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A REMARK ON DIFFERENTIAL OPERATORS OF INFINITE ORDER

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ABSTRACT. In this paper we give a necessary and sufficient condition for a linear differential operator of infinite order to act invariantly in the space of distributions with compact support.

Let $P(\xi)$ be a polynomial in ξ. The solvability and related problems of the differential equation

$$P(D)h = f$$

in the space $\mathcal{E}' = \mathcal{E}'(\mathbf{R}^n)$ of distributions with compact support have been studied by B. Malgrange, L. Hörmander, V.P. Palamodov and others (see, for example, [1-3]).

The aim of this paper is to study the same problems for differential operators of infinite order. Let $\{a_\alpha\}$ be a sequence of complex numbers. We put

$$A(D) = \sum_{\alpha \geq 0} a_\alpha D^\alpha$$

and consider the solvability in $\mathcal{E}'$ of the equation

$$A(D)h = f\ . \tag{1}$$

To study (1) we have to understand the action of the symbol $A(D)h$ on test functions $\varphi \in C^\infty(\mathbf{R}^n)$. The first thought came to our mind is that:

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$$< A(D)h, \varphi > = < h, \sum_{\alpha \geq 0} (-1)^{|\alpha|} a_\alpha D^\alpha \varphi > ,$$

but the series $\sum_{\alpha \geq 0} (-1)^{|\alpha|} a_\alpha D^\alpha \varphi(x)$, in general, does not converge. Therefore, we have to define $< A(D)h, \varphi >$ by other ways. We present here one of the possible solutions:

Definition 1. By $A(D)h$ we denote the weak limit of the sequence

$$\sum_{|\alpha| \leq N} a_\alpha D^\alpha h, \quad N \to \infty , \tag{2}$$

i.e., if it converges in $\sigma(\mathcal{E}', \mathcal{E})$ to some $g \in \mathcal{E}'$.

Define now conditions on $\{a_\alpha\}$ so that sequence (2) converges in $\sigma(\mathcal{E}', \mathcal{E})$ for all $h \in \mathcal{E}'$. It follows from the definition of the Fourier transform that

$$\Big\langle \sum_{|\alpha| \leq N} a_\alpha D^\alpha h, \varphi \Big\rangle = \Big\langle \big(\sum_{|\alpha| \leq N} a_\alpha \xi^\alpha \big) \hat{h}(\xi) \ , \ (F^{-1}\varphi)(\xi) \Big\rangle$$

for any $\varphi \in C^\infty$. Therefore, if sequence (2) weakly converges to g, then

$$\Big\langle \big(\sum_{|\alpha| \leq N} a_\alpha \xi^\alpha \big) \hat{h}(\xi) \ , \ (F^{-1}\varphi)(\xi) \Big\rangle \to \big\langle \hat{g}(\xi), (F^{-1}\varphi)(\xi) \big\rangle \ , \ N \to \infty$$

for all $\varphi \in C^\infty(\mathbf{R}^n)$. Hence, for $\varphi(x) = e^{-ix\eta}, \eta \in \mathbf{R}^n$ we get

$$(F^{-1}\varphi)(\xi) = (2\pi)^{-n} \delta(\xi - \eta)$$

and then

$$\big(\sum_{|\alpha| \leq N} a_\alpha \eta^\alpha \big) \hat{h}(\eta) \ \to \ \hat{g}(\eta) \ , \ N \to \infty \ .$$

Putting $h = \delta$, i.e., $\hat{h}(\eta) \equiv 1$, we obtain

$$\lim_{N \to \infty} \Big(\sum_{|\alpha| \leq N} a_\alpha \eta^\alpha \Big) = \hat{g}(\eta)$$

for any $\eta \in \mathbf{R}^n$. Then $A(\xi) = \sum_{\alpha \geq 0} a_\alpha \xi^\alpha$ is an entire function.

So, in order that the sequence (2) converge s in $\sigma(\mathcal{E}', \mathcal{E})$ for all $h \in \mathcal{E}'$, it is necessary that $A(\xi) = \sum_{\alpha \geq 0} a_\alpha \xi^\alpha$ is an entire function, and if $g \in \mathcal{E}'$ is the weak limit of the sequence $\sum_{|\alpha| \leq N} a_\alpha D^\alpha h, N \to \infty$, then $A(\xi)\hat{h}(\xi) = \hat{g}(\xi)$ for all $\xi \in \mathbf{R}^n$.

