

## ON THE TWO-SIDED PREDICTABLE APPROXIMATION FOR STOCHASTIC PROCESSES

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ABSTRACT. The two-sided predictable stochastic processes are introduced to show that they can approximate every bounded measurable stochastic process  $(X_t(\omega), 0 < t \leq 1)$   $P$ -almost surely uniformly in  $t$ . Consequently, it is proved that the two-sided predictable algebra also generates the product  $\sigma$ -field  $\mathcal{B}((0, 1]) \otimes \mathcal{F}$ .

### I. NOTATIONS AND NOTIONS

Let  $(\Omega, \mathcal{F}, P)$  be the standard Wiener space with the canonical Brownian motion  $(W_t, 0 \leq t \leq 1)$ ,  $\mathcal{F}_t$  and  $\mathcal{F}^t$  the completion of  $\sigma\{W_s, s \leq t\}$  and  $\sigma\{W_1 - W_s, 1 - t \leq s \leq 1\}$  respectively, and  $\mathcal{F}_{(s,t]^c} = \mathcal{F}_s \vee \mathcal{F}^{1-t}$ .

Put

$$\Omega^* = (0, 1] \times \Omega,$$

$\mathcal{F}$  = the product  $\sigma$ -field, that is  $\mathcal{B}((0, 1]) \otimes \mathcal{F}$ ,

$\mathcal{T}$  = algebra generated by  $\{(s, t] \times G, s < t, G \in \mathcal{F}_{(s,t]^c}\}$  which is called the two-sided predictable algebra,

$S$  = algebra generated by  $\{(s, t] \times G, s < t, G \in \mathcal{F}\}$ , which generates the product  $\sigma$ -field  $\mathcal{F}^*$ .

Let  $(X_t, 0 < t \leq 1)$  be an integrable stochastic process, then  $(X_t, 0 < t \leq 1)$  induces a signed measure on  $\mathcal{T}$  defined by

$$\lambda_X((s, t] \times G) = \int_{\Omega} (X_t - X_s) 1_G dP.$$

Then  $X$  is said to be an  $S$ -martingale (resp. an  $S$ -quasi-martingale) iff  $\lambda_X \equiv 0$  (resp.,  $\lambda_X$  is a measure of bounded variation) (see [D-N-S]).

**Definition 1.1.** The two-sided predictable stochastic processes are the simple stochastic processes of the form

$$(1) \quad X_t = \sum_{i \geq 1} \lambda_i 1_{A_i}(t, \omega),$$

where  $\lambda_i$ 's are real numbers,  $A_i \in \mathcal{T}$  for all  $i$ , and  $A_i$ 's are disjointed.

In this short note we shall show that every bounded measurable stochastic process  $(X_t(\omega), 0 < t \leq 1)$  can be approximated  $P$ -almost surely uniformly in  $t$  by two-sided predictable stochastic processes. This result will be used to prove that  $\mathcal{T}$  also generates the product  $\sigma$ -field  $\mathcal{F}^*$ .

## II. THE MAIN RESULTS

For  $C \subset \Omega^*$  we denote by  $\Pi(C)$  the projection of  $C$  on  $\Omega$  and  $A\Delta B$  denotes the symmetric difference of the two sets  $A$  and  $B$ . First of all we shall prove the following lemma.

**Lemma 2.1.** *For any  $A \in S$  and  $\varepsilon > 0$ , there exists a  $B \in \mathcal{T}$  such that*

$$P(\Pi(A\Delta B)) < \varepsilon.$$

*Proof.* Since every set in  $S$  can be written as a finite union of sets of form:  $(s, t] \times G$ ,  $s < t$ ,  $G \in \mathcal{F}$ , we need only to consider the case  $A = (s, t] \times G$ , with  $G \in \mathcal{F}$ . Since  $\mathcal{F}$  is generated by  $\{W_u - W_v; u \geq v \geq t\} \cup \{W_t - W_u, W_u - W_s, s \leq u \leq t\} \cup \{W_u - W_v, v \leq u \leq s\}$ , we can take without loss of generality  $G = \bigcap_{0 \leq i \leq n} (W_{t_{i+1}} - W_{t_i} \in I_i)$ , where  $0 = t_0 < t_1 < \dots < t_n = 1$  such that  $s = t_\ell$ ,  $t = t_{k+1}$  for some  $\ell < k$ , and  $I_i$  are some open intervals with rational boundaries. We can write

