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ON A CLASS OF NONLINEAR ELLIPTIC EQUATIONS AND BOUNDARY VALUE PROBLEMS IN THE LIMIT DOMAIN

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Abstract. In this paper we study the boundary value problem for a class of high order nonlinear elliptic equations in the bounded domain G_0 with smooth boundary Γ_0 in the space $\mathbf{R}^n$ where the domain G_0 is considered as a limit (in some sense) of the family of domain $\{G_t\}$ which depend smoothly on a parameter $t \in [0,T]$ in S. G. Krein's sense when t tends to 0.

1. Introduction

Let G_0 be a bounded domain with sufficiently smooth boundary γ_0 in the space R^n. Let us consider a differential elliptic operator $L(x,D)$ of order $2m$ and a system $\{B_j(x,D)\}$ of linear differential expressions of order m_j, $j = 1,2,\dots,m$, $m_j \leq 2m-1$, with smooth coefficients in the domain G_0. We consider the following boundary-value problem:

$$L(x,D)u(x) = h(x,u) + f(x) \quad \text{on} \quad G_0 \tag{1.1}$$

$$B_j(x,D)u(x) = g_j(x) \quad \text{on} \quad \Gamma_0 \ (j = 1,2,\dots,m) \tag{1.2}$$

where $h(x,u)$ is an expression of $u(x)$ and x, $f(x)$ and $g_j(x)$ $(j = 1,2,\dots,m)$ are functions given on G_0.

In the linear case, results on boundary value elliptic problems in general form are well-known. These results and their methods have been extended to classes of nonlinear equations. Among the methods used for nonlinear differential equations, the iterative method is the best known one which has often worked for "regularization" of the problems. Results of this nature may be found in [1-5].

The aim of the present paper is to study the existence and the uniqueness of solution of the problem (1.1) (1.2) under suitable hypothesis for $h(x,u)$ and $f(x)$ is a function of a space with weight $H^s_a(G_0)$ (see [10]).

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First of all let us mention some results which will be applied to our arguments. We consider in the domain G_0 a family of domains $\{G_t\}$ with boundaries $\{\Gamma_t\}$ smoothly depending on a parameter $t \in [0, T]$ in S. G. Krein's sense (see [6] and [7]), where $0 < T < 1$. Assume further that

$$G_t \to G_0 \quad \text{as} \quad t \to 0 \tag{1.3}$$

(Therefore (1.1) (1.2) is called the boundary value problem in the limit domain!).

Assumption I. Assume that in each domain G_t, for $t \in [0, T]$, the boundary value problem

$$L(x, D)u(x) = f(x) \quad \text{on} \quad G_t \tag{1.4}$$

$$B_j(x, D)u(x) = g_j(x) \quad \text{on} \quad \Gamma_t \tag{1.5}$$
$$(j = 1, 2, \ldots, m)$$

is a boundary value elliptic problem. Furthermore, for $u(x) \in H^{2m+s}(G_t)$, $s \geq 0$, the following a priori estimate

$$\|u\|_{H^{2m+s}(G_t)} \leq C\Big\{\|Lu\|_{H^s(G_t)} + \sum_{j=1}^{m} \|B_j u\|_{H^{2m+s-m_j-\frac{1}{2}}(\Gamma_t)}\Big\} \tag{1.6}$$

holds, where C is a constant.

It follows readily from the a priori estimate (1.6) that the problem (1.4)-(1.5) in each domain G_t for $t \in [0, T]$ has a unique solution.

Let us first make the following remark.

Remark. In general, the constant C in the a priori estimate (1.6) depends on $t \in [0, T]$, i.e. $C = C(t)$ for $t \in [0, T]$.

Throughout this paper we assume that the function $C(t)$ in (1.6) is constant.

