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ON THE GEOMETRIC COMPOSED VARIABLE AND THE ESTIMATE OF THE STABLE DEGREE OF THE RENYI'S CHARACTERISTIC THEOREM

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1. Introduction

Let $X_1, X_2, \dots$ be nonnegative independent identically distributed random variable, $P\{X_j > x\} = \overline{F}(x)$, $F(x) = 1 - \overline{F}(x)$, and $E(X_j) = \int_R x dF(x) < +\infty$, $j = 1, 2, \dots$ and let N be independent of X_j, $j = 1, 2, \dots$ with the geometric distribution function, i.e.

$$P\{N = k\} = p(1-p)^{k-1}, \quad k = 1, 2, \dots \quad (0 < p < 1).$$

In [1], the random variable $z = \sum_{j=1}^{N} X_j$ is called the *geometric composed variable* of X_{j-s}. Put

$$G(x) = P\{z \le x\}, \quad G_p(x) = P\{pz \le x\} \text{ and } \overline{G}_p(x) = P\{pz > x\}. \tag{1.1}$$

Renyi [3] characterized the exponential distribution by proving the following two assertions:

(i) $\lim_{p \to 0} \overline{G}_p(x) = e^{-x}$,

(ii) $\overline{G}_p(x) = \overline{F}(x) \leftrightarrow \overline{F}(x) = e^{-x}$.

We will consider the stability of this theorem.

Suppose that $\varphi(t)$, $\varphi_z(t)$ and $\varphi_{fz}(t)$ are characteristic functions of $F(x)$, $G(x)$, $G_p(x)$, respectively. Then, if $a(z) = \dfrac{pz}{1 - qz}$, $(q = 1 - p)$ is the generating function of N, we have (see [2])

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$$
(1.2) \qquad
\begin{aligned}
\varphi_z(t) &= a[\varphi(t)] = \frac{p\varphi(t)}{1-q\varphi(t)}\ , \\
\varphi_{pz}(t) &= \varphi_z(pt) = a[\varphi(pt)] = \frac{p\varphi(pt)}{1-q\varphi(pt)}\ \cdot
\end{aligned}
$$

We will distinguish two cases:

1. $F(x)$ is a ε-exponential distribution, i.e., $\exists T(\varepsilon) > 0$, $T(\varepsilon) \to \infty$ when $\varepsilon \to 0$, such that

$$
(1.3) \qquad |\varphi(t) - \varphi_0(t)| \leq \varepsilon, \quad \forall t : |t| \leq T(\varepsilon),
$$

where

$$
(1.4) \qquad \varphi_0(t) = \frac{1}{1-it}\ \cdot
$$

2. $G_p(x) = P\{pz \leq x\}$ is the ε-exponential distribution function, i.e., $\exists T(\varepsilon) > 0$, $T(\varepsilon) \to \infty$ when $\varepsilon \to 0$, such that

$$
(1.5) \qquad |\varphi_{pz}(t) - \varphi_0(t)| \leq \varepsilon \quad \forall t; |t| \leq T(\varepsilon),
$$

where $\varphi_{pz}(t)$ and $\varphi_0(t)$ are the characteristic functions from (1.2) and (1.4).

2. Stability theorems

Theorem 2.1. *Assume that $F(x)$ is a ε-exponential distribution function. Then we have*

$$
(2.1) \qquad
\begin{aligned}
&\text{(i)} \quad |\varphi_{pz}(t) - \varphi_0(t)| \leq \frac{\varepsilon}{p}\,, \quad \forall t : |t| \leq \frac{T(\varepsilon)}{p}\,; \\
&\text{(ii)} \quad \lambda(G_p; F_0) = \max\Big\{\frac{\varepsilon}{2f}\,; \frac{1}{T(\varepsilon)}\Big\} \ \textit{(with } T(\varepsilon) \textit{ as above)},
\end{aligned}
$$

where $F_0(x)$ is the exponential distribution function and $G_p(x)$ as in (1.1) *and*

$$
\lambda(G_p; F_0) = \min_{T>0}\Big\{ \max\Big[\max_{|t|\leq T(\varepsilon)} \frac{1}{2}|\varphi_{pz} - \varphi_0(t)|; \frac{1}{T(\varepsilon)}\Big]\Big\}
$$

Proof. (i) From (1.2), (1.4), we have the following estimations

$$|\varphi_{pz}(t)-\varphi_0(t)| = |\varphi_z(pt)-\varphi_0(t)| = \Big|\frac{p\varphi(pt)}{1-q\varphi(pt)}-\varphi_0(t)\Big| = \Big|\frac{p\varphi(pt)}{1-q\varphi(pt)}-\frac{1}{1-it}\Big| = \Big|\frac{p\varphi(pt)-it.p\varphi(pt)-1+q\varphi(pt)}{[1-q\varphi(pt)](1-it)}\Big| \,. \tag{2.2}$$

