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A NOTE ON THE HILBERT-SAMUEL FUNCTION IN A TWO-DIMENSIONAL LOCAL RING

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1. Introdution

Let $(A, \mathbf{m})$ be a local Cohen-Macaulay ring of dimension $d > 0$ and I an $\mathbf{m}$-primary ideal. We assume throughout the paper that $A/\mathbf{m}$ is an infinite field. If we denote the Hilbert-Samuel function $\lambda(A/I^n)$ by $H_I(n)$ and the corresponding polynomial by $P_I(n)$, then $P_I(n)$ can be written in the form:

$$P_I(n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I),$$

where $e_i(I) \in \mathbf{Z}$ and $e_0(I) > 0$ is the multiplicity of I. We will omit I in the notation if there is no confusion.

There are some classes of ideals I for which suitable relations between some coefficients e_i and $\lambda(A/I)$ determine the whole Hilbert-Samuel function and force the associated graded ring $G(I) = A/I \oplus I/I^2 \oplus \cdots$ of I to have good properties. A classical example is that if $e_0 = 1$, then $e_i = 0$ for $i > 0$, $I = \mathbf{m}$, and A is a regular local ring. Inspired by Kubota [Ku], Huneke [Hu] and, independently, Ooishi [O] showed that if $\lambda(A/I) = e_0 - e_1$, then $G(I)$ is a Cohen-Macaulay ring, $e_i = 0$ for $i > 1$ and $H_I(n) = P_I(n)$ for all $n \geq 1$. Recently, Sally [Sa1, Sa2] showed that if $\lambda(A/I) = e_0 - e_1 + e_2$ and $e_2 \leq 2$, then the reduction number $r(I)$ of I is less than or equal to 2, $\operatorname{depth} G(I) \geq d-1$, and $H_I(n) = P_I(n)$ for all $n \geq 1$, too. In fact, the relation $\lambda(A/I) = e_0 - e_1$ (resp. $\lambda(A/I) = e_0 - e_1 + e_2$) can be written in the form $H_I(1) = P_I(1)$ if $d = 1$ (resp. $d = 2$). Their proofs are based on the case $d = 1$ and $d = 2$, respectively. Another result by Huneke [Hu, Theorem 2.11] says that if $d = 2$ and I agrees with its Ratliff-Rush closure $\tilde{I}$, then $H_I(n) = P_I(n)$ for all $n \geq 1$ is equivalent to $\operatorname{depth} G(I) \geq 1$ and $r(I) \leq 2$. From these phenomena we think that the following question is of interest:

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Let $d = 2$ and assume that $H_I(n) = P_I(n)$ for all $n = 1, \dots, n_0$, where n_0 is a given positive integer. When is $H_I(n) = P_I(n)$ for all $n \geq 1$?

In this note we give some partial results to this question in the case $I = \tilde{I}$. Then we can prove that the Hilbert-Samuel function and Hilbert-Samuel polynomial agree for all $n \geq 1$ if they agree at $n = 1$, 2 (Theorem 3.3). If we only assume $H_I(1) = P_I(1)$, then in the case $e_2 = 3$ we can show that $H_I(n) = P_I(n)$ for all $n \geq 3$ and $r(I) \leq 3$ (Proposition 3.6). Since we could not find any counterexample, we think that in this case $H_I(2) = P_I(2)$, too. As the main tool we use the local cohomology module theory, a part of which was developed in [Sa2].

2. Background

In this section we recall some basic facts and introduce some notations.

A Noetherian graded ring $S = \oplus_{n \geq 0} S_n$ is called standard if S is generated by S_1 over S_0. Set $S_+ = \oplus_{n>0} S_n$. If S_0 is an artinian local ring, we denote the Hilbert function giving the length $\lambda(S_n) = \lambda_{S_0}(S_n)$ by $h_S(n)$ and the corresponding Hilbert polynomial by $p_S(n)$. Then we have the following useful formula given by Serre:

Lemma 2.1 $\quad p_S(n) - h_S(n) = \sum_{i \geq 0} (-1)^{i+1} \lambda(H^i_{S_+}(S)_n).$

We set

$$a_i(S) = \max\{n \in \mathbf{Z}; \quad H^i_{S_+}(S)_n \neq 0\},$$

where $a_i(S) := -\infty$ if $H^i_{S_+}(S) = 0$. Recall that a homogeneous element z of a graded ring S is said to be *filter-regular* if $[0 : z]_n = 0$ for all $n \gg 0$. The proof of the following result is a modification of that of [Na], Proposition 4.2 and Lemma 4.3.