Let A_0 denote a unbounded operator defined by the problem with homogeneous boundary value condition in the limit domain G_0:

$$L(x, D)u(x) = f(x) \quad \text{on} \quad G_0,$$
$$B_j(x, D)u(x) = 0 \quad \text{on} \quad \Gamma_0, \ (j = 1, 2, \ldots, m).$$

For $s \geq 0$ we set

$$H^{2m+s}_{\Gamma_0}(G_0) = \Big\{u(x) \in H^{2m+s_0}(G_0) : B_j(x, 0)u(x) = 0 \quad \text{on } \Gamma_0\Big\}.$$

Then A_0 is an operator mapping from $H^{2m+s}_{\Gamma_0}(G_0)$ to $H^s(G_0)$ according to the following formula

$$H^{2m+s}_{\Gamma_0}(G_0) \ni u(x) \mapsto A_0 u = Lu \in H^s(G_0). \tag{1.7}$$

From that the operator A_0 is invertible, and we denote by A_0^{-1} its bounded inverse operator. There is the following estimate:

$$\|u\|_{H^{2m+s}(G_0)} \le \|A_0^{-1}\|_S \cdot \|Lu\|_{H^s(G_0)}, \tag{1.8}$$

for $u(x) \in H^{2m+s}_{\Gamma_0}(G_0)$ and $s \ge 0$, where $\|A_0^{-1}\|_S$ is the norm of the operator A_0^{-1}.

For $a \ge 0$ we denote by $H^S_a(G_0)$, $s \ge 0$, the set of all functions $f(x)$ defined in G_0 such that their restrictions to each domain G_t for $t \in (0,T]$ belong to $H^S(G_t)$ (see [9] and [10]) and the following condition holds:

$$\|f\|_{H^s_a(G_0)} = \sup_{0<t\le T} t^a \|f\|_{H^s(G_t)} < +\infty. \tag{1.9}$$

We have proved (see [10]) that for $f(x) \in H^s_a(G_0)$, $a \ge 0$, $s \ge 0$, the quantity $\|f\|_{H^S_a(G_0)}$ defines a norm in $H^S_a(G_0)$. Moreover, we have

(i) $H^s(G_0) \subset H^s_a(G_0), \quad a \ge 0,\ \forall s \ge 0,$

(ii) $f(x) \in H^s(G_t)$ if $f(x) \in H^s_a(G_0)\ \ t \in (0,T]$.

It may happen that as $t \to 0$, $\|f\|_{H^s(G_t)}$ tends to the infinity provided that

$$\|f\|_{H^s(G_t)} \le G.t^{-a} \quad (t \to 0).$$

Thus, in a neighborhood of the boundary Γ_0 these functions may be unbounded.

Furthermore, the space $H^S_a(G_0)$ defined as above can be considered as a Sobolev-Slobodeski's space with weight.

2. Boundary-value problem in the limit domain

$$\text{Let}\quad f(x) \in H^s_a(G_0), \quad a > 0, \quad g_j(x) = H^{2m+s-m_j}(G_0),$$

$j = 1, 2, \ldots, m$, $s \ge 0$. We consider the following boundary-value problem in the domain G_0:

$$L(x,D)u(x) = f(x) \text{ in } G_0, \tag{2.1}$$

$$B_j(x,D)u(x) = g_j(x) \text{ on } \Gamma_0,\ (j = 1,2,\dots,m). \tag{2.2}$$

Definition 2.1. A function $u(x) \in H^{2m+s}(G_0)$, $s \geq 0$, is said to be a solution of the problem (2.1)-(2.2) if the following conditions are satisfied:

$$\lim_{t\to 0} \|Lu - f\|_{H^s(G_t)} = 0,$$

$$\|B_j u - g_j\|_{H^{2m+s-m_j-\frac{1}{2}}(\Gamma_c)} = 0, \quad (j = 1,2,\dots,m). \tag{2.3}$$

Recall that if $f(x) \in H^s_a(G_0)$, then its restriction to G_t belongs to $H^s(G_t)$ for $t \in (0,T]$.

Let us denote by $u(t,x)$ the unique solution of the problem (1.4) (1.5) in the domain G_t for $t \in (0,T]$:

$$\begin{aligned} L(x,D)u(t,x) &= f(x) \quad \text{in } G_t,\\ B_j(x,D)u(t,x) &= g_j(x) \quad \text{on } \Gamma_t,\ (j = 1,2,\dots,m). \end{aligned}$$

Then $u(t,x) \in H^{2m+s}(G_t)$ and the following estimate holds:

$$\|u\|_{H^{2m+s}(G_t)} \leq C\Big\{\|f\|_{H^s(G_t)} + \sum_{j=1}^{m} \|g_j\|_{H^{2m+s-m_j-\frac{1}{2}}(\Gamma_t)}\Big\}, \tag{2.4}$$

where C is a constant, $s \geq 0$.

