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CHARACTERIZATIONS OF BANACH SPACES VIA CONVEX AND OTHER LOCALLY LIPSCHITZ FUNCTIONS

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Dedicated to Hoang Tuy on the occasion of his seventieth birthday

Abstract. Various properties of Banach spaces, including the reflexivity and the Schur property of a space, are characterized in terms of properties of corresponding classes of locally Lipschitz functions on those spaces.

1. Introduction: "Sequentially reflexive" spaces and motivating results.

We will work with real Banach spaces and let B_X and S_X denote the closed unit ball and unit sphere of the Banach space X respectively. As in [2], we will say that a Banach space is *sequentially reflexive* if Mackey and norm convergence coincide sequentially in its dual space. The following result was proved in [22]; see also [3, Theorem 5].

Theorem 1.1. *For a Banach space X, the following are equivalent.*

(a) *X does not contain an isomorphic copy of ℓ_1.*

(b) *X is sequentially reflexive.*

Let us recall that a function f is *Gateaux differentiable* at x if there is a $\Lambda \in X^*$ such that

$$\lim_{t\to 0}[f(x+th) - f(x) - \Lambda(th)]/t = 0$$

for each $h \in S_X$. If the above limit is uniform over weakly compact sets, then f is *weak Hadamard differentiable* at x; if the limit is uniform for

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$h \in B_X$, then f is *Fréchet differentiable* at x. We will denote these notions as G-differentiability, WH-differentiability and F-differentiability. In reflexive spaces, it is clear that WH-differentiability and F-differentiability are the same because closed balls are weakly compact. Surprisingly, these two notions of differentiability coincide for continuous convex functions in a much wider class of spaces as the following theorem from [3] demonstrates.

Theorem 1.2. *For a Banach space X the following are equivalent.*

(a) *X is sequentially reflexive.*

(b) *WH-differentiability and F-differentiability coincide for continuous convex functions on X.*

This result motivated us to consider the following question. In which Banach spaces do Fréchet differentiability and weak Hadamard differentiability coincide for general Lipschitz functions? The following observation adds to the intrigue of this question.

Theorem 1.3. *Suppose a sequentially reflexive space X admits a Lipschitz WH-differentiable bump function, then X is an Asplund space.*

Proof. By the smooth variation principle for bump functions (see e.g. [23, Theorem 4.10]) every continuous convex function on X has a point of weak Hadamard differentiability. According to Theorem 1.2, every continuous convex function has a point of Fréchet differentiability. Thus X is an Apslund space. □

In spite of the (misleading) evidence provided by Theorem 1.3 that WH-differentiability of Lipschitz bump functions on sequentially reflexive Banach space has structural implications similar to those of F-differentiable bump function, we will provide an elementary proof of the following result in the next section.

Theorem 1.4. *For a Banach space X the following are equivalent.*

(a) *X is reflexive.*

(b) *WH-differentiability and F-differentiability coincide for Lipschitz functions on X.*

Since we first proved Theorem 1.4, results have been established showing that WH-differentiability and F-differentiability are distinct notions on nonreflexive spaces for functions much closer to convex functions than just Lipschitz functions (see Theorem 2.1). Nonetheless, these stronger results rely on significantly deeper structural theorems in Banach space theory, and as such, the essence as to why these notions of differentiability are different is not as apparent in their proofs.

In addition to the implications of sequential reflexivity in the study of differentiability of convex functions, it also has applications to other notions of fundamental importance in convex analysis as illustrated in the following result which comes from [5] and [7].

Theorem 1.5. *For a Banach space X the following are equivalent.*

(a) *X is sequentially reflexive.*

(b) *Every continuous convex function bounded on weakly compact subsets of X is bounded on bounded subsets of X.*

(c) *If a sequence of lsc convex functions converges uniformly on weakly compact sets to a continuous affine function, then the convergence is uniform on bounded sets.*

Let us remark that there are several characterizations of Banach spaces not containing ℓ_1 in terms of functions that are weakly sequentially continuous. For a flavour of results in this direction, we are content to refer the reader to Gutiérrez's paper [16] and the references therein.

A major focus of this note is the natural question that arises from a comparison of Theorems 1.2, 1.4 and 1.5. That is, do Theorems 1.2 and 1.5 remain valid for differences of continuous convex functions? (These are variously called *difference-convex functions* or *dc-functions*.) Our aim is to provide an account of the main results of which we are aware that address this and some related questions, including those pertaining to "one-sided" or "directional differentiability". We will say that a function f is *directionally Fréchet differentiable at x* if

$$\lim_{t\downarrow 0}[f(x+th)-f(x)]/t$$

exists uniformly for h in B_X. If the above limit is uniform over weakly compact sets, then f is *weak Hadamard directionally differentiable* at x.