Lemma 2.2. *Let S be a standard graded ring over an artinian local ring S_0.*

(i) *If* $\dim S = 1$, *then for all* $n \geq 0$,

$$\lambda(H^1_{S_+}(S)_n) \leq \max\{0, \lambda(H^1_{S_+}(S)_{n-1}) - 1\}.$$

(ii) *If* $\dim S = 2$, *then for every filter-regular element* $z \in S_1$,

$$\lambda(H^1_{S_+}(S)_{n+1}) \leq \max\{0, \lambda(H^1_{S_+}(S)_n) - 1\} \quad \text{if} \quad n \geq 1 + a_1(S/zS),$$
$$\lambda(H^1_{S_+}(S)_{n+1}) \leq \lambda(H^1_{S_+}(S)_n) \quad \text{if} \quad n = a_1(S/zS).$$

Proof. (i) Let $z \in S_1$ be a filter-regular element of S. Set $R = S/zS$. Note that if for some $i > 0$, $R_i = 0$, then $R_j = 0$ for all $j > i$. Moreover, without loss of generality, we may assume that $H^0_{S_+}(S) = 0$. Hence (i) immediately follows from the following exact sequence:

$$0 \to H^0_{S_+}(R)_n \cong R_n \to H^1_{S_+}(S)_{n-1} \xrightarrow{\cdot z} H^1_{S_+}(S)_n \to 0.$$

(ii) Set $S' = S/H^0_{S_+}(S)$. Then $H^1_{S_+}(S) \cong H^1_{S'_+}(S')$. If $z \in S_1$ is a filter-regular element of S, then its image z' in S' is also a filter-regular element of S'. Again set $R = S/zS$. From the exact sequence

$$\frac{zS + H^0_{S_+}(S)}{zS} \to R \to S'/z'S' \cong \frac{S}{H^0_{S_+}(S) + zS} \to 0,$$

it follows that $H^1_{S'_+}(S'/zS') \cong H^1_{S_+}(R)$. This means, replacing S by S' we may assume that $H^0_{S_+}(S) = 0$. Then for $n \geq a_1(R)$ we have the exact sequence

$$(1) \qquad 0 \to H^0_{S_+}(R)_{n+1} \to H^1_{S_+}(S)_n \to H^1_{S_+}(S)_{n+1} \to 0.$$

Let $w \in R_1$ be a filter-regular element of R. Consider the exact sequence

$$H^0_{R_+}(R)_{n-1} \xrightarrow{\cdot w} H^0_{R_+}(R)_n \to (R/zR)_n \to H^1_{R_+}(R)_{n-1} \xrightarrow{\cdot z} H^1_{R_+}(R)_n \to 0.$$

Since $1+a_1(R) \geq 0$, if $H^0_{R_+}(R)_i = 0$ and $i \geq 2+a_1(R)$, then $H^0_{R_+}(R)_j = 0$ for all $j > i$. Putting this in the exact sequence (1) we get the statement (ii).

Now let A be a local Cohen-Macaulay ring. For any $\mathbf{m}$-primary ideal I there is the largest ideal $\tilde{I}$ with the same Hilbert-Samuel polynomial as I. This ideal is called the *Ratliff-Rush closure* of I and is defined as follows (cf. [RR]):

$$\tilde{I} = \bigcup_{i \geq 1} (I^{i+1} : I^i).$$

Then it was shown that

$$\widetilde{I^n} = \bigcup_{i \geq 1} (I^{n+i} : I^i),$$

and $\widetilde{I^n} = I^n$ for $n \gg 0$.

Using this notion, Sally gave in [Sa2] a very interesting formula for computing the components of the local cohomology $H^2_{R_+}(R)$ of the Rees algebra $R = A[It] = A \oplus It \oplus I^2t^2 \oplus \cdots$ for 2-dimensional rings.