Moreover, for $a > 0$ we have

$$\|u\|_{H^{2m+s}(G_t)} \leq C.t^{-a}\Big\{t^a\|f\|_{H^s(G_t)} + \sum_{j=1}^{m} \|g_j\|_{H^{2m+s-m_j-\frac{1}{2}}(\Gamma_t)}\Big\},$$

for $t \in (0,T]$.

Under the hypothese $f(x) \in H^s_a(G_0)$, $a > 0$, from the last inequality we obtain the following estimate for the solution $u(t,x)$

$$\|u\|_{H^{2m+s}(G_t)} \leq \mathcal{C}(t)\Big\{\|f\|_{H^s_a(G_0)} + \sum_{j=1}^{m} \|g_j\|_{H^{2m+s-m_j}(G_0)}\Big\}, \tag{2.5}$$

for $s \geq 0$, where $\mathcal{C}(t)$ satisfies the asymptotic estimate

$$\mathcal{C}(t) = 0(t^{-a}),\ t \to 0,\ a > 0. \tag{2.6}$$

On the other hand, as indicated in [7], the solution $u(t,x)$ can be extended to a function in $H^{2m+s}(G_0)$

$$u_t(t,x) = R_t u(t,x)$$

by an operator of extension R_t. Moreover, the operator of extension R_t can be always chosen as linear operator mapping from $H^{2m+s}(G_t)$ to $H^{2m+s}(G_0)$ which is uniformly bounded in norm for all $t \in (0,T]$, $0 \leq 2m+s \leq N$, where N is a sufficiently large natural number.

Under the stated assumption we have prove (see in [9], [10]) that if

$$f(x) \in H^s_a(G_0), \quad g_j(x) \in H^{2m+s-m_j}(G_0),$$

$j = 1, 2, \ldots, m$, for $s \geq k_0+1$, $2m+s \leq N$ and $k_0 = [\frac{a}{1-a}]+1$, $0 < a < 1$, then $u_t(t,x)$ is considered as an abstract function defined in the domain $(0,T] \times G_0$ with values in the space $H^{2m+s-(k_0+1)}(G_0)$ which satisfies the following estimate

$$\begin{aligned} \left\|u_\tau - u_t\right\|_{H^{2m+s-(k_0+1)}(G_0)} &\leq \mathcal{C}.|t-\tau|\Big\{\left\|f\right\|_{H^s_a(G_0)} \\ &\qquad + \sum_{j=1}^{m} \left\|g_j\right\|_{H^{2m+s-m_j}(G_0)}\Big\}, \end{aligned} \tag{2.7}$$

for all $\tau, t \in (0,T]$, where $\mathcal{C}$ is constant.

It follows readily from (2.7) that there exists a limit

$$\lim_{t\to 0} u_t(t,x) = u_0(x) \text{ in } H^{2m+s-(k_0+1)}(G_0) \tag{2.8}$$

i.e.

$$\lim_{t\to 0} \left\|u_t - u_0\right\|_{H^{2m+s-(k_0+1)}(G_0)} = 0.$$

Finally, we have the following theorem on the existence and the uniqueness of the solution of the problem (2.1)-(2.2) (see [9], [10]).

Theorem 2.1. *Let* $f(x) \in H^s_a(G_0)$, $g_j(x) \in H^{2m+s-m_j}(G_0)$, $j = 1, 2, \ldots, m$, $s \geq k_0 + 1$, $k_0 = [\frac{a}{1-a}] + 1$, $0 < a < 1$. *Then the function* $u_0(x) \in H^{2m+s-(k_0+1)}(G_0)$ *defined by* (2.8) *is the unique solution of the problem* (2.1)-(2.2), *and there is the following estimates:*

$$\|u_0\|_{H^{2m+s-(k_0+1)}(G_0)} \leq C\Big\{\|f\|_{H^s_a(G_0)} + \sum_{j=1}^{m} \|g_j\|_{H^{2m+s-m_j}(G_0)}\Big\}, \tag{2.9}$$

where C *is a constant.*

3. Boundary-value problem for nonlinear elliptic equations

Let $f(x) \in H^s_a(G_0)$, $g_j(x) \in H^{2m+s-m_j}(G_0)$,

$j = 1, 2, \ldots, m$, $s \geq k_0 + 1$, $k_0 = [\frac{a}{1-a}] + 1$, $0 < a < 1$.

