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MODULES WITH FLAT SOCLES AND ALMOST EXCELLENT EXTENSIONS

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Abstract. Modules (resp. rings) with flat socles are called FS-modules (resp. FS-rings) which are preserved by Morita equivalences and (almost) excellent extensions. One result of Xue Weimin [7] is generalized. We show that when S is an almost excellent extension of R, if either ring is (i) left SF-ring, or (ii) right coherent, or (iii) left semi-hereditary, or (iv) left perfect, then so is another. This answers a question of Xue Weimin [8] in the affirmative.

According to Nicholson and Watters [5], a module M is called a PS-module if its socle $\mathrm{Soc}(M)$ is projective, and a ring R is called a (left) PS-ring if ${}_RR$ is a PS-module. Examples of PS-modules include nonsingular left $R-$modules, regular left $R-$modules in the sense of Zelmanowitz, and left $R-$modules with zero socle. In [7], Xue Weimin proved that PS-modules are preserved by Morita equivalences and excellent extensions. As a generalization of PS-modules and PS-rings, Liu Zhongkui [3] gave the following definition.

Definition. *A left R-module M is called an FS-module if every simple submodule is flat; equivalently if Soc(${}_RM$) is flat. A ring R is called a left FS-ring if ${}_RR$ is an FS-module.*

Liu Zhongkui [3] proved, among other things, that the notion of FS-rings is preserved by a Morita equivalence (cf. [3, Theorem 3.7]) or an excellent extension (cf. [3, Theorem 3.2]). The purpose of this paper is to prove that FS-modules are preserved by Morita equivalences and (almost) excellent extensions. We also show that a weakly duo reduced PS-ring must be a right PS-ring. Finally, for an almost excellent extension S of R, the relation between properties of the ring S and the ring R are studied.

Throughout the paper, all rings have a unity and all modules are left unitary. We freely use the terminologies and notions of [1] and [3].

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Modifying the proof of [3], Theorem 2.4, we first have

Proposition 1. *The following are equivalent for an R-module* ${}_RM$*:*

(i) ${}_RM$ *is a FS-module;*

(ii) *If* L *is a maximal left ideal of* R *then either* $r_M(L) = 0$ *or* $a \in aL$ *for every* $a \in L$*;*

(iii) *Every simple left* $R-$*module* ${}_RK$ *is either flat or Hom(K,M)=0.*

Proof. (i) $\Rightarrow$ (ii). Let L be a maximal left ideal of R and $r_M(L) \neq 0$. Let $0 \neq m \in r_M(L)$. Since $L \subseteq l_R(m) \neq R$ and L is a maximal left ideal, we have $L = l_R(m)$. Now $R/L \cong Rm$ is flat by hypothesis and so $a \in aL$ for every $a \in L$.

(ii) $\Rightarrow$ (iii). Let ${}_RK = Rk$ be simple. Then $K = Rk \cong R/L$ where $L = l_R(k)$ is a maximal left ideal of R. If $a \in aL$ for every $a \in L$, then ${}_RK \cong R/L$ is flat. If $r_M(L) = 0$, let $f \in Hom_R(K, M)$. If $f(k) = m \in M$, then $Lm = f(Lk) = f(0) = 0$, so $m = 0$ and $f = 0$.

(iii) $\Rightarrow$ (i). If K is a simple submodule of ${}_RM$ then $Hom_R(K, M) \neq 0$ and so K is flat by the hypothesis.

Theorem 2. *Let* F : *R-Mod*$\longrightarrow$ $S-$*Mod define a Morita equivalence. Then an R-module* ${}_RM$ *is a FS-module if and only if the* $S-$*module* $F(M)$ *is a FS-module.*

Proof ($\Leftarrow$). Let ${}_RK$ be a simple module which is not flat. By [1, Proposition 21.6 and p. 268, Ex. 12], $F(K)$ is a simple $S-$module which is not flat. Then $Hom_S(F(K), F(M)) = 0$ by Proposition 1. By [1, Proposition 21.2], $Hom_R(K, M) = 0$. Hence ${}_RM$ is a FS-module by Proposition 1. The implication ($\Rightarrow$) can be proved similarly.