From now on, if not otherwise stated, let A be a 2-dimensional local Cohen-Macaulay ring and I an $\mathbf{m}$-primary ideal. Then we have

Lemma 2.3 [Sa2, Proposition 2.4]. *For all* $n \geq 0$,

$$\lambda(H^2_{R_+}(R)_n) = P_I(n) - \lambda(A/\widetilde{I^n}).$$

Lemma 2.4 [Sa2, Corollary 2.7]. *For any minimal reduction* $\underline{x}$ *of* I *and for all* $n \geq 0$*, we have*

$$\lambda(I^{n+2}/(\underline{x}\widetilde{I^{n+1}} \cap I^{n+2})) \leq P_I(n) - \lambda(A/\widetilde{I^n}).$$

In the next section we need, as in [Sa2], some fundamental ideas from Section 2 of Huneke's paper [Hu]. For a minimal reduction $\underline{x}$ of I and $n \geq 1$ set

$$v_n = \lambda(I^{n+1}/\underline{x}I^n) - \lambda((I^n : \underline{x}A)/I^{n-1}). \tag{2}$$

Then

Lemma 2.5. (i) *For all* $n \geq 1$,

$$v_n = [P_I(n+1) - H_I(n+1)] + [P_I(n-1) - H_I(n-1)] - 2[P_I(n) - H_I(n)].$$

(ii) $\lambda(A/I) - (e_0 - e_1) = \sum_{n\geq 1} v_n$, *and* $e_2 = \sum_{n \geq 1} nv_n$.

Finally recall that the reduction number of I is defined as follows. Let J be a minimal reduction of I. Set $r_J(I) = \min\{n \geq 0;\ I^{n+1} = JI^n\}$ and $r(I)$, the *reduction number* of I, $= \min\{r_J(I);\ J$ is a minimal reduction of $I\}$.

3. Results

Recall that $(A, \mathbf{m})$ is always assumed to be a 2-dimensional local Cohen-Macaulay ring and I an $\mathbf{m}$-primary ideal. For short, we also denote $G(I)$ by G.

Lemma 3.1. *Assume that* $I = \tilde{I}$ *and* $\lambda(A/I) = e_0 - e_1 + e_2$. *Then*

(i) $a_2(R) = a_2(G) \leq 0$ *and* $\lambda(H^2_{G_+}(G)_0) = e_2$.

(ii) $P_I(n) = \lambda(A/\widetilde{I^n})$ *for all* $n \geq 1$.

(iii) *For all* $n \geq 0$,

$$\lambda(H^0_{G_+}(G)_n) = \lambda((\widetilde{I^{n+1}} \cap I^n)/I^{n+1}),$$
$$\lambda(H^1_{G_+}(G)_n) = \lambda(\widetilde{I^n}/(\widetilde{I^{n+1}} + I^n)).$$

Proof. (i) By Lemma 2.3, $\lambda(H^2_{R_+}(R)_1) = 0$ and $\lambda(H^2_{R_+}(R)_0) = e_2$. It is easy to show that $\lambda(H^2_{R_+}(R)_{n+1}) \leq \lambda(H^2_{R_+}(R)_n)$ for all n (see, e. g., [Sa2, 2.8]. Hence $a_2(R) \leq 0$. Further, note that $H^i_{R_+}(A)_n = 0$ for $n \neq 0$ and $H^i_{R_+}(G) \cong H^i_{G_+}(G)$ (A is considered as a graded R-module concentrated in degree 0). From the exact sequences

$$0 \to R_+ \to R \to A \to 0,$$

and

$$0 \to R_+(1) \to R \to G \to 0,$$

we get the exact sequence

$$0 = H^2_{R_+}(R)_{n+1} \to H^2_{R_+}(R)_n \to H^2_{G_+}(G)_n \to 0,$$

for all $n \geq 0$. Hence $H^2_{G_+}(G)_n = 0$ for all $n > 0$ and $\lambda(H^2_{G_+}(G)_0) = \lambda(H^2_{R_+}(R)_0) = e_2$.

(ii) follows from (i) and Lemma 2.3.

(iii) The first equality follows from the definition of $\widetilde{I^{n+1}}$. In order to prove the second equality we use Lemma 2.1. Note that $h_G(n) = \lambda(I^n/I^{n+1})$ and $p_G(n) = P_I(n+1) - P_I(n)$. Since $\tilde{I} = I$, putting $n = 0$ in Lemma 2.1 we obtain

$$-e_2 = \lambda(H^1_{G_+}(G)_0) - \lambda(H^2_{G_+}(G)_0) = \lambda(H^1_{G_+}(G)_0) - e_2.$$

Hence $\lambda(H^1_{G_+}(G)_0) = 0$. For $n \geq 1$, we have

$$\begin{aligned}\lambda(H^1_{G_+}(G)_n) - \lambda(H^0_{G_+}(G)_n) &= P_I(n+1) - P_I(n) - \lambda(I^n/I^{n+1}) \\ &= \lambda(\widetilde{I^n}/I^n) - \lambda(\widetilde{I^{n+1}}/I^{n+1}).\end{aligned}$$