In many respects, this paper is a much updated version of the unpublished manuscript [4], and as such presents many of the theorems and proofs of results therein. In this revision, we have endeavoured to state many related results obtained subsequent to our original version. Except in a few instances where we have included complete proofs that are surprisingly simple, we will be content to give references to the proofs of published results.

2. Characterizations of reflexive spaces

We state the characterizations in terms of nonreflexivity. One should also note that Theorem 2.1(c) and (e) show very sharp limitations on the possibilities of extending (b) in Theorem 1.2, whereas (b) and (d) show limitations to possible extensions of (b) and (c) in Theorem 1.5.

Theorem 2.1. *For a Banach space X the following are equivalent.*

(a) *X is not reflexive.*

(b) *There is a non-increasing sequence of equi-Lipschitz norms converging uniformly on weakly compact sets to a norm, but such that the convergence is not uniform on bounded sets.*

(c) *There is a Lipschitz convex function ϕ such that ϕ is directionally weak Hadamard differentiable at some point at which it is not directionally Fréchet differentiable.*

(d) *There are continuous convex functions f and h on X such that $h - f$ is bounded on weakly compact sets, but unbounded on B_X.*

(e) *There are norms μ and ν on X such that $\mu - \nu$ is weak Hadamard but not Fréchet differentiable at some point.*

(f) *There is a Lipschitz function ϕ such that ϕ is weak Hadamard but not Fréchet differentiable at some point.*

(g) *There is a sequence $\{x_n\} \subset B_X$ and a $\delta > 0$ such that any sequence $\{y_n\}$ satisfying $\|y_n - x_n\| < \delta$ has no weakly convergent subsequence.*

Proof. The equivalence of (a) through (e) was established in [8, Theorem 1 and Corollary 5]. We will present the elementary proof of the equivalence of (a), (f) and (g) from [4].

(a) $\Rightarrow$ (g): Because X is not reflexive, by the Eberlein-Smulian theorem there is a sequence $\{x_n\} \subset B_X$ with no weakly convergent subsequence. We will suppose no subsequence of $\{x_n\}$ satisfies the property in (g) and arrive at a contradiction by producing a weakly convergent subsequence of $\{x_n\}$.

Given $\epsilon = 1$, by our supposition, we choose $N_1 \subset \mathbf{N}$ and $\{z_{1,i}\}_{i\in N_1}$ such that $\|x_i - z_{1,i}\| < 1$ for $i \in N_1$ and $w\text{-}\lim_{i\in N_1} z_{1,i} = z_1$. Supposing N_{k-1} has been chosen, we choose $N_k \subset N_{k-1}$ and $\{z_{k,i}\}_{i\in N_k} \subset X$ satisfying

$$\|x_i - z_{k,i}\| < \frac{1}{k} \quad \text{for} \quad i \in N_k \quad \text{and} \quad w\text{-}\lim_{i\in N_k} z_{k,i} = z_k. \tag{2.1}$$

In this manner we construct $\{z_{k,i}\}_{i\in N_k}$ and N_k for all $k \in \mathbf{N}$.

Notice that $z_n - z_m = w\text{-}\lim_{i\in N_n}(z_{n,i} - z_{m,i})$ for $n > m$. Thus by the w-lower semicontinuity of $\|\cdot\|$ and (2.1) we obtain

$$\begin{aligned}\|z_n - z_m\| &\le \liminf_{i\in N_n}\|z_{n,i} - z_{m,i}\| \le \liminf_{i\in N_n}(\|z_{n,i} - x_i\| + \|x_i - z_{m,i}\|)\\ &\le \frac{1}{n} + \frac{1}{m} \le \frac{2}{m}.\end{aligned}$$

Thus z_n converges in norm to some $z_\infty \in X$.

Now for each $n \in \mathbf{N}$ choose integers $i_n \in N_n$ with $i_n > n$. We will show $x_{i_n} \xrightarrow{w} z_\infty$. So let $\Lambda \in B_{X^*}$ and $\epsilon > 0$ be given. We select an $n_0 \in \mathbf{N}$ which satisfies

$$\frac{1}{n_0} < \frac{\epsilon}{3} \quad \text{and} \quad \|z_m - z_\infty\| < \frac{\epsilon}{3} \quad \text{for} \quad m \ge n_0. \tag{2.2}$$

Because $z_{n_0,i} \xrightarrow{w} z_{n_0}$, we can select m_0 so that

$$\big|\langle\Lambda, z_{n_0,i} - z_{n_0}\rangle\big| < \frac{\epsilon}{3} \quad \text{for all} \quad i \ge m_0. \tag{2.3}$$