Corollary 3. *If* F : $R-$*Mod*$\longrightarrow$ $S-$*Mod defines a Morita equivalence, then* R *is a FS-ring if and only if* S *is a FS-ring.*

Proof. If R is a FS-ring then the faithful $S-$module $F({}_RR)$ is a FS-module by Theorem 2, hence S is a FS-ring by [3], Theorem 2.4.

Xue [7, Proposition 4] proved that for each $R-$module ${}_RM$, the power series module $M[[x]]$ is a PS-module over the power series ring $R[[x]]$. In particular, $R[[x]]$ is a PS-ring for any ring R. Thus we have

Proposition 4. *For each* $R-$*module* ${}_RM$*, the power series module* $M[[x]]$ *is a FS-module over the power series ring* $R[[x]]$*. In particular,* $R[[x]]$ *is a FS-ring for any ring* R*.*

A ring R is duo if each one-sided ideal of R is a two-sided ideal. As a generalization of left duo rings, a ring R is called weakly left duo if for every $r \in R$ there is a natural number $n(r)$ such that $Rr^{n(r)}$ is a two-sided ideal of R. A local ring with nil radical is weakly left duo, but not necessarily left duo. A ring R is weakly duo if it is weakly right and left duo.

The notion of PS-rings is not left-right symmetric (cf. [5]). Recently, Xue Weimin [7] proved that a duo ring R is a PS-ring if and only if it is right PS-ring. We generalize this result to weakly duo rings.

Proposition 5. *A weakly duo reduced ring R is a PS-ring if and only if R is a right PS-ring.*

Proof. Let R be a weakly duo PS-ring. If rR is a minimal right ideal, then there is a natural number $n(r)$ such that $r^{n(r)}R$ is a two-sided ideal of R, thus $r^{n(r)}R = rR$ and $Rr^{n(r)} = r^{n(r)}R$ is also a minimal left ideal since $r^{n(r)}R \subseteq rR$ and rR is a minimal right ideal. Hence $Rr^{n(r)}$ is projective and $l_R(r^{n(r)}) = Re$ for some $e^2 = e \in R$ since e lies in the center of R, we see $e \in r_R(r^{n(r)})$. Since Re is a maximal left ideal, $eR = Re$ is also a maximal right ideal. Now $eR \subseteq r_R(r^{n(r)}) \neq R$, so $eR = r_R(r^{n(r)})$ and rR is a projective right $R-$module.

Let R and S be rings with the same unity, $R \subseteq S$. The ring S is an excellent extension of R if the following conditions are satisfied:

(i) If ${}_SM$ is an $S-$module with an $S-$submodule ${}_SN$ and N is a direct summand of M as an $R-$module, then N is a direct summand of M as an $S-$module.

(ii) There is a finite set $\{1 = s_1, s_2, ..., s_n\} \subseteq S$ such that S is free left and right $R-$module with basis $\{1 = s_1, s_2, ..., s_n\}$ and $Rs_i = s_iR$ for all $i = 1, ..., n$.

Examples include finite matrix rings, and crossed product RG where G is a finite group with $|G|^{-1} \in R$. Xue Weimin [8] weakened condition (ii) as follows:

(iii) $S = \sum_{i=1}^{n} Rs_i \supseteq R$ is a finite normalizing extension such that ${}_RS$ is a projective $R-$modules.

Following [8], the ring extension $R \subseteq S$ is called an almost excellent extension if the conditions (i) and (iii) are satisfied. See [4, 8] for further information about excellent extensions and almost excellent extensions.