Hence,

$$\begin{aligned}\lambda(H^1_{G_+}(G)_n) &= \lambda((\widetilde{I^{n+1}} \cap I^n)/I^{n+1}) - \lambda(\widetilde{I^{n+1}}/I^{n+1}) + \lambda(\widetilde{I^n}/I^n)\\ &= \lambda(\widetilde{I^n}/I^n) - \lambda(\widetilde{I^{n+1}}/(\widetilde{I^{n+1}} \cap I^n))\\ &= \lambda(\widetilde{I^n}/I^n) - \lambda((\widetilde{I^{n+1}} + I^n)/I^n) = \lambda(\widetilde{I^n}/(\widetilde{I^{n+1}} + I^n)).\end{aligned}$$

Lemma 3.2. *If $\widetilde{I^n} = \widetilde{I^{n+1}} + I^n$ for all $n \geq p$, then $\widetilde{I^n} = I^n$ for all $n \geq p$.*

Proof. For all $n \geq p$ we have $\widetilde{I^n} = \widetilde{I^{n+1}} + I^n = \widetilde{I^{n+2}} + I^n = \cdots$. Since $\widetilde{I^i} = I^i$ for $i \gg 0$, we get $\widetilde{I^n} = I^n$.

Let I be an $\mathbf{m}$-primary ideal of a d-dimensional local ring A. An element $x \in I$ is called a *superficial element* for I if there exists an integer p such that $(I^n : x) \cap I^p = I^{n-1}$ for all $n \gg 0$. A system of elements $x_1, \ldots, x_t \in I,\ t \leq d$, is called a *superficial sequence* for I if the image of x_i is a superficial element for $I/(x_1, ..., x_{i-1})A$, $1 \leq i \leq t$. From the proof of [ZS, Lemma 8.8.5] it follows that $x \in I$ is a superficial element for I if and only if its initial form x^* in $G(I)$ is a filter-regular element of degree 1.

The following theorem gives an answer to the question posed in the introduction and improves [Hu, Theorem 2.11].

Theorem 3.3. *Let $(A, \mathbf{m})$ be a 2-dimensional Cohen-Macaulay ring and I an $\mathbf{m}$-primary ideal. Assume that $\tilde{I} = I$. Then the following conditions are equivalent:*

(1) $H_I(n) = P_I(n)$ *for* $n = 1, 2$.

(2) $H_I(n) = P_I(n)$ *for all* $n \geq 1$.

(3) $\operatorname{grade} G(I)_+ \geq 1$ *and* $r_J(I) \leq 2$ *for any minimal reduction* J *of* I.

Proof. Huneke [Hu] has proven the equivalence of (2) and (3). (2) $\Rightarrow$ (1) is trivial. We give here a proof of (1) $\Rightarrow$ (2) and a new proof of (2) $\Rightarrow$ (3). Assume (1). Then by Lemma 3.1 (iii) $H^1_{G_+}(G)_1 = H^1_{G_+}(G)_2 = 0$. Let $y \in I$ be a superficial element for I. Then we have the exact sequence of local cohomology:

$$H^1_{G_+}(G)_{n-1} \to H^1_{G_+}(G)_n \to H^1_{G_+}(G/y^*G)_n \to H^2_{G_+}(G)_{n-1} = 0,$$

for all $n \geq 2$. Hence $H^1_{G_+}(G/y^*G)_2 = 0$. By Lemma 2.2, $H^1_{G_+}(G/y^*G)_n = 0$ for all $n \geq 2$ and $H^1_{G_+}(G)_n = 0$ for all $n \geq 0$. By Lemma 3.1 (ii) and Lemma 3.2, it follows that $\widetilde{I^n} = I^n$ for all $n \geq 1$. That means $P_I(n) = H_I(n)$ for all $n \geq 1$ and $H^0_{G_+}(G) = 0$. By Lemma 2.4, it follows that $I^3 = JI^2$, i.e. $r_J(I) \leq 2$.

As a consequence we get the following improvement of [Hu, Theorem 4.6(i)].