For $m \ge \max\{n_0, m_0\}$, we have

$$\begin{aligned}\big|\langle\Lambda, x_{i_m} - z_\infty\rangle\big| &\le \big|\langle\Lambda, x_{i_m} - z_{n_0,i_m}\rangle\big| + \big|\langle\Lambda, z_{n_0,i_m} - z_{n_0}\rangle\big| + \big|\langle\Lambda, z_{n_0} - z_\infty\rangle\big|\\ &< \|x_{i_m} - z_{n_0,i_m}\| + \frac{\epsilon}{3} + \|z_{n_0} - z_\infty\|\\ &\qquad\qquad [\text{by (2.3) since } i_m > m \ge m_0]\\ &< \frac{1}{n_0} + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon. \qquad [\text{by (2.2) and (2.1)}]\end{aligned}$$

Therefore $x_{i_n} \xrightarrow{w} z_\infty$ contradicting the fact that $\{x_n\}$ has no weakly convergent subsequence.

(g) $\Rightarrow$ (f): By (g) we choose $\{x_n\} \subset B_X$, and a $\Delta \in (0,1)$ such that $\{z_n\} \subset X$ has no weakly convergent subsequence whenever $\|z_n - x_n\| < \Delta$. By passing to another subsequence if necessary we may assume $\|x_n - x_m\| > \delta$ for all $n \ne m$, with some $0 < \delta < 1$.

For $n = 1, 2, \ldots$, let $B_n = \{x \in X : \|x - 4^{-n}x_n\| \le \delta\Delta 4^{-n-1}\}$ and put $C = X \setminus \cup_{n=1}^\infty B_n$. Because $4^{-m} + \delta\Delta 4^{-m-1} < 4^{-n} - \delta\Delta 4^{-n-1}$ for $m > n$, we have that $B_n \cap B_m = \emptyset$ whenever $n \ne m$. For $x \in X$, let $f(x)$ be the distance of x from C. Thus f is a Lipschitz function on X with $f(0) = 0$. We will check that f is WH-differentiable at 0 but not F-differentiable at 0.

Let us now observe that f is G-differentiable at 0. Fix any $h \in X$ with $\|h\| = 1$. Then $[0, +\infty)h$ meets at most one ball B_n. In fact assume $t_m, t_n > 0$ are such that $\|t_i h - 4^{-i} x_i\| < \delta\Delta 4^{-i-1}$ for $i = n, m$. Then $|4^i t_i - 1| < \frac{\delta\Delta}{4}$ for $i = n, m$ and

$$\begin{aligned}\|x_n - x_m\| &\leq \|x_n - 4^n t_n h\| + \|4^n t_n h - 4^m t_m h\| + \|4^m t_m h - x_m\| \\ &< \frac{\delta\Delta}{4} + \frac{2\delta\Delta}{4} + \frac{\delta\Delta}{4} = \delta\Delta < \delta.\end{aligned}$$

Because $\|x_n - x_m\| \geq \delta$ for $m \neq n$, this shows that $n = m$. It thus follows that for $t > 0$ small enough, we have $f(th) = 0$. Therefore f is G-differentiable at 0, with $f'(0) = 0$. Let us further check that f is not F-differentiable at 0. Indeed, $\dfrac{f(4^{-n}x_n)}{\|4^{-n}x_n\|} = \dfrac{\delta\Delta}{4}$ for all n while $\|4^{-n}x_n\| \to 0$.

Finally, assume that f is not WH-differentiable at 0. Then there are a weakly compact set $K \subset B_X, \epsilon > 0$, and sequences $\{k_m\} \subset K, t_m \downarrow 0$ such that $\dfrac{f(t_m k_m)}{t_m} > \epsilon$ for all $m \in \mathbf{N}$. Hence, as f is 1-Lipschitz, we have $\inf \|k_n\| \geq \epsilon > 0$. Further, because $f(t_m k_m) > 0$, there are $n_m \in \mathbf{N}$ such that $\|t_m k_m - 4^{-n_m} x_{n_m}\| < \Delta\delta 4^{-m_n - 1}$, $m = 1, 2, \dots$. Consequently,

$$\|4^{n_m} t_m k_m - x_{n_m}\| < \frac{\Delta\delta}{4} < \Delta \quad \text{and} \quad |4^{n_m} t_m \|k_m\| - 1| < \frac{\Delta\delta}{4}. \tag{2.4}$$

By our initial selection of $\{x_n\}$ and Δ, the first inequality in (2.4) says that $\{4^{n_m} t_m k_m\}$ does not have a weakly convergent subsequence. However the second inequality in (2.4) together with $\inf \|k_n\| > 0$ ensures that $4^{n_m} t_m$ is bounded. Now because $\{k_m\}$ is relatively weakly compact, $4^{n_m} t_m k_m$ has a weakly convergent subsequence, a contradiction. This proves f is WH-differentiable at 0.