We define now conditions so that $A(D)$ acts invariantly in $\mathcal{E}'$.

Theorem 1. *Let $A(\xi)$ be an entire function. Then $A(D)$ acts invariantly in $\mathcal{E}'$ if and only if there exist numbers $C, M, r < \infty$ such that*

$$|A(z)| \leq C(1 + |z|)^M \exp(r|\mathrm{Im} z|) \, , z \in \mathbf{C}^n \; . \tag{3}$$

Proof. ($\Rightarrow$). Put $h(x) = (2\pi)^{-n}\delta(x)$. Then $F[A(D)h] = A(\xi)\hat{h}(\xi) = A(\xi) \in F(\mathcal{E}')$. Therefore, by Paley-Wiener-Schwartz theorem [1, p. 220], we get (3).

($\Leftarrow$). Let the entire function $A(z)$ satisfy (3). Then for any $h \in \mathcal{E}'$ it follows from the Paley-Wiener-Schwartz theorem that

$$|\hat{h}(z)| \leq C_1(1 + |z|)^{M_1} \exp(r_1|\mathrm{Im} z|) \; , \; z \in \mathbf{C}^n$$

for some numbers $C_1, M_1, r_1 < \infty$. Therefore, taking into account that

$$|A(z)\hat{h}(z)| \leq CC_1(1 + |z|)^{M+M_1} \exp((r + r_1)|\mathrm{Im} z|) \; , \; z \in \mathbf{C}^n$$

and Paley-Wiener-Schwartz theorem we get what we have to show. The proof is complete.

Further, let K be a compact in $\mathbf{R}^n$. Denote by $\mathcal{E}'(K)$ the space of distributions with support contained in K.

We define now conditions on entire function $A(\xi)$ so that $A(\xi)\hat{h}(\xi) \in F[\mathcal{E}'(K)]$ for all $h \in \mathcal{E}'(K)$.

Theorem 2. *Let $A(\xi)$ be an entire function and K - a compact in $\mathbf{R}^n$. In order that $A(\xi)\hat{h}(\xi) \in F[\mathcal{E}'(K)]$ for all $h \in \mathcal{E}'(K)$, it is necessary and sufficient that $A(\xi)$ is a polynomial.*

Proof. We only need to prove the necessity. We shall begin by showing that for any $r > 0$ there exist numbers $L, M < \infty$ such that

$$|A(z)| \leq L(1 + |z|)^M \exp(r|\mathrm{Im} z|) \, , z \in \mathbf{C}^n \; . \tag{4}$$

Assume the contrary, that there is a number $r > 0$ such that for any $m \geq 1$ there exists a point $z^m \in \mathbf{C}^n$ such that

$$(5) \qquad |A(z^m)| \geq m(1+|z^m|)^m \exp(r|\mathrm{Im} z^m|)\ ,\ m \geq 1\ ,$$

where $z^m = (z_1^m, \dots, z_n^m)$, $z_k^m = x_k^m + iy_k^m$, $k = 1, \dots, n$.

On the other hand, it follows from $A(\xi)\hat{h}(\xi) \in F[\mathcal{E}'(K)]$ and Paley-Wiener-Schwartz theorem that

$$(6) \qquad |A(z)\hat{h}(z)| \leq C(1+|z|)^N \exp H(\mathrm{Im} z)\ ,\ z \in \mathbf{C}^n$$

for some constants $C, N < \infty$, where $H(t)$ – the support function of K.

Further, by virtue of the continuity of $A(z)$ we can assume that (5) holds for $|y^m| > 0, m \geq 1$. We represent

$$z^m = x^m + iy^m\ , y^m = |y^m| y^m/|y^m|\ ,\ m \geq 1$$

and take from $\{y^m/|y^m|\}$ a convergent subsequence. For simplicity of notation we may assume that $\{y^m/|y^m|\}$ converges to some point y^*. Clearly, $|y^*| = 1$.