Corollary 3.4. *Let A and I be as above. If $H_I(n) = P_I(n)$ for $n = 1, 2$, then $G(I)$ is a Cohen-Macaulay ring if and only if $I^2 \cap J = JI$ for one (or all) minimal reduction J of I.*

Example 3.5. Of course, $\operatorname{grade} G_+ \geq 1$ is equivalent to $\widetilde{I^n} = I^n$ for all $n \geq 1$. However, the condition $H_I(n) = P_I(n)$ for all $n \geq 1$ does not imply that $\tilde{I} = I$. Let us consider the following example of Sally [Sa2, p. 546]: Let $A = k[[x, y]]$, where k is a field. Set $I = (x^4y, x^6, xy^5, y^6)$. It was shown in [Sa2] that $P_I(1) = H_I(1)$ and $I \neq \tilde{I}$. In fact one can immediately check that $P_I(n) = H_I(n)$ for all $n \geq 1$.

What happens if we only assume $H_I(1) = P_I(1)$ in Theorem 3.3 ? As mentioned in the introduction, Huneke [Hu], Ooishi [O] (for $e_2 = 0$) and Sally showed that if $H_I(1) = P_I(1)$ and $e_2 \leq 2$ then $\operatorname{grade} G(I)_+ \geq 1$, $r_J(I) \leq 2$ and $H_I(n) = P_I(n)$ for all $n \geq 1$. The above example by Sally shows that $e_2 = 2$ is the largest value of e_2 for this kind of results. In this example, $I \neq \tilde{I}$. Can we get the same result for larger e_2 if we additionally assume that $I = \tilde{I}$? Is that true for all e_2 if $I = \mathbf{m}$? Below we give a result towards an answer to this question in the case $e_2 = 3$.

Proposition 3.6. *Let $(A, \mathbf{m})$ be a 2-dimensional Cohen-Macaulay ring and I an $\mathbf{m}$-primary ideal. Assume that $I = \tilde{I}$ and $\lambda(A/I) = e_0 - e_1 + e_2$ with $e_2 = 3$. Let $\underline{x}$ be a minimal reduction of I such that $\underline{x}$ is a superficial sequence for I. Then either*

(i) $r_{\underline{x}}(I) \leq 2$ *and* $\operatorname{grade} G(I)_+ \geq 1$ *and* $H_I(n) = P_I(n)$ *for all* $n \geq 1$, *or*

(ii) $r_{\underline{x}}(I) = 3$, $\widetilde{I^n} = I^n$ *for all* $n \geq 3$, $\lambda(I^2/\underline{x}I) = \lambda(I^3/\underline{x}I^2) = 2$ *and* $\lambda(I^3 : \underline{x}/I^2) = 1$, *and* $H_I(n) = P_I(n)$ *for all* $n \geq 3$.

Proof. By Lemma 3.1 (i) and Lemma 3.1 we have

$$\lambda(A/\widetilde{I^2}) = P_I(2) = 3e_0 - 2e_1 + e_2 = e_0 + 2\lambda(A/I) - 3.$$

By [V], $\lambda(A/I^2) = e_0 + 2\lambda(A/I) - \lambda(I^2/\underline{x}I)$. Hence

$$\lambda(\widetilde{I^2}/I^2) = 3 - \lambda(I^2/\underline{x}I).$$

Since $e_2 \neq 0$, $\lambda(I^2/\underline{x}I) > 0$ (see [Hu,O]). If $\lambda(I^2/\underline{x}I) = 3$, then $\widetilde{I^2} = I^2$. By Lemma 3.1 and Theorem 3.3, I satisfies (i). We consider two other cases separately.

Case 1. $\lambda(I^2/\underline{x}I) = 1$. Then $\lambda(\widetilde{I^2}/I^2) = 2 \geq \lambda(H^1_{G_+}(G)_2)$ (by Lemma 3.2 (iii)). From the exact sequence

$$0 = H^1_{G_+}(G)_1 \to H^1_{G_+}(G/x_1^*G)_1 \to H^2_{G_+}(G)_0 \to H^2_{G_+}(G)_1 = 0,$$

it follows that $\lambda(H^1_{G_+}(G/x_1^*G)_1) = e_2 = 3$. By Lemma 2.2 (i) we obtain

$$\lambda(H^1_{G_+}(G/x_1^*G)_n) \leq \begin{cases} 3 & \text{if} \quad n = 1, \\ 2 & \quad\quad n = 2, \\ 1 & \quad\quad n = 3, \\ 0 & \quad\quad n \geq 4. \end{cases} \tag{3}$$