(f) $\Rightarrow$ (a): This is clear since the unit ball cannot be weakly compact. □

One should notice that it is also easy to derive (g) from (f) in the above theorem. Indeed, because f is not Fréchet differentiable, we can choose $\{x_k\}$ in the unit sphere S_X of X and $t_k \downarrow 0$ which satisfy

$$\frac{|f(t_k x_k) - f(0)|}{t_k} \geq \epsilon \quad \text{for some} \quad \epsilon > 0.$$

Using the fact that f is Lipschitz and WH-differentiable at 0, one can easily show that $\{x_k\}$ satisfies (g).

We should also remark that the sequence as given in (g) has been used to characterize reflexivity in terms of comparing two forms of set convergences of bounded weakly closed sets (see [6, Theorem 4.2(b)]). For a thorough account on set convergences and their applications, we recommend Beer's monograph [1].

Unlike the proof presented above that (f) holds in nonreflexive spaces, the proof given in [8] that (b) through (e) are valid in nonreflexive spaces uses a decidedly nonelementary result. Indeed, it relies on a result from [17] that uses Rosenthal's ℓ_1 theorem and has the Josefson-Nissenzweig theorem as a direct consequence. The following remark justifies the use of the Josefson-Nissenzweig theorem in the proof that (d) occurs in nonreflexive spaces.

Remark 2.2. The condition in Theorem 2.1(d) directly implies the existence of a sequence in S_{X^*} that converges weak* to 0.

Proof. Since each continuous convex function is bounded below on B_X, we know that one of the functions, say h, is unbounded above on B_X and so the remark follows from [5, Lemma 2.3]. We include a very simple proof of this. Let $F_n = \{x : h(x) \leq n\}$ and choose $x_n \in B_X$ such that $h(x_n) > n$. Using the separation theorem we choose $\phi_n \in S_{X^*}$ such that $\phi_n(x_n) \geq \sup_{F_n} \phi_n$. If ϕ_n did not converge weak* to 0 we could choose $\bar{x} \in X$ such that $\phi_n(\bar{x}) > 1$ for infinitely many n. However, $\phi_n(x_n) \leq 1$, and so $\bar{x} \notin F_n$ for each n which means $h(\bar{x}) > n$ for each n, a contradiction. $\square$

Sadly, we do not know if the Josefson-Nissensweig theorem can likewise be directly derived from Theorem 2.1(b),(c) or (e).

Notice that Theorem 2.1 illustrates that, with many properties, differences of convex functions have more in common with Lipschitz or locally Lipschitz functions than they do with convex functions. Relatedly, however, it is not clear that there are any non superreflexive Banach spaces on which arbitrary Lipschitz functions can be approximated uniformly on bounded sets by differences of convex functions.

While we have shown that WH-differentiability and F-differentiability are distinct notions for Lipschitz functions (and hence for continuous functions), the implications of the existence of WH-differentiable bump functions on sequentially reflexive Banach spaces have not been completely clarified. In particular, we mention the following questions arising from Theorem 1.3 (we refer the reader to the monograph [9] for an exposition on Banach spaces admitting smooth bump functions).

Question 2.3. (a) *If X is sequentially reflexive and admits a Lipschitz WH-differentiable bump function, does X admit a Lipschitz F-differentiable bump function?*

(b) *If X is sequentially reflexive and admits a continuous WH-differentiable bump function, is X an Asplund space?*

The existence of norms that are directionally Fréchet differentiable has also received considerable attention; see for example [13], [15] and the references therein. The following question, which is somewhat related to Question 2.3(b), was posed by Giles and Sciffer in [14], where it was first shown that directional WH-differentiability need not imply directional Fréchet differentiability for convex functions on c_0.

Question 2.4. If the norm on X is directionally WH-differentiable and $X \not\supset \ell_1$, is X an Asplund space?

Concerning this question, let us note that [15] shows any space with a directionally F-differentiable norm is an Asplund space. However, Theorem 2.1 shows that directional WH-differentiability need not imply directional F-differentiability for convex functions on spaces that do not contain ℓ_1. For some further questions related to directional differentiability, we refer the reader to [15].

3. Spaces in whose duals weak* and Mackey convergence coincide sequentially

If X is such that weak* and Mackey convergence coincide sequentially in X^*, we will say X has the *DP* property.* Our interest in this property stems from the following theorem.