Let $r(t)=\varphi(t)-\varphi_0(t)$. According to (1.3), there exists $T(\varepsilon)$ with $T(\varepsilon)\to+\infty$ when $\varepsilon\to 0$ such that $|r(t)|\le\varepsilon\ \forall t:|t|\le T(\varepsilon)$. Therefore

$$|r(pt)| = |\varphi(pt)-\varphi_0(pt)|\le\varepsilon \quad \forall t:|t|\le\frac{T(\varepsilon)}{p}\,. \tag{2.3}$$

Hence, from (2.2), we get

$$|\varphi_{pz}(t)-\varphi_0(t)| = \Big|\frac{(1-ipt)[\varphi_0(pt)+r(pt)]-1}{(1-it)[1-q\varphi(pt)]}\Big| = \Big|\frac{1-ipt}{1-it}\Big|\cdot\Big|\frac{r(pt)}{1-q\varphi(pt)}\Big| = \frac{\sqrt{1+p^2t^2}}{\sqrt{1+t^2}}\cdot\frac{|r(pt)|}{|1-q\varphi(pt)|}\,.$$

Notice that $\sqrt{1+p^2t^2}\le\sqrt{1+t^2}$ and $\forall z\in C$, $1-q\le 1-q|z|\le|1-qz|$. So,

$$0<1-q\le 1-q|\varphi(pt)|\le|1-q\varphi(pt)|.$$

Thus,

$$\frac{\sqrt{1+p^2t^2}}{\sqrt{1+t^2}}\cdot\frac{|r(pt)|}{|1-q\varphi(pt)|}\le\frac{|r(t)|}{1-q}\le\frac{\varepsilon}{1-q}=\frac{\varepsilon}{p}\quad\forall t:|t|\le\frac{T(\varepsilon)}{p}\,. \tag{2.4}$$

(ii) Since $F(x)$ is a ε-exponential distribution function, by (1.3) we can find $T(\varepsilon)$ such that

$$\max_{|t|\le T(\varepsilon)}|\varphi(t)-\varphi_0(t)|\le\varepsilon. \tag{2.5}$$

Using (2.4), we obtain

$$\begin{aligned}
&\max\Big\{\max_{|t|\le T(\varepsilon)}\frac{1}{2}|\varphi_{pz}(t)-\varphi_0(t)|\,;\frac{1}{T(\varepsilon)}\Big\}\\
&\le \max\Big\{\max_{|t|\le \frac{T(\varepsilon)}{p}}\frac{1}{2}|\varphi_{pz}(t)-\varphi_0(t)|\,;\frac{1}{T(\varepsilon)}\Big\}\\
&\le \max\Big\{\frac{\varepsilon}{2p}\,;\frac{1}{T(\varepsilon)}\Big\}\,.
\end{aligned}$$

Therefore,

$$\lambda(G_p;F_0) < \max\Big\{\frac{\varepsilon}{2p}\,,\frac{1}{T(\varepsilon)}\Big\}\,.$$

This completes the proof of Theorem 2.1.

Theorem 2.2. *Assume that $\mu_0 = E|x_j| < +\infty$ and $F(x)$ is the ε-exponential distribution function with $T(\varepsilon)$ as in (2.5) which satisfies the condition $T(\varepsilon) = 0(\varepsilon^{-\alpha})$ (for some α and ε sufficiently small). Then*

$$\rho(G_p;F_0) < C_1\varepsilon^{\alpha} + C_2\varepsilon|ln\varepsilon|,$$

where C_1, C_2 are the constants independent of ε and

$$\rho(G_p,F_0) = \sup_{x\in R^1}|F_p(x)-F_0(x)|.$$

Proof. At first, since $F_0(x)$ is exponential distribution function,

$$F_0'(x) = \begin{cases} e^{-x}, & x\ge 0,\\ 0, & x<0.\end{cases}$$

Hence, $\sup\limits_x F_0'(x) = 1$. Using Esseen's inequality (see [1]) with $q(x)$ and $F_0(x)$ we get

$$\begin{aligned}
\rho(G_p;F_0) &< \frac{1}{\pi}\int\limits_{-T(\varepsilon)}^{T(\varepsilon)}\Big|\frac{\varphi_{pz}(t)-\varphi_0(t)}{t}\Big|dt + \frac{24}{\pi T(\varepsilon)}\sup_{x\in\Omega}|F_0'(x)|\\
&= I + \frac{24}{\pi T(\varepsilon)},
\end{aligned}$$

where $T(\varepsilon)$ is defined by (1.3). Hence

$$\begin{aligned}
I &= \frac{1}{\pi} \int_{-T(\varepsilon)}^{T(\varepsilon)} \Big| \frac{\varphi_{pz}(t) - \varphi_0(t)}{t} \Big| dt \\
&= \frac{1}{\pi} \Big[\int_{|t| \le \delta} \ldots dt + \int_{\delta < |t| < T(\varepsilon} \ldots dt \Big] \\
&= \frac{1}{\pi} (I_1 + I_2),
\end{aligned}$$

for some number δ, $0 < \delta < T(\varepsilon)$, which will be chosen later.