Theorem 6. *Let S be an almost excellent extension of R. If ${}_SM$ is an $S-$module then*

(i) ${}_SM$ *is flat if and only if* ${}_RM$ *is flat.*

(ii) ${}_SM$ *is a FS-module if and only if* ${}_RM$ *is a FS-module.*

Proof. (i) One direction follows from [8], Lemma 6. Now assume that ${}_RM$ is flat . Thus $(M^*)_R$ is injective, where M^* is the character module of M. Obviously, when the character module $(M^*)_S$ of $S-$module ${}_SM$ is regarded as an $R-$module, it coincides with the character module $(M^*)_R$ of $R-$module ${}_RM$. Thus $(M^*)_S$ is injective by [8, Theorem 1 (1)]. Therefore ${}_SM$ is flat.

(ii) By [8, Theorem 1 (4)], $Soc({}_SM) = Soc({}_RM)$. It follows from (i) that ${}_SSoc({}_SM)$ is flat if and only if ${}_RSoc({}_RM)$ is flat.

Corollary 7. *Let S be an excellent extension of R. If ${}_SM$ is an $S-$module then*

(i) ${}_SM$ *is flat if and only if* ${}_RM$ *is flat.*

(ii) ${}_SM$ *is a FS-module if and only if* ${}_RM$ *is a FS-module.*

Corollary 8 [3]. *If S is an excellent extension of R then S is a FS-ring if and only if R is a FS-ring.*

A ring R is called a left (right) SF-ring if all simple left (right) $R-$modules are flat. Xue Weimin [8] proved that for an almost excellent extension S of R, if S is a left SF-ring then R is a left SF-ring. He also asked whether or not the converse is true. The following result answers this question in the affirmative.

Theorem 9. *Let S be an almost excellent extension of R. Then S is a left SF-ring if and only if R is a left SF-ring.*

Proof. One direction follows from [8, Proposition 7]. Now assume that R is a left SF-ring and ${}_SM$ is a simple module. By [8, Theorem 1 (4)], ${}_RM$ is semisimple. Thus ${}_RM$ is flat and ${}_SM$ is also flat by Theorem 6(1). Therefore S is a left SF-ring.

Recall that a ring R is called a right coherent ring if every direct product of $R-$modules is flat. For right coherent rings we have

Theorem 10. *If S is an almost excellent extension of R then S is right coherent if and only if R is right coherent.*

Proof. Let R be a right coherent ring. Suppose that $\{M_i : i \in I\}$ is a collection of flat $S-$modules. By Theorem 6(1), M_i is a flat $R-$module for every $i \in I$. Thus $\prod_{i\in I} M_i$ is a flat $R-$module by the left coherence of R. Obviously, when the product $\prod_{i\in I} {}_SM_i$ of $S-$modules ${}_SM_i$ is regarded as an $R-$module, it coincides with the direct product $\prod_{i\in I} M_i$ of $R-$modules ${}_RM_i$. Again by Theorem 6(1), $\prod_{i\in I} {}_SM_i$ is a flat $S-$module. Thus S is right coherent.

Conversely, if S is right coherent and let $\{N_j : j \in J\}$ be a collection of flat $R-$modules. It is easy to see that $S \otimes_R N_j$ is a flat $S-$module for every $R-$module N_j. Thus the direct product $\prod_{j\in J}(S \otimes_R N_j)$ is a flat $S-$module by the coherence of S. Obviously, S is a finitely presented right $R-$module. Therefore, there exists an isomorphism of $S-$modules:

$$S \otimes_R (\prod_{j\in J} N_j) \cong \prod_{j\in J}(S \otimes_R N_j).$$

Thus $S \otimes_R (\prod_{j\in J} N_j)$ is a flat $S-$module and $\prod_{j\in J} N_j$ is a flat $R-$module by [6, Proposition 2.1]. Hence R is right coherent.

Corollary 11. *Let R be a ring and G a finite group such that $|G|^{-1} \in R$. Then the crossed product $R * G$ is right coherent if and only if R is right coherent.*

Recall that R is a left semi-hereditary ring if and only if every finitely generated left ideal of R is projective if and only if each finitely generated submodule of a projective left $R-$module is projective.