Theorem 3.1. *Let X be a Banach space. Then the following are equivalent.*

(a) *X has the DP* property.*

(b) *WH-differentiability and G-differentiability coincide for all lsc convex functions on X.*

(c) *Every continuous convex function is bounded on weakly compact subsets of X.*

(d) *A sequence of lsc convex functions $\{f_n\}$ converges uniformly to a continuous affine function ϕ on weakly compact sets provided $\{f_n\}$ converges pointwise to ϕ on X.*

Proof. The equivalence of (a) and (b) was established in [3], while the equivalence of (a) and (c) is shown in [5]. The techniques of [5] were later used in [7] to show that (a) implies (d) (while (d) implies (a) is completely clear). □

Let us recall some standard definitions. A Banach space is said to have the *Schur property* if weakly convergent sequences are norm convergence (equivalently weakly compact sets are norm compact). A Banach space X is said to have the *Grothendieck property* if weak* and weak convergence agree sequentially in X^*. A Banach space X has the Dunford-Pettis property if $x_n^*(x_n) \to 0$ whenever $x_n^* \xrightarrow{w^*} 0$ and $x_n \xrightarrow{w} 0$. Notice that the DP* property differs from the Dunford-Pettis property only in that one merely requires $x_n^* \xrightarrow{w^*} 0$ rather than $x_n^* \xrightarrow{w} 0$ (and hence the notation 'DP* property').

Remark 3.2. (a) Every Banach space with the Schur property has the DP* property.

(b) If X has the Dunford-Pettis property and the Grothendieck property, then X has the DP* property.

(c) ℓ_∞ is a space with the DP* property that does not have the Schur property.

Proof. Both (a) and (b) follow directly from the definitions involved. For (c) recall that ℓ_∞ has both the DP property (see [10, p. 103] and Grothendieck property (see [1, p. 177]) but does not have the Schur property. □

It is natural to ask when the Schur and DP* properties are equivalent. The next result and its corollaries will show that they are equivalent for relatively small spaces. First let us recall that a linear operator is said to be *completely continuous* if it maps weakly convergent sequences to norm convergent sequences.

Theorem 3.3. *For a Banach space X, the following are equivalent.*

(a) *X has the DP* property.*

(b) *If B_{Y^*} is weak*-sequentially compact, then each continuous linear $T : X \to Y$ is completely continuous.*

Proof. (a) ⇒ (b): We will prove this by contraposition. Suppose (b) fails, that is, there is an operator $T : X \to Y$ which is not completely continuous for some Y with B_{Y^*} weak*-sequentially compact. Hence we choose $\{x_n\} \subset X$ such that $x_n \xrightarrow{w} 0$ but $\|Tx_n\| \not\to 0$. Because $Tx_n \xrightarrow{w} 0$, we know that $\{Tx_n\}$ is not relatively norm compact. Hence letting $E_n = \text{span}\,\{y_k : k \leq n\}$ with $y_k = Tx_k$ we know there is an $\epsilon > 0$ such that $\sup_k d(y_k, E_n) > \epsilon$ for each n.

By passing to a subsequence, if necessary, we assume $d(y_n, E_{n-1}) > \epsilon$ for each n. Now choose $\Lambda_n \in B_{Y^*}$ such that $\langle \Lambda_n, x \rangle = 0$ for all $x \in E_{n-1}$ and $\langle \Lambda_n, x_n \rangle \geq \epsilon$. Because B_{Y^*} is weak$*$-sequentially compact, there is a subsequence Λ_{n_k} such that $\Lambda_{n_k} \xrightarrow{w^*} \Lambda \in B_{Y^*}$. Observe that $\langle \Lambda_n, y_k \rangle = 0$ for $n > k$ and consequently $\langle \Lambda, y_k \rangle = 0$ for all k. Now let $z_k^* = T^*(\Lambda_{n_k} - \Lambda)$ and $z_k = x_{n_k}$. Certainly $z_k^* \xrightarrow{w^*} 0$ and $z_k \xrightarrow{w} 0$ while $\langle z_k^*, z_k \rangle = \langle \Lambda_{n_k} - \Lambda, Tx_{n_k} \rangle = \langle \Lambda_{n_k} - \Lambda, y_{n_k} \rangle \geq \epsilon$ for all k. This shows that X fails the DP^* property.

(b) $\Rightarrow$ (a): This follows, for example, from [3] which shows that weak Hadamard and Gateaux differentiability agree for convex functions on X if each operator $T : X \to c_0$ is completely continuous; and hence X has the DP* property in this case. $\square$

Corollary 3.4. *If X has a weak*-sequentially compact dual ball or, more generally, if every separable subspace of X is a subspace of a complemented subspace with weak*-sequentially compact dual ball, then the following are equivalent.*

(a) *X has the DP* property.*

(b) *X has the Schur property.*

Proof. Because (b) $\Rightarrow$ (a) is true more generally, we show (a) $\Rightarrow$ (b). If B_{X^*} is weak*-sequentially compact and X is not Schur, then $I : X \to X$ is not completely continuous and Theorem 3.3 applies. More generally, suppose $x_n \xrightarrow{w} 0$ but $\|x_n\| \not\to 0$ and $\overline{\text{span}}\{x_n\} \subset Y$ with B_{Y^*} weak*-sequentially compact. Supposing there is a projection $P : X \to Y$, then P is not completely continuous since $P|_Y$ is the identity on Y. $\square$

We can say more in the case that X is weakly countably determined (WCD); see [21] and [9, Chapter VI] for the definition and further properties of WCD spaces.