In particular, $a_1(G/x_1^*G) \leq 3$. By Lemma 2.2 (ii) and from the exact sequence

$$H^1_{G_+}(G)_{n-1} \to H^1_{G_+}(G)_n \to H^1_{G_+}(G/y^*G)_n, \tag{4}$$

we then get:

$$\lambda(H^1_{G_+}(G)_n) \leq \begin{cases} 0 & \text{if} \quad n = 1, \\ 2 & \quad\quad n = 2, \\ 3 & \quad\quad n = 3, \\ 3 & \quad\quad n = 4, \\ 2 & \quad\quad n = 5, \\ 1 & \quad\quad n = 6, \\ 0 & \quad\quad n \geq 7. \end{cases} \tag{5}$$

By Lemma 3.2 it follows that $\widetilde{I^n} = I^n$ for all $n \geq 7$. Therefore, by Lemma 2.5 (i) and Lemma 3.1 (ii), $v_n = 0$ for all $n \geq 8$. We claim that

$\lambda(I^i/\underline{x}I^{i-1}) \leq 1$ for all $i \geq 2$, $v_1 = v_2 = v_3 = v_4 = 1$ and $v_i \leq 1$.

(The proof is the same as the middle of the proof of [Sa2, Theorem 3.1]). If $I^2 = (\underline{x}I, uw)$, with $u, w \in I$ and $\mathbf{m}uw \subseteq \underline{x}I$, then for $i > 2$, $I^i = (\underline{x}I^{i-1}, u^{i-1}w)$ and $\mathbf{m}u^{i-1}w \subseteq \underline{x}I^{i-1}$. Hence $\lambda(I^i/\underline{x}I^{i-1}) \leq 1$ for all

$i > 2$ and $v_i \leq 1$. Let $^-$ denote the reduction mod $x_1 A$. We need to show that $\lambda(\overline{I}^2/\underline{x}\overline{I}) = \lambda(\overline{I}^3/\underline{x}\overline{I}^2) = \lambda(\overline{I}^4/\underline{x}\overline{I}^3) = 1$. From the exact sequence

$$0 \to \overline{I}^{i+1}/\underline{x}\overline{I}^i \to \overline{I}^i/\underline{x}\overline{I}^i \to \overline{I}^i/\overline{I}^{i+1} \to 0,$$

we get that $\lambda(\overline{I}^i/\overline{I}^{i+1}) = e_0 - j_i$, where $j_i = \lambda(\overline{I}^{i+1}/\underline{x}\overline{I}^i)$. So $0 \leq j_i \leq 1$ and $j_i = 0$ implies $j_{i+1} = 0$. Thus, for large n,

$$\lambda(\overline{A}/\overline{I}^n) = e_0 n - (e_0 - \lambda(A/I) + \sum j_i) = e_0 n - e_1 = e_0 n - (e_0 - \lambda(A/I) + 3).$$

It follows that $j_1 = j_2 = j_3 = 1$, which is the desired conclusion. From this we immediately obtain that $\lambda(I^3/\underline{x}I^2) = \lambda(I^4/\underline{x}I^3) = 1$ and $x_1 A \cap I^i \subseteq \underline{x}I^{i-1}$ for $i = 2, 3, 4$. Therefore $I^i : \underline{x} = I^{i-1}$ for $i = 2, 3, 4$ and $v_2 = v_3 = 1$. If $\lambda(I^5/\underline{x}I^4) = 0$, then $v_4 = 0$ and $v_n \leq 0$ for $n \geq 5$. This contradicts to the second equality in Lemma 2.5 (ii). Thus $\lambda(I^5/\underline{x}I^4) = 1 = v_4$.

From Lemma 2.5 (ii) we now have $v_5 + v_6 + v_7 = -1$ and $v_6 + 2v_7 = -2$. Since $0 \geq v_7 = -\lambda(\widetilde{I^6}/I^6) = -\lambda(H^1_{G_+}(G)_6) \geq -1$ (by Lemma 2.5 (i) and (5), there are two possibilities: $v_5 = v_6 = 0, v_7 = -1$ or $v_5 = 1, v_6 = -2$ and $v_7 = 0$.