In order to estimates I_1, we put

$$\begin{aligned}
I_1^* &= \int_{|t| \le \delta} \Big| \frac{\varphi_{pz}(t) - 1}{t} \Big| dt, \\
I_1^{**} &= \int_{|t| < \delta} \Big| \frac{\varphi_0(t) - 1}{t} \Big| dt.
\end{aligned} \tag{2.7}$$

Then we have

$$I_1 \le I_1^* + I_1^{**}. \tag{2.8}$$

Since there exist the moments $\mu_0 = E|x_j| = 1$; $j = 1, 2, \ldots$, there exist also the moments $\mu_z = E|z|$ and $\mu_{pz} = E|pz|$. Hence

$$\begin{aligned}
|\varphi_z(t) - 1| &\le \mu_z |t|, \quad \forall t \in \mathcal{R}; \\
|\varphi_{pz}(t) - 1| &\le \mu_{pz} \quad \forall t \in \mathcal{R}.
\end{aligned}$$

Therefore

$$\begin{aligned}
I_1^* &\le \int_{|t| \le \delta} \frac{\mu_{pz}|t|}{|t|} dt = 2\mu_{pz}\delta, \\
I_1^{**} &\le \int_{|t| \le \delta} \frac{\mu_0 |t|}{|t|} dt = 2\delta \quad (\mu_0 = 1).
\end{aligned} \tag{2.9}$$

Using (2.8), (2.9), we obtain

$$I_1 < 2(1 + \mu_{pz})\delta.$$

In order to estimate I_2, we notice that if $F(x)$ is a ε-exponential function, then from (2.1) with $T'(\varepsilon) = \dfrac{1}{p}T(\varepsilon)$, and $T'(\varepsilon) > T(\varepsilon)$ (so that $|\varphi_{pz}(t) - \varphi_0(t)| \leq \dfrac{\varepsilon}{p} \ \ \forall |t| \leq T'(\varepsilon)$), we get

$$\begin{aligned} I_2 &\leq \int\limits_{\delta<|t|<T(\varepsilon)} |\ldots| dt \\ &\leq \frac{\varepsilon}{p} \int\limits_{\delta<|t|<T'(\varepsilon)} \frac{1}{|t|} dt \\ &= \frac{2\varepsilon}{p} \int\limits_{\delta}^{T'(\varepsilon)} \frac{dt}{t} = \frac{2}{p}\varepsilon \ln\Big(\frac{T'(\varepsilon)}{\delta}\Big) \cdot \end{aligned}$$

If we choose $\delta = \varepsilon^\beta$ for some $\beta > 0$, we will have the following estimations:

$$I_1 \leq 2(1+\mu_{pz})\varepsilon^\beta, \quad I_2 \leq \frac{2}{p}\varepsilon \Big|\ln \frac{T'(\varepsilon)}{\varepsilon^\beta}\Big| \cdot$$

By using (2.6), we conclude that

$$\begin{aligned} \rho(G_p; F_0) &\leq I + \frac{24}{\pi T(\varepsilon)} \\ &\leq \frac{1}{\pi}(I_1 + I_2) + \frac{24}{\pi + T(\varepsilon)} \\ &\leq \frac{1}{\pi}\big[2(1+\mu_{pz})\varepsilon^\beta\big] + \frac{2}{p\pi}\varepsilon\Big|\ln\frac{T'(\varepsilon)}{\varepsilon^\beta}\Big| + \frac{24}{\pi T(\varepsilon)} \\ &= K(\varepsilon), \end{aligned}$$

where $K(\varepsilon) \to 0$ when $\varepsilon \to 0$.