Corollary 3.5. *For a Banach space X, the following are equivalent.*

(a) *X is WCD and has the DP* property.*

(b) *X is separable and has the Schur property.*

Proof. It is obvious that (b) $\Rightarrow$ (a), so we prove (a) $\Rightarrow$ (b). First, since B_{X^*} is weak*-sequentially compact (see e.g. [21, Corollary 4.9] and [19, Theorem 11], it follows from Corollary 3.4 that X has the Schur property. But WCD Schur spaces are separable (see e.g. [21, Theorem 4.3]). $\square$

Remark 3.6. The following items contain facts concerning the DP* property.

(a) Corollary 3.4 is satisfied, for instance, by Gateaux differentiability spaces see [19, 23] for definition and [19, Theorem 11] for a proof that such spaces have weak* sequentially compact dual balls. A wide class of spaces for which every separable subspace is a subspace of a complemented separable subspace is the class of spaces with countably norming M-basis; see [24, Lemma 1], see also [26] and the references therein for further properties and definition of M-bases. Notice that $\ell_1(\Gamma)$ has a countably norming M-basis for any Γ, thus spaces with countably norming M-bases and the Schur property need not be separable.

(b) Let X be a space such that X has the Schur property but X^* does not have the Dunford-Pettis property (cf. [11, p. 178]). Then X has the DP* property, but X^* does not have the DP* property.

(c) There are spaces with the DP^* property that have neither the Schur nor Grothendieck properties; for example $\ell_1 \times \ell_\infty$.

(d) It is well-known that ℓ_∞ has ℓ_2 as a quotient (see [20, p. 111]). Thus quotients of spaces with the DP^* need not have the DP^*. It is clear that superspaces of spaces with the DP^* need not have the DP^*; the example $c_0 \subset \ell_\infty$ shows that subspaces need not inherit the DP^*.

(e) In [18], Haydon has constructed a nonreflexive Grothendieck $C(K)$ space that does not contain ℓ_∞. Using the continuum hypothesis, Talagrand constructed a nonreflexive Grothendieck $C(K)$ space X such that ℓ_∞ is neither a subspace nor a quotient of X (see [25]). Since $C(K)$ spaces have the Dunford-Pettis property (see [10, p. 113]), both these spaces have the DP^*.

Given the previous corollaries and remark, it seems natural to ask what natural (widely studied) properties of Banach spaces in conjunction with the DP* property imply that the Banach space has the Schur property. For example, as far as we know, the answer to the following question is not known.

Question 3.7. If a Banach space has an M-basis and the DP property, does it have the Schur property?*

We close this section with a curious example concering the extension of convex functions where smoothness cannot be preserved.

Example 3.8. Let X be a space with the Grothendieck and Dunford-Pettis properties such that X does not have the Schur property (e.g. ℓ_∞). Then there is a separable subspace Y (e.g. c_0) of X and a continuous convex function f on Y such that f is G-differentiable at 0 (as a function on Y), but no continuous convex extension of f to X is G-differentiable at 0 (as a function on X); there also exist $y_0 \in Y\backslash\{0\}$ and an equivalent norm $\|\cdot\|$ on Y whose dual norm is strictly convex but no extension of $\|\cdot\|$ to X is G-differentiable at y_0.

Proof. Let Y be a separable non-Schur subspace of X. By Corollary 3.4, there is a continuous convex function f on Y which is G-differentiable at 0, but is not WH-differentiable at 0. Since any extension $\tilde{f}$ of f also fails to be WH-differentiable at 0, it follows that $\tilde{f}$ is not G-differentiable at 0 because X has the DP^*. Because Y fails the DP^* property, there is a sequence $\{\Lambda_n\} \subset X^*$ such that Λ_n converges w^* but not Mackey to 0. By the proof of [3, Theorem 3], there is a norm $\|\cdot\|$ on Y whose dual is strictly convex that fails to be WH-differentiable at some $y_0 \in Y\backslash\{0\}$; as above, no extension of $\|\cdot\|$ to X can be G-differentiable at y_0. $\square$

For additional results concerning the extension of smooth norms the reader may wish to consult [27]; see also [9].

4. Characterizations of spaces with Schur or Dunford-Pettis properties

The following theorem provides some characterizations of Banach spaces with the Schur property.