Case 1a. $v_5 = v_6 = 0$ and $v_7 = -1$. Then $\lambda(H^1_{G_+}(G)_6) = 1$ and, therefore, we must have all equalities in (3) and (5). By Lemma 3.1 (ii), $P_I(n) - H_I(n) = -\lambda(\widetilde{I^n}/I^n)$. Using Lemma 2.5 (i) one can deduce that $\lambda(\widetilde{I^6}/I^6) = 1 = \lambda(H^1_{G_+}(G)_6)$,..., $\lambda(\widetilde{I^3}/I^3) = 3 = \lambda(H^1_{G_+}(G)_3)$. It then follows from Lemma 3.1 (iii) that $\widetilde{I^{n+1}} \subseteq I^n$ for $n \geq 2$ and $\lambda(H^0_{G_+}(G)_n) = \lambda(H^1_{G_+}(G)_{n+1})$ for $n \geq 2$. Note that for any standard graded ring S with ht $S_+ > 0$ we always have the following exact sequence:

$$(6)\qquad 0 \to H^0_{S_+}(S)_n \to H^0_{S_+}(S/zS)_n \to H^1_{S_+}(S)_{n-1} \to H^1_{S_+}(S)_n \to H^1_{S_+}(S/zS)_n,$$

where $z \in S_1$ is a filter-regular element of S.

Applying this exact sequence to G with $n = 5$ we obtain the exact sequence

$$0 \to H^0_{G_+}(G)_5 \to H^0_{G_+}(G/x_1^*G)_5 \to H^1_{G_+}(G)_4 \to H^1_{G_+}(G)_5 \to 0.$$

Hence $\lambda(H^0_{G_+}(G)_5) = 2$. Applying (6) to G/x_1^*G with $n = 5$, we have

$$0 \to H^0_{G_+}(G/x_1^*G)_5 \to H^0_{G_+}(G/(x_1^*, x_2^*)G)_5 \to H^1_{G_+}(G/x_1^*G)_4 = 0.$$

Therefore $\lambda(I^5/(\underline{x}I^4 + I^6)) = \lambda(H^0_{G_+}(G/(x_1^*, x_2^*)G)_5) = 2$ which contradicts to the inequality $\lambda(I^5/\underline{x}I^4) \leq 1$.

Case 1b. $v_5 = 1, v_6 = -2$ and $v_7 = 0$. Then $H^1_{G_+}(G)_6 = 0$ and $\lambda(H^1_{G_+}(G)_5) = 2$. Now, as in Case 1a, applying (6) to G and to G/x_1^*G with $n = 6$ we will get a contradiction that $\lambda(I^6/(\underline{x}I^5 + I^7)) = 2$.

Summing up, Case 1 does not occur.

Case 2. $\lambda(I^2/\underline{x}I) = 2$. Then $\lambda(\widetilde{I^2}/I^2) = 1 \geq \lambda(H^1_{G_+}(G)_2)$ (by Lemma 3.1 (iii)). From the exact sequence (4) it follows that $\lambda(H^1_{G_+}(G/x_1^*G)_2) = \lambda(H^1_{G_+}(G)_2) \leq 1$. Applying Lemma 2.2 we obtain

$$\lambda(H^1_{G_+}(G/x_1^*G)_n) \leq \begin{cases} 3 & \text{if} \quad n = 1, \\ 1 & \quad\quad n = 2, \\ 0 & \quad\quad n \geq 3, \end{cases} \tag{7}$$

and

$$\lambda(H^1_{G_+}(G)_n) \leq \begin{cases} 0 & \text{if} \quad n = 1, \\ 1 & \quad\quad n = 2, \\ 1 & \quad\quad n = 3, \\ 0 & \quad\quad n \geq 4. \end{cases} \tag{8}$$

If $\lambda(H^1_{G_+}(G)_3) = 1$, we must have all equalities in (7) and (8). Then applying (6) to G and to G/x_1^*G with $n = 2$ we get a contradiction that $\lambda(I^2/(\underline{x}I + I^3)) = 3 \leq \lambda(I^2/\underline{x}I) = 2$. Thus $\lambda(H^1_{G_+}(G)_n) = 0$ for $n \geq 3$. By Lemma 3.2, $\widetilde{I^n} = I^n$ for all $n \geq 3$. Computing v_2 and v_3 by Lemma 2.5 (ii) and (2) we get $v_2 = 2$ and $v_3 = -1$. Hence $\lambda(I^3/\underline{x}I^2) = 2$ and $\lambda(I^3 : \underline{x}/I^2) = 1$. Now applying Lemma 3.1 and [T, Proposition 3.2] we get the statement (ii).

Remark. In fact, in (ii) of the above proposition we have $\lambda(H^1_{G_+}(G)_2) = \lambda(H^0_{G_+}(G)_1) = 1$ and $H^1_{G_+}(G)_{n+1} = H^0_{G_+}(G)_n = 0$ for $n \neq 1$.

