{\alpha_2}{2}\cot^2\frac{\alpha_3}{3} \\
&\qquad = \frac{\tan^2\frac{\alpha_1}{2} + \tan^2\frac{\alpha_2}{2} + \tan^2\frac{\alpha_3}{2}}{\tan^2\frac{\alpha_1}{2}\ \tan^2\frac{\alpha_2}{2}\ \tan^2\frac{\alpha_3}{2}} \geq 3^3.
\end{aligned} \tag{3.64}$$

Using the abbreviations

$$q := t_1 + t_2,\ r := t_2 + t_3,\ s := t_1 + t_3,\ v := t_1 + t_2 + t_3,$$

one obtains by the cosine-theorem

$$\cos\alpha_1 = \frac{q^2 + s^2 - r^2}{2qs}\,;$$

together with $\cos 2\alpha_1 = 2\cos^2\alpha_1 - 1$ it follows

$$\cos^2\frac{\alpha_1}{2} = \frac{1+\cos\alpha_1}{2} = \frac{vt_1}{qs},\quad \sin^2\frac{\alpha_1}{2} = \frac{1-\sin\alpha_1}{2} = \frac{t_2t_3}{qs}\,.$$

Thus we have

(3.65)
$$\tan^2\frac{\alpha_1}{2} = \frac{t_2t_3}{vt_1} \text{ and analogously, } \tan^2\frac{\alpha_2}{2} = \frac{t_1t_3}{vt_2}\,,\ \tan^2\frac{\alpha_3}{2} = \frac{t_1t_2}{vt_3}\,.$$

Then (3.64) becomes

$$A = v^2\,\frac{t_1t_2t_3(t_1^{-2}+t_2^{-2}+t_3^{-2})}{t_1t_2t_3} = (t_1+t_2+t_3)^2(t_1^{-2}+t_2^{-2}+t_3^{-2}) = g(t).$$

The result $t_1^0 = t_2^0 = t_3^0$ in (3.63) is equivalent to $\alpha_1 = \alpha_2 = \alpha_3$ which means that the triangle is equilateral. For that case in (3.62) equality holds.

Remark 3.4. Inequality (3.62) can be found in [28], p. 183, by modifying and combining 6.23 and 6.24, or by using a comment of W. Janous in [28], p. 169, (2'), for the case $n = 0$, $p = 2$. Of course, (3.62) and thus (3.64) can be generalized to the inequality ([28], p. 169, (2'))

$$\text{(3.66)}\qquad \frac{\sum\limits_{j=1}^{3}\tan^p\dfrac{\alpha_j}{2}}{\prod\limits_{j=1}^{3}\tan^p\dfrac{\alpha_j}{2}} \geq 3^{p+1},\quad p \geq 1.$$

The proof of (3.66) can be given like that one of (3.62). Taking in (3.65)

$$\tan^p\frac{\alpha_j}{2}\,,\quad j = 1,2,3,\ p \geq 1,$$

we conclude from (3.66)

$$g(t) := (t_1 + t_2 + t_3)^p (t_1^{-p} + t_2^{-p} + t_3^{-p}) \geq 3^{p+1}. \tag{3.67}$$

Because in 2.2., Example 2.4

$$h(t) := \Big(\sum_{j=1}^{m} t_j\Big)^p \Big(\sum_{j=1}^{m} t_j^p\Big), \quad m \in \mathbf{N},$$

was shown to be a c-orthogonal posynomial, the validity of (3.67) follows immediately by Theorem 3.1.

Moreover, using Theorem 3.1 once again, it follows

$$h(t) \geq m^{p+1}, \quad m \in \mathbf{N},\ p \geq 1. \tag{3.68}$$

Inequality (3.68), rewritten, yields

$$\Big(\frac{t_1 + \cdots + t_m}{m}\Big)^p \geq \frac{m}{t_1^{-p} + \cdots + t_m^{-p}}\ , \tag{3.69}$$

which means that for any $p \geq 1$ the p-th power of the arithmetic mean of the variables $t_j > 0$, $j = 1, \dots, m$, is not less than the harmonic mean of their negative p-th powers.

Example 3.5. To prove the inequality

$$3t_1^2 + 2t_2^3 + t_3^6 \geq 6t_1t_2t_3, \quad \forall t_i > 0,\ i = 1, 2, 3, \tag{3.70}$$

or, equivalently

$$g_0(t) := 3t_1t_2^{-1}t_3^{-1} + 2t_1^{-1}t_2^2t_3^{-1} + t_1^{-1}t_2^{-1}t_3^5 \geq 6, \quad \forall t > 0, \tag{3.71}$$

one has to check whether the posynomial $g_0(t)$ is c-orthogonal. Since

$$A_{[0]}^T c_{[0]} = 0, \quad \text{where } A_{[0]}^T = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 5 \end{pmatrix},\ c_{[0]}^T = (3, 2, 1),$$

this property is fulfilled according to Definition 2.1. Therefore, by Theorem 3.1 we conclude $\min\limits_{t>0} g_0(t) = 6$, which yields (3.71) and so (3.70).

Example 3.6. To prove the well-known geometric mean-arithmetic mean inequality in its most familiar form

$$(3.72) \qquad (x_1 x_2 \dots x_m)^{\frac{1}{m}} \le \frac{1}{m}(x_1 + x_2 + \cdots + x_m), \; x_i > 0, \; m \in \mathbf{N},$$

first we set $x_j^{\frac{1}{m}} := t_j$, $m \in \mathbf{N}$. Thus, (3.72) is equivalent to the inequality

$$m \le \frac{\sum\limits_{j=1}^{m} t_j^m}{\prod\limits_{j=1}^{m} t_j} := g(t), \quad m \in \mathbf{N}.$$

Since $g(t)$ is a c-orthogonal posynomial (see Example 2.5), by Theorem 3.1 we conclude $\min\limits_{t>0} g(t) = m$.

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