Theorem 4.1. *For a Banach space X, the following are equivalent.*

(a) *X has the Schur property.*

(b) *G-differentiability and F-differentiability coincide for w^*-lsc continuous convex functions on X^*.*

(c) *G-differentiability and F-differentiability coincide for dual norms on X^*.*

(d) *Each continuous weak*-lsc convex function on X^* is bounded on bounded subsets of X^*.*

(e) *G-differentiability and WH-differentiability agree for Lipschitz func-*

tions on X.

(f) *G-differentiability and WH-differentiability coincide for differences of Lipschitz convex functions.*

(g) *Every continuous convex function on X is weak Hadamard directionally differentiable.*

Proof. Let us outline the equivalence of (a) and (e) first. Clearly (a) implies (e), so we will show (e) implies (a).

(e) $\Rightarrow$ (a): We suppose X is not Schur and proceed by contraposition. Let $\{x_n\} \subset S_X$ be such that $x_n \xrightarrow{w} 0$. Since $\{x_n\}$ is not relatively norm compact, we may assume by passing to a subsequence if necessary that $\|x_i - x_j\| > \delta$ for some $\delta \in (0,1)$ whenever $i \neq j$.

As in the proof of Theorem 2.1, let $B_n = \{x \in X : \|x - 4^{-n}x_n\| \leq \delta 4^{-n-1}\}$, $C = X \backslash \cup_{n=1}^{\infty} B_n$ and let $f(x) = d(x, C)$. Now $f(0) = 0$ and the argument of Theorem 2.1, shows that f is G-differentiable at 0 with $f'(0) = 0$. However,

$$\frac{f(4^{-n}x_n)}{4^{-n}} = \frac{\delta}{4} \quad \text{for all} \quad n \in \mathbf{N}.$$

Since $\{x_n\} \cup \{0\}$ is weakly compact, it follows that f is not WH-differentiable at 0.

(a) $\Leftrightarrow$ (d): This was outlined in the remark in [5, p. 67]; see also Remark 2.2 and Theorem 4.2.

(a) $\Rightarrow$ (b): Suppose (b) does not hold. Then for some continuous convex weak*-*lsc* f on X^*, there exists $\Lambda_0 \in X^*$ such that f is G-differentiable at Λ_0 but f is not F-differentiable at Λ_0. Let $f'(\Lambda_0) = x^{**} \in X^{**}$. We also choose $\delta > 0$ and $K > 0$ such that for $x_1^*, x_2^* \in B(\Lambda_0, \delta)$ we have $|f(x_1^*) - f(x_2^*)| \leq K\|x_1^* - x_2^*\|$ (since f is locally Lipschitz). Because f is not F-differentiable at Λ_0, there exist $t_n \downarrow 0, t_n < \frac{\delta}{2}, \Lambda_n \in S_{X^*}$ and $\epsilon > 0$ such that

$$f(\Lambda_0 + t_n\Lambda_n) - f(\Lambda_0) - \langle x^{**}, t_n\Lambda_n\rangle \geq \epsilon t_n. \tag{4.1}$$

Because f is convex and weak*-*lsc*, using the separation theorem we can choose $x_n \in X$ satisfying

$$\langle x_n, x^*\rangle \leq f(\Lambda_0 + t_n\Lambda_n + x^*) - f(\Lambda_0 + t_n\Lambda_n) + \frac{\epsilon t_n}{2} \quad \text{for all} \quad x^* \in X^*; \tag{4.2}$$

Putting $x^* = -t_n\Lambda_n$ in (4.2) and using (4.1) one obtains

$$\begin{aligned}\langle x_n, t_n\Lambda_n\rangle &\geq f(\Lambda_0 + t_n\Lambda_n) - f(\Lambda_0) - \frac{\epsilon t_n}{2}\\ &\geq \langle x^{**}, t_n\Lambda_n\rangle + \frac{\epsilon t_n}{2}\cdot\end{aligned}$$

And hence, $\|x_n - x^{**}\| \geq \frac{\epsilon}{2}$ for all n.

Let $\eta > 0$ and fix $x^* \in S_{X^*}$. Since f is G-differentiable at Λ_0, there is a $0 < t_0 < \frac{\delta}{2}$ such that for $|t| \leq t_0$ we have

$$\langle x^{**}, tx^*\rangle - f(\Lambda_0 + tx^*) + f(\Lambda_0) \geq -\frac{\eta}{2}t_0. \tag{4.3}$$

Using (4.2) with the fact that f has Lipschitz constant K on $B(\Lambda_0, \delta)$, for $|t| \leq t_0$ we obtain

$$\begin{aligned}\langle x_n, tx^*\rangle &\leq f(\Lambda_0 + t_n\Lambda_n + tx^*) - f(\Lambda_0 + t_n\Lambda_n) + \frac{\epsilon t_n}{2}\\ &\leq f(\Lambda_0 + tx^*) - f(\Lambda_0) + \frac{\epsilon t_n}{2} + 2Kt_n.\end{aligned}$$