$$\begin{aligned} A &= \bigcup_{1 \leq i \leq k} (t_i, t_{i+1}] \times G \\ &= \bigcup_{1 \leq i \leq k} \{(t_i, t_{i+1}] \times (G_i \cap (W_{t_{i+1}} - W_{t_i} \in I_i))\}, \end{aligned}$$

where

$$G_i = \bigcap_{j \neq i} (W_{t_{j+1}} - W_{t_j} \in I_j) \in \mathcal{F}_{(t_i, t_{i+1}]^c}.$$

Thus it suffices to prove our lemma for a set  $A$  of the form

$$A = (s, t] \times G, \text{ with } G = \{W_t - W_s \in I\},$$

where  $I = (c, d)$  is an open interval.

Let  $s = t_0 < t_1 < \dots < t_{m+1} = t$ , and define

$$\begin{aligned} G^0 &= \{W_t - W_{t_1} \in I\}, \\ G^m &= \{W_{t_m} - W_s \in I\}, \\ G^i &= \{(W_t - W_{t_{i+1}} + W_{t_i} - W_s) \in I\}, \quad 0 \leq i \leq m - 1. \end{aligned}$$

We want to show

$$(2) \quad P\left(\bigcup_{0 \leq i \leq m} (G^i \Delta G)\right) \rightarrow 0,$$

as  $\delta := \max_i (t_{i+1} - t_i) \rightarrow 0$ .

Since for any  $u > v$ , the probability for  $\{W_u - W_v = c, \text{ or } d\}$  is zero and the paths of  $W$  are continuous, we have for  $P$ -a.s.  $W$  with  $\delta$  small enough

$$W \in G \iff W \in G^i, \quad \text{for all } 0 \leq i \leq m.$$

Therefore  $W \notin \bigcup_{0 \leq i \leq m} \{G^i \Delta G\}$   $P$ -a.s., if  $\delta$  is small enough. This implies

$$1 - \bigcup_{0 \leq i \leq m} \{G^i \Delta G\}(W) \rightarrow 0, \quad P - a.s.$$

as  $\delta \rightarrow 0$ . This proves (2).

To prove our lemma, we take

$$B = \bigcup_{0 \leq i \leq m} ((t_i, t_{i+1}] \times G^i).$$

Obviously  $B \in \mathcal{T}$  since  $G^i \in \mathcal{F}_{(t_i, t_{i+1}]^c}$  and

$$P(\Pi(A \Delta B)) = P\left(\bigcup_{0 \leq i \leq m} (G^i \Delta G)\right) \rightarrow 0$$

as  $\delta \rightarrow 0$ . q.e.d.

*Remark.* The idea, which is used in the proof of Lemma 2.1 and can be seen as an advantage of the two-sided predictable case, is that if we make  $(t - s)$  smaller and smaller, then  $\mathcal{F}_{(s, t]^c}$  approximates  $\mathcal{F}$ .

The main result of this note is the following.

**Theorem 2.2.** *Suppose that  $(X_t, 0 < t \leq 1)$  is a bounded measurable stochastic process. Then there exists a sequence of two-sided predictable stochastic processes of the form (1)*

$$(X_t^{(n)}(\omega), 0 < t \leq 1), \quad n = 1, 2, \dots$$

such that

$$(3) \quad \lim_{n \rightarrow +\infty} X_t^{(n)}(\omega) = X_t(\omega) \quad P\text{-almost surely, uniformly in } t \in (0, 1].$$