Theorem 12. *If S is an almost excellent extension of R then S is left semi-hereditary if and only if R is left semi-hereditary.*

Proof. Let R be a left semi-hereditary ring. Suppose that M is a projective $S-$module and N a finitely generated submodule of $S-$module M. Assume that $N = Sx_1 + Sx_2 + ...Sx_m$ where $x_i \in N$. Set $y_{ij} = s_ix_j, i = 1, 2, ..., n; j = 1, 2, ..., m$, where the letters $s_1, ..., s_n$ come from the definition of almost excellent extensions. Obviously, $N = \sum_{i=1}^{n}\sum_{j=1}^{m} Ry_{ij}$, i.e. ${}_RN$ is finitely generated $R-$module. By [8, Theorem 1(2)], ${}_RM$ is projective, hence ${}_RN$ is projective since R is left semi-hereditary. Again by [8, Theorem 1 (2)], ${}_SN$ is projective. Therefore S is left semi-hereditary.

Conversely, suppose that S is left semi-hereditary and let M be a projective $R-$module and N a finitely generated submodule of $R-$module M. By [6, Corollary 3.4 (2)], $S \otimes_R M$ is a projective $S-$module. Obviously, $S \otimes_R N$ is a finitely generated $S-$module, and thus it is a projective $S-$module since S is left semi-hereditary. Again by [6, Corollary 3.3], N is a projective $R-$module. Thus R is left semi-hereditary.

Corollary 13. *Let R be a ring and G a finite group such that $|G|^{-1} \in R$. Then the crossed product $R * G$ is left semi-hereditary if and only if R is left semi-hereditary.*

Recall that R is left perfect if and only if every flat left $R-$module is projective. For left perfect rings we have

Theorem 14. *If S is an almost excellent extension of R then S is left perfect if and only if R is left perfect.*

Proof. Suppose that S is left perfect and M a flat $R-$module. Then by [6, Corollary 3.4 (1)], $S \otimes_R M$ is a flat $S-$module. Thus $S \otimes_R N$ is a projective $S-$module since S is left perfect. Again by [6, Corollary 3.3], M is a projective $R-$module. Thus R is left perfect.

Conversely, assume that R is a left perfect ring. Suppose that M is a flat $S-$module. By Theorem 6 (1), ${}_RM$ is flat. Thus ${}_RM$ is projective since R is left perfect. Therefore ${}_SM$ is projective by [8, Theorem 1]. This proves that S is left perfect.

Corollary 15. *Let R be a ring and G a finite group such that $|G|^{-1} \in R$. Then the crossed product $R*G$ is left perfect if and only if R is left perfect.*

Let S be an almost excellent extension of R. It is proved in [8] that S is regular if and only if R is regular. Here we give a new method to prove this result by using the concept of direct summand sum property of modules.

Recall from [2] that an $R-$module M is said to have the direct summand sum property if the sum of two direct summands of M is again a direct summand of M. Garcia [2, Proposition 1.7] proved that R is a regular ring if and only if every finitely generated projective $R-$module has the direct summand sum property.

Theorem 16. *If S is an almost excellent extension of R then S is regular if and only if R is regular.*

Proof. If S is regular, then it is clear that R is regular. Let R be a regular ring. Suppose that M is a finitely generated projective $S-$module and N, L are direct summands of M. It is clear that ${}_RM$ is a projective module by [8, Theorem 1 (2)]. Suppose $M = Sx_1 + ... + Sx_m$. Set $y_{ij} = s_ix_j, i = 1, 2, ..., n; j = 1, 2, ..., m$, where the letters $s_1, ..., s_n$ come from the definition of almost excellent extensions. Then $\{y_{ij} | 1 \leq i \leq n, 1 \leq j \leq m\}$ is the generating set of ${}_RM$. This means that ${}_RM$ is finitely generated. Obviously, ${}_RN, {}_RL$ are also direct summands of ${}_RM$. Thus ${}_R(N + L)$ is a direct summand of ${}_RM$ by [2, Proposition 1.7]. Since S is an almost excellent extension of R, it follows that ${}_S(N + L)$ is a direct summand of ${}_SM$. This means that ${}_SM$ has the direct summand sum property. Thus, by [2, Proposition 1.7], S is a regular ring.

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