Further we want to complete an observation by Huneke related to the independence of reduction numbers. Let

$$c(I) = \max\{n;\ \widetilde{I^n} \neq I^n\},$$

and

$$n(I) = \max\{n;\ P_I(n) \neq H_I(n)\}.$$

$n(I)$ is called by Ooishi the postulation number of I. (We set $c(I) = -\infty$ if $\widetilde{I^n} = I^n$ for all n). Huneke [Hu, Proposition 2.14] proved that if $n(I) \geq c(I) + 1$, then $r(I)$ does not depend on the choice of minimal reduction. Now we will show that this is also true if $n(I) < c(I)$. Thus if $r_J(I)$ depends on J we must have $n(I) = c(I)$.

Proposition 3.7. *Let A be a 2-dimensional Cohen-Macaulay ring and I an $\mathbf{m}$-primary ideal. If $n(I) \neq c(I)$, then $r(I)$ does not depend on the choice of minimal reduction.*

Proof. By Huneke's result we may assume that $n(I) < c(I)$. Let $n \geq c := c(I) > 0$. By Lemma 2.3 we then get that $\lambda(H^2_{R_+}(R)_n) = \lambda(\widetilde{I}/I^n)$. Hence $a_2(R) = c$. Similarly to the proof of Lemma 3.1 (i) we can show that $a_2(G) = c$. Now let $n > c$. Applying Lemma 2.1 to G we obtain

$$\begin{aligned}0 &= (P_I(n+1) - P_I(n)) - \lambda(I^n/I^{n+1}) = p_G(n) - h_G(n)\\ &= \lambda(H^1_{G_+}(G)_n) - \lambda(H^0_{G_+}(G)_n).\end{aligned}$$

Since $H^0_{G_+}(G)_n \cong (\widetilde{I^{n+1}} \cap I^n)/I^{n+1} = 0$, we must have $H^1_{G_+}(G)_n = 0$ for all $n > c$. Hence, by [T, Proposition 3.2], $r_J(I) = a_2(G) + 2 = c + 2$ for any minimal reduction J of I.

Finally, let us give a partial result in the d-dimensional case. Using Theorem 3.3 and reducing to the 2-dimensional case, as it was done in the proof of [Sa2, Theorem 4.4], we can prove the following result

Corollary 3.8. *Let $(A, \mathbf{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$. Then the following conditions are equivalent:*

(1) $e_0(\mathbf{m}) - e_1(\mathbf{m}) + e_2(\mathbf{m}) = 1$ *and* $\lambda(A/\mathbf{m}^2) = e_0(\mathbf{m})(d+1) - e_1(\mathbf{m})d + e_2(\mathbf{m})(d-1)$.

(2) $\lambda(A/\mathbf{m}^n) = P_{\mathbf{m}}(n)$ *for all* $n > 0$ *and* $e_3(\mathbf{m}) = \cdots = e_d(\mathbf{m}) = 0$.

(3) $r(\mathbf{m}) = 2$ *and* $G(\mathbf{m})$ *is a Cohen-Macaulay ring.*

Remarks and acknowledgement

We don't know if the above corollary can be extended to any $\mathbf{m}$-primary ideal with $I = \tilde{I}$. This paper is a revised version of an author's manuscript entitled "*Two notes on coefficients of the Hilbert-Samuel polynomial*" (Preprint 1993). The above corollary was written in that manuscript as one of main results. But Valla has pointed to the author that

this corollary can be also deduced from [EV, Theorem 2.1]. The author is grateful to Elias and Valla for this and for pointing him a mistake in the first version of this corollary.

In that version the following result was also proven: if $(A, \mathbf{m})$ is an arbitrary local ring of positive dimension then $e_1(\mathbf{m}) \leq e_0(\mathbf{m})(e_0(\mathbf{m}) - 1)/2$. After completing the paper the author has learnt from Elias that the same result and the same proof (using Gotzmann's representation [Go] of the Hilbert polynomial!) were independently discovered by Valla et al..

When this paper has been submitted, the author received a reprint of the paper [I]. In that paper, using a completely different method, Itoh proved Corollary 3.8 for all integrally closed ideals and a version of Theorem 3.3 (see [I, Proposition 16 and Theorem 17]). In the preprint [B] Blancafort was able to prove the inequality $e_1(I) \leq e_0(I)(e_0(I) - 1)/2$ for any $\mathbf{m}$-primary ideal $I \subset A$ such that A/I is equicharacteristic.

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