We consider in G_0 the following problem:

$$L(x, D)u(x) = h(x, u) + f(x) \quad \text{in } G_0, \tag{3.1}$$

$$B_j(x, D)u(x) = g_j(x) \quad \text{on } \Gamma_0, \ (j = 1, 2, \ldots, m) \tag{3.2}$$

under suitable hypotheses for $h(x, u)$ as follows.

Assumption II. Assume that the expression $h(u, x)$ satisfies the following conditions:

(i) For $u(x) \in H^s(G_0)$, $h(x, u) \in H^s(G_0)$, $s \geq 0$,

(ii) For $u(x)$, $v(x) \in H^s(G_0)$,

$$\|h(., u) - h(., v)\|_{H^s(G_0)} \leq M.\|u - v\|_{H^s(G_0)}, \tag{3.3}$$

where M is a positive constant such that

$$\mathcal{E} = M\|A_0^{-1}\|_s < 1. \tag{3.4}$$

Definition 3.1. A function $u(x) \in H^{2m+s}(G_0)$ is called a solution of the problem (3.1)-(3.2) if

$$\lim_{t \to 0} \|Lu - h(., u) - f\|_{H^s(G_t)} = 0,$$

$$\|B_j u - g_j\|_{H^{2m+s-m_j-\frac{1}{2}}(G_0)} = 0 \quad (j = 1, 2, \ldots, m).$$

In the sequel, the solution $u(x)$ of the problem (3.1) (3.2) will be written in the following form

$$u(x) = v(x) + u_0(x),$$

where $u_0(x)$ is the unique solution of the problem (2.1) (2.2), and $v(x)$ is a solution of the following problem:

$$L(x, D)v(x) = h(x, v + u_0) \quad \text{on } G_0, \tag{3.5}$$

$$B_j(x, D)v(x) = 0 \quad \text{on } \Gamma_0, \ (j = 1, 2, \ldots, m). \tag{3.6}$$

Recall that the existence and the uniqueness of the solution $u_0(x)$ have been proved by Theorem 2.1 and this solution is defined by (2.8).

Now we will prove analogous result for the problem (3.5)-(3.6).

Theorem 3.1. *Under assumption II the problem* (3.5)-(3.6) *has a unique solution* $v(x) \in H^{2m+s-(k_0+1)}(G_0)$.

Proof. The main tool for our proof will be the iterative method. By Theorem 2.1, the function $u_0(x)$ which has been defined by (2.8) is the unique solution of the problem (2.1) (2.2) in $H^{2m+s-(k_0+1)}(G_0)$, $s-(k_0+1) \geq 0$, $k_0 = [\frac{a}{1-a}] + 1$.

For simplicity of notation we put $s_0 = s - (k_0 + 1)$. Then $s_0 \geq 0$, and $u_0(x) \in H^{2m+s_0}(G_0)$, hence $u_0(x) \in H^{s_0}(G_0)$. Therefore, under assumption II, $h(x, u_0) \in H^{s_0}(G_0)$.

Let $v_1(x)$ be a unique solution of the following boundary-value elliptic problem.

$$L(x, D)v_1(x) = h(x, u_0) \quad \text{on } G_0,$$
$$B_j(x, D)v_1(x) = 0 \quad \text{on } \Gamma_0, \ (j = 1, 2, \ldots, m).$$

Then $v_1(x) \in H^{2m+s_0}(G_0)$. Therefore, $v_1(x)$ together with $v_1(x) + u_0(x)$ belong to $H^{s_0}(G_0)$.

Let $v_2(x)$ be the unique solution of the problem:

$$L(x, D)v_2(x) = h(x, v_1 + u_0) \quad \text{on } G_0,$$
$$B_j((x, D)v_2(x) = 0 \quad \text{on } \Gamma_0, \ (j = 1, 2, \ldots, m).$$

Similarly as above, we inductively define a sequence $\{v_n(x)\}$ $(n = 1, 2, \dots)$, where $v_n(x)$ is the unique solution of the problem:

$$L(x, D)v_n(x) = h(x, v_{n-1} + u_0) \quad \text{in } G_0,$$
$$B_j(x, D)v_n(x) = 0 \quad \text{on } \Gamma - 0, \ (j = 1, 2, \dots, m).$$