*Proof.* Since  $S$  generates the product  $\sigma$ -field  $\mathcal{F}^*$  and the process  $(X_t, 0 < t \leq 1)$  is bounded, it is enough to prove (3) in the case

$$X_t(\omega) = 1_A(t, \omega),$$

where  $A = (a, b] \times G$ ,  $0 \leq a < b \leq 1$  and  $G \in \mathcal{F}$ . By Lemma 2.1 for every  $n = 1, 2, \dots$  there exists a subset  $B_n \in \mathcal{T}$  such that  $P(\Pi(B_n \Delta A)) < \frac{1}{2^n}$ . Put

$$X_t^{(n)} = 1_{B_n}(t, \omega).$$

It is clear that  $X_t^{(n)}(\omega) = X_t(\omega)$  for all  $(t, \omega) \notin B_n \Delta A$ . On the other hand, since  $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$  is finite, by the Borell-Cattelli law, it follows that for  $P$ -a.s. every  $\omega$  belongs only to a finite number of sets  $B_n \Delta A$ . Therefore for  $P$ -a.s. there exists  $n_0 \in \{1, 2, \dots\}$  for every  $\omega$  such that for all  $n \geq n_0$  we have

$$X_t^{(n)}(\omega) = X_t(\omega) \quad \text{for all } 0 < t \leq 1$$

From Theorem 2.2 we can derive the following interesting result.

**Corollary 2.3.** *The two-sided predictable algebra  $\mathcal{T}$  also generates the product  $\sigma$ -field  $\mathcal{F}^*$ .*

*Proof.* Put  $\sigma(\mathcal{T}) =$  the  $\sigma$ -field generated by the algebra  $\mathcal{T}$ . Clearly,  $\mathcal{T} \subset S$  implies that  $\sigma(\mathcal{T}) \subset \mathcal{F}^*$ . For every  $A \in S$ , it follows from Theorem 2.2 that the process  $(1_A(t, \omega), 0 < t \leq 1)$  is  $\sigma(\mathcal{T})$ -measurable. That means  $A \in \sigma(\mathcal{T})$ .

Therefore  $S \subset \sigma(\mathcal{T})$  and thus  $\mathcal{F}^* = \sigma(\mathcal{T})$ .

*Remark.* Suppose that  $(X_t, 0 \leq t \leq 1)$  and  $(Y_t, 0 \leq t \leq 1)$  are right continuous, integrable processes with increasing paths, and  $X_0 = Y_0 = 0$ .

These processes induce two nonnegative measures on the product space  $\Omega^*$  defined by

$$\lambda_X((s, t] \times G) = \int_{\Omega} (X_t - X_s) 1_G dP,$$

$$\lambda_Y((s, t] \times G) = \int_{\Omega} (Y_t - Y_s) 1_G dP,$$

for every  $0 \leq s < t \leq 1$ ,  $G \in \mathcal{F}$ .

Assume that  $\lambda_X = \lambda_Y$  on  $\mathcal{T}$ , i.e.  $(X_t - Y_t, 0 \leq t \leq 1)$  is an  $S$ -martingale. Since  $\mathcal{T}$  generates the product  $\sigma$ -field  $\mathcal{F}^*$ , it follows from Corollary 2.3 that  $\lambda_X = \lambda_Y$  on  $\mathcal{F}^*$ . Therefore for all  $0 < t \leq 1$  and for all  $G \in \mathcal{F}$  we have

$$\int_G X_t dP = \int_G Y_t dP.$$

It means that  $X_t = Y_t$   $P$ - a.s.,

Thus for  $P$ - a.s. we have  $X_t = Y_t$  for all  $0 \leq t \leq 1$ . So, by methods of measure theory we have showed the uniqueness of the Doob-Meyer decomposition for anticipating processes which was formulated and proved in [D-N-S] Theorem 3.3 by using the smooth Wiener functionals.

#### REFERENCES

1. N. M. Duc, D. Nualard and M. Sanz, *The Doob-Meyer decomposition for anticipating processes*, Stochastics and Stochastics Reports **34** (1991), 221-239.

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