We choose $\eta \in K$ such that

$$(7) \qquad H(y^*) = \sup_{t \in K} t y^* = \eta y^*\ .$$

Taking into account of $h(x) = (2\pi)^{-n}\delta(x-\eta) \in \mathcal{E}'(K)$ and $\hat{h}(\xi) = \exp(-i\eta\xi)$, and combining (5) and (6), we get

$$\begin{aligned} & m(1+|z^m|)^m \exp(r|\mathrm{Im} z^m|)\exp(-i\eta z^m) \\ & = m(1+|z^m|)^m \exp(r|y^m|)\exp(\eta y^m) \\ & \leq |A(z^m)\hat{h}(z^m)| \\ & \leq C(1+|z^m|)^N \exp H(y^m) \\ & = C(1+|z^m|)^N \exp\Big(\sup_{t \in K} |y^m| \frac{y^m}{|y^m|} t\Big) \\ (8) \qquad & = C(1+|z^m|)^N \exp\Big(\sup_{t \in K} |y^m|(y^* t + (\frac{y^m}{|y^m|} - y^*)t)\Big)\ . \end{aligned}$$

Put $m_0 = \max\{C, N\}$. Then it follows from (8) that for $m \geq m_0$

$$\exp\ (r|y^m| + \eta y^m) \leq \exp\Big(\sup_{t \in K} |y^m|(y^* t + (\frac{y^m}{|y^m|} - y^*)t)\Big)\ ,$$

which is equivalent to

$$r|y^m| + \eta y^m \leq \sup_{t\in K} |y^m|\Big(y^* t + (\frac{y^m}{|y^m|} - y^*)t\Big)$$
$$= |y^m| \sup_{t\in K}\Big(y^* t + (\frac{y^m}{|y^m|} - y^*)t\Big)\ ,\ m \geq m_0\ .$$

Therefore, by virtue of (7) we get for $m \geq m_0$

$$r + \frac{y^m}{|y^m|} \leq \sup_{t\in K}\big(y^* t + (\frac{y^m}{|y^m|} - y^*)t\big)$$
$$\leq \sup_{t\in K} y^* t + \sup_{t\in K}(\frac{y^m}{|y^m|} - y^*)t$$
$$= \eta y^* + \sup_{t\in K}(\frac{y^m}{|y^m|} - y^*)t\ .$$

By letting $m \to \infty$ we have

$$r + \eta y^* \leq \eta y^*$$

because of

$$\sup_{t\in K}\big(\frac{y^m}{|y^m|} - y^*\big)t \leq \big|\frac{y^m}{|y^m|} - y^*\big| \sup_{t\in K}|t| \to 0\ ,\ m \to \infty\ .$$

Therefore, $r \leq 0$, which is impossible. So we have proved (4).

It follows from (4) and Paley-Wiener-Schwartz theorem that $A(\xi)$ is the Fourier transform of a distribution with the support contained in the ball $\mathcal{B}(0, r)$. Then since $r > 0$ is arbitrarily chosen, we get that the distribution $F^{-1}A$ concentrates on the origin $\{0\}$ of coordinates. Therefore,

$$< F^{-1}A, \varphi >= \sum_{|\alpha|\leq k} b_\alpha D^\alpha \varphi(x)\ ,\ \varphi \in C^k(\mathbf{R}^n)$$

by virtue of Theorem 2.3.4 [1, p. 64], where k is the order of the distribution $F^{-1}A$. Hence,

$$A(\xi) = \sum_{|\alpha|\leq k} b_\alpha \xi^\alpha\ ,$$

i.e., $a_\alpha = b_\alpha, |\alpha| \leq k$ and $a_\alpha = 0, |\alpha| > k$, because $A(\xi)$ is an entire function. So we have proved the necessity.

Remark. Let K be an arbitrary compact in $\mathbf{R}^n$. It follows from the obtained results that if $A(D)$ is a differential operator of infinite order then $A(D)$ cannot act invariantly in $\mathcal{E}'(K)$. Furthermore, let $f \in \mathcal{E}'(K)$ (or $\in \mathcal{E}'$) and $A(\xi)$ be an entire function of exponential type. Then, clearly, equation (1) has a solution in $\mathcal{E}'(K)$ ($\mathcal{E}'$ resp.) if and only if $\hat{f}(\xi) = A(\xi)g(\xi)$, where $F^{-1}g \in \mathcal{E}'(K)$ ($\mathcal{E}'$ resp.).

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