We remark that for all n, $v_n(x) \in H^{2m+s_0}(G_0)$, hence $v_n - v_{n-1} \in H^{2m+s_0}(G_0)$. Therefore, from the estimate (1.8) we have

$$\|v_n - v_{n-1}\|_{H^{2m+s_0}(G_0)} \leq \|A_0^{-1}\|_{s_0} \|L(v_n - v_{n-1})\|_{H^{s_0}(G_0)}, \tag{3.7}$$

for $n = 2, 3, 4, \dots$. Applying (3.7) and (3.3) for $n = 2$ one sees that

$$\begin{aligned}
&\|v_2 - v_1\|_{H^{2m+s_0}(G_0)} \\
&\leq \|A_0^{-1}\|_{s_0}.\|L(v_2 - v_1)\|_{H^{s_0}(G_0)} \\
&\leq \|A_0^{-1}\|_{s_0}.\|h(., v_1 + u_0) - h(., u_0)\|_{H^{s_0}(G_0)} \\
&\leq \|A_0^{-1}\|_{s_0}.M.\|(v_1 + u_0) - u_0\|_{H^{s_0}(G_0)}
\end{aligned}$$

i.e. we have

$$\|v_2 - v_1\|_{H^{2m+s_0}(G_0)} \leq M.\|A_0^{-1}\|_{s_0}.\|v_1\|_{H^{s_0}(G_0)}. \tag{3.8}$$

Applying the estimates (3.7), (3.3) and (3.8) successively we have

$$\begin{aligned}
&\|v_3 - v_2\|_{H^{2m+s_0}(G_0)} \leq \|A_0^{-1}\|_{s_0}.\|L(v_3 - v_2)\|_{H^{s_0}(G_0)} \\
&\leq \|A_0^{-1}\|_{s_0}.\|h(., v_2 + u_0) - h(., v_1 + v_0)\|_{H^{s_0}(G_0)} \\
&\leq \|A_0^{-1}\|_{s_0}.M.\|v_2 - v_1\|_{H^{s}(G_0)} \leq (M.\|A_0^{-1}\|_{s_0})^2.\|v_1\|_{H^{s_0}(G_0)}.
\end{aligned}$$

In the same way, for $n = 4, 5, \dots$, we obtain the estimate

$$\|v_n - v_{n-1}\|_{H^{2m+s_0}(G_0)} \leq (M.\|A_0^{-1}\|_{s_0})^n.\|v_1\|_{H^{s_0}(G_0)}. \tag{3.9}$$

Observe further that:

$$\|v_1\|_{H^{2m+s_0}(G_0)} \leq \|A_0^{-1}\|_{s_0}.\|Lv_1\|_{H^{s_0}(G_0)} = \|A_0^{-1}\|_{s_0}.\|h(., u_0)\|_{H^{s_0}(G_0)}.$$

Then the estimate (3.9) can be written as follows:

$$\|v_n - v_{n+1}\|_{H^{2m+s_0}(G_0)} \leq \mathcal{E}^n \|A_0^{-1}\|_{s_0}.\|h(., u_0)\|_{H^{s_0}(G_0)}, \tag{3.10}$$

for $n = 1, 2, \dots,$

where

$$\mathcal{E} = M\|A_0^{-1}\|_{s_0} < 1. \tag{3.11}$$

The inequality (3.10) allows us to estimate differences $v_n - v_m$ of the functions v_n and v_m of the sequence $\{v_n\}$ according to the norm in the space $H^{2m+s_0}(G_0)$.

First of all, it holds that

$$\begin{aligned}
&\|v_n - v_m\|_{H^{2m+s_0}(G_0)} \\
&\le \|v_n - v_{n+1}\|_{H^{2m+s_0}(G_0)} + \|v_{n+1} - v_{n+2}\|_{H^{2m+s_0}(G_0)} \\
&+ \dots + \|v_{m-1} - v_m\|_{H^{2m+s_0}(G_0)}.
\end{aligned}$$