If $T(\varepsilon) = 0(\varepsilon^{-\alpha})$, then $T'(\varepsilon) = \dfrac{T(\varepsilon)}{p} = 0(\varepsilon^{-\alpha})$ and

$$K(\varepsilon) = C_1\varepsilon^\beta + C_2\varepsilon\Big|\ln\frac{C_3}{\varepsilon^{\alpha+\beta}}\Big| + C_4\varepsilon^\alpha,$$

where C_1, C_2, C_3, C_4 are the constants independent of ε. Since $0 < \delta < T(\varepsilon)$ we can choose $\beta > \alpha$. Then

$$K(\varepsilon) < C_1\varepsilon^\alpha + C_2\varepsilon|\ln\varepsilon^{-2\beta}| + C_4\varepsilon^\alpha < \xi_1\varepsilon^\alpha + \xi_2\varepsilon|\ln\varepsilon|.$$

This completes the proof of Theorem 2.2.

Theorem 2.3. *Assume that $G_{pz}(x)$ is a ε-exponential distribution function for some sufficiently small ε. Then*

$$\text{(2.10)}\qquad \begin{aligned} &\text{(i)}\quad F(x)\ \textit{is a}\ \Big(\frac{\varepsilon}{p-q\varepsilon}\Big)\text{-}\ \textit{exponential distribution function}\\ &\text{(ii)}\quad \lambda(F;F_0)\le \max\Big\{\frac{\varepsilon}{2(p-q\varepsilon)}\,;\frac{1}{pT(\varepsilon)}\Big\}\ , \end{aligned}$$

where $T(\varepsilon)$ is defined as in the definition of the ε-exponential function $G_{pz}(x)$.

Proof. (i) According to the hypothesis, for a given ε, there exists $T = T(\varepsilon)$ such that

$$\text{(2.11)}\qquad \begin{aligned} &|\varphi_{pz}(t)-\varphi_0(t)| = |r(t)| \le \varepsilon \quad \forall t : |t| \le T(\varepsilon),\\ &\Big|r\Big(\frac{t}{p}\Big)\Big| \le \varepsilon \quad \forall t : |t| \le pT(\varepsilon). \end{aligned}$$

By (1.2) we have

$$\varphi_{pz}(t) = \frac{p\varphi(pt)}{1-q\varphi(pt)}\ ,$$
$$\varphi(pt) = \frac{\varphi_{pz}(t)}{p+q\varphi_{pz}(t)}\ ,$$

and

$$\varphi(u) = \frac{\varphi_{pz}\big(\frac{u}{p}\big)}{p+q\varphi_{pz}\big(\frac{u}{p}\big)}\ .$$

Hence,

$$|\varphi(u)-\varphi_0(u)| = \Big|\frac{\varphi_{pz}\big(\frac{u}{p}\big)}{p+q\varphi_{pz}\big(\frac{u}{p}\big)} - \frac{1}{1-iu}\Big|$$
$$= \Big|\frac{\varphi_{pz}\big(\frac{u}{p}\big) - iu\varphi_{pz}\big(\frac{u}{p}\big) - p - q\varphi_{pz}\big(\frac{u}{p}\big)}{(1-iu)\big[p+q\varphi_{pz}\big(\frac{u}{p}\big)\big]}\Big|,$$

(2.12)

$$\Big|\frac{\big[r\big(\frac{u}{p}\big)+\varphi_0\big(\frac{u}{p}\big)\big](p-iu)-p}{(1-iu)\big[p+q\varphi_{pz}\big(\frac{u}{p}\big)\big]}\Big| = \frac{|p-iu|\big|r\big(\frac{u}{p}\big)\big|}{(1-iu)\big[p+q\varphi_{pz}\big(\frac{u}{p}\big)\big]}$$
$$\leq \frac{\big|r\big(\frac{u}{p}\big)\big|}{\big|p+q\varphi_{pz}\big(\frac{u}{p}\big)\big|} .$$

We notice that for all complex numbers u,

$$|u| \geq \max\{|\mathrm{Im}u|; |\mathrm{Re}u|\}.$$

Therefore,

$$\begin{aligned}
\big|p+q\varphi_{pz}\big(\frac{u}{p}\big)\big| &= \big|p+q\big[r\big(\frac{u}{p}\big)+\varphi_0\big(\frac{u}{p}\big)\big]\big| \\
&= \big|p+\frac{qp^2}{p^2+u^2}+qr\big(\frac{u}{p}\big)+iu\frac{pq}{p^2+u^2}\big| \\
&\geq \mathrm{Re}\big\{p+\frac{qp^2}{p^2+q^2}+qr\big(\frac{u}{p}\big)+iu\frac{pq}{p^2+u^2}\big\}\big| \\
&\geq p-q\big|\mathrm{Re}r\big(\frac{u}{p}\big)\big| \qquad (2.13)\\
&\geq p-q\varepsilon, \quad \forall u: |u| \leq pT(\varepsilon).
\end{aligned}$$

From (2.11), (2.12) and (2.13) we can derive (2.10).

(ii) This follows directly from (2.10) and the definition of the metric $\lambda(.,.)$.

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