Choosing n_0 so large that $\dfrac{\epsilon t_n}{2} + 2Kt_n < \dfrac{\eta}{2}t_0$ for $n \geq n_0$, the above inequality yields

$$f(\Lambda_0 + tx^*) - f(\Lambda_0) - \langle x_n, tx^*\rangle \geq -\frac{\eta}{2}t_0 \quad \text{for} \quad n \geq n_0,\ |t| \leq t_0. \tag{4.4}$$

Adding (4.3) and (4.4) results in

$$\langle x^{**} - x_n, tx^*\rangle \geq -\eta t_0 \quad \text{for} \quad n \geq n_0,\ |t| \leq t_0.$$

Hence $|\langle x^{**} - x_n, x^*\rangle| \leq \eta$ for $n \geq n_0$. This shows that $x_n \xrightarrow{w^*} x^{**}$. Combining this with the fact that $\|x_n - x^{**}\| \not\to 0$ shown above, we conclude that for some $\delta > 0$ and some subsequence we have $\|x_{n_i} - x_{n_{i+1}}\| > \delta$ for all i. However $x_{n_i} - x_{n_{i+1}} \xrightarrow{w} 0$ (in X) because $x_{n_i} - x_{n_{i+1}} \xrightarrow{w^*} 0$ (in X^{**}). This shows that X is not Schur.

Since (b) $\Rightarrow$ (c) is obvious, so we show that (c) $\Rightarrow$ (a). Write $X = Y \times \mathbf{R}$ and suppose that X is not Schur. Then we can choose $\{y_n\} \subset Y$ such that $y_n \xrightarrow{w} 0$ but $\|y_n\| = 1$ for all n. Let $\{\gamma_n\} \subset (\frac{1}{2}, 1)$ be such that $\gamma_n \uparrow 1$ and define $|||\cdot|||$ on $X^* = Y^* \times \mathbf{R}$ by

$$|||(\Lambda, t)||| = \sup\big\{|\langle \Lambda, y_n\rangle + \gamma_n t|\big\} \vee \frac{1}{2}(\|\Lambda\| + |t|).$$

This norm is dual since it is a supremum of weak*-*lsc* functions and the proof of [3, Theorem 1] shows that $|||\cdot|||$ is Gateaux but not Fréchet differentiable at $(0,1)$.

Finally, (a) implies (g) follows from Dini's monotone convergence theorem, while (a) implies (f) is clear from the definitions; [8, Proposition 8] shows that both (f) and (g) fail in spaces without the Schur property. $\square$

The Schur property can also be characterized by comparing pointwise and uniform convergence of equi-Lipschitz functions; see [7, Proposition 3.5].

Our final theorem provides a characterization of the Dunford-Pettis property via functions on the dual space.

Theorem 4.2. *For a Banach space X, the following are equivalent.*

(a) *X has the Dunford-Pettis property.*

(b) *G-differentiability and WH-differentiability coincide for real-valued w^*-lsc convex functions on X^*.*

(c) *G-differentiability and WH-differentiability coincide for dual norms on X^*.*

(d) *Each continuous weak*-lsc convex function on X^* is bounded on weakly compact subsets of X^*.*

Proof. The equivalence of (a), (b) and (c) can be proved using results from [3] and Smulian's test in a fashion similar to Theorem 4.1.

(a) $\Rightarrow$ (d): Although this is outlined in [5, p. 67], we will present a simpler proof which does not rely on a weak* variation of the technical lemma [5, Lemma 2.6].

Suppose there is a continuous weak*-lsc function f that is unbounded on a weakly compact set W. Choose $x_n^* \in W$ such that $f(x_n^*) > n$ and let $F_n := \{x^* \in X^* : f(x^*) \leq n\}$. Because F_n is weak* closed and convex, we use the separation theorem to choose $x_n \in X$ such that

$$x_n(x_n^*) \geq \sup_{F_n} \phi_n. \tag{4.5}$$

As in Remark 2.2, it follows that $x_n \xrightarrow{w^*} 0$ in X^{**} and hence $x_n \xrightarrow{w} 0$ in X. Because W is weakly compact, by passing to a subsequence we may assume x_n^* converges weakly. Now the Dunford-Pettis property implies $x_n^*(x_n) \to 0$. However, this cannot occur because (4.5) and the fact that there is a neighborhood V of 0 in X^* and $K > 0$ such that $f(x^*) \leq K$ for all $x^* \in V$.

(d) ⇒ (a): This follows from [5, Lemma 2.1]. □

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