Applying the estimates (3.10) to each member in the right hand side of the last inequality we obtain, for $m > n$,

$$\begin{aligned}
&\|v_n - v_m\|_{H^{2m+s_0}(G_0)} \\
&\le \mathcal{E}^n\|A_0^{-1}\|_{s_0}.\|h(.,u_0)\|_{H^{s_0}(G_0)} + \mathcal{E}^{n+1}\|A_0^{-1}\|_{s_0}.\|h(.,u_0)\|_{H^{s_0}(G_0)} \\
&+ \dots + \mathcal{E}^{m-1}\|A_0^{-1}\|_{s_0}.\|h(.,u_0)\|_{H^{s_0}(G_0)} \\
&\le \mathcal{E}^n(1 + \mathcal{E} + \dots + \mathcal{E}^{m-n-1} + \dots).\|A_0^{-1}\|_{s_0}.\|h(.,u_0)\|_{H^{s_0}(G_0)}.
\end{aligned}$$

By condition (3.4) it follows that

$$\|v_n - v_m\|_{H^{2m+s_0}(G_0)} \le \frac{\|A_0^{-1}\|_{s_0}}{1-\mathcal{E}}.\mathcal{E}^n\|h(.,u_0)\|_{H^{s_0}(G_0)} \tag{3.12}$$

for all n, m $(m > n)$.

The estimate (3.12) shows that $\{v_n(x)\}$, $n = 1, 2, \dots,$ is a fundamental sequence in the space $H^{2m+s_0}(G_0)$. Therefore, there exists a limit

$$\lim_{n\to+\infty} v_n = v \quad in \quad H^{2m+s_0}(G_0), \tag{3.13}$$

i.e.

$$\lim_{n\to+\infty} \|v_n - v\|_{H^{2m+s_0}(G_0)} = 0.$$

In the sequel we will prove that the function $v(x)$ defined by (3.13) is a unique solution of the problem (3.5) (3.6).

It is clear that

$$\left\|h(., v_n + u_0) - h(., v + u_0)\right\|_{H^{s_0}(G_0)} \leq M.\left\|v_n - v\right\|_{H^{s_0}(G_0)}.$$

Further, we have

$$L(x, D)v_n(x) = h(x, v_{n-1} + u_0),$$

for $n = 2, 3, \ldots$. Therefore,

$$\begin{aligned}
&\left\|L(x, D)v(x) - h(x, v + u_0)\right\|_{H^{s_0}(G_0)} \\
&\leq \left\|L(x, D)v(x) - L(x, D)v_n\right\|_{H^{s_0}(G_0)} \\
&+ \left\|h(., v_{n-1} + u_0) - h(., v + u_0)\right\|_{H^{s_0}(G_0)} \\
&\leq C\left\|v_n - v\right\|_{H^{s_0+2m}(G_0)} + M.\left\|v_{n-1} - v\right\|_{H^{s_0}(G_0)},
\end{aligned}$$

where C is a constant.

Letting $n \to +\infty$ and using (3.13) we obtain from the last estimate the following equality

$$L(x, D)v(x) = h(x, v + u_0) \quad \text{in } H^{s_0}(G_0).$$

In other words, $v(x)$ is a solution of the equation (3.5). We now verify the boundary condition (3.6).

First, it is obvious that

$$\left\|B_j(x, D)v_n(x) - B_j(x, D)v(x)\right\|_{H^{2m+s_0-m_j-\frac{1}{2}}(\Gamma_0)} \leq C\left\|v_n - v\right\|_{H^{2m+s_0}(G_0)},$$

where C is a constant, $n = 1, 2, \ldots$; $j = 1, 2, \ldots, m$.

Therefore, from these inequalities we get, as $n \to +\infty$,

$$\lim_{n\to+\infty} \left\|B_j(x, D)v_n(x) - B_j(x, D)v(x)\right\|_{H^{2m+s_0-m_j-\frac{1}{2}}(\Gamma_0)} = 0.$$

Since $B_j(x, D)v_n(x) = 0$ on Γ_0 for all $n \geq 1$, it follows that

$$B_j(x, D)v(x) = 0 \quad \text{on } \Gamma_0.$$

In other words, $v(x)$ satisfies the conditions (3.6).

Thus, we have proved that $v(x)$ is a solution of the boundary value problem (3.5)-(3.6).

To complete the proof it remains to show the uniqueness of the solution $v(x)$.

To this end, suppose that $\tilde{v}(x) \in H_{\Gamma_0}^{2m+s_0}(G_0)$ is a solution of the equation (3.5), i.e.,

$$L(x,D)\tilde{v}(x) = h(x, \tilde{v} + u_0) \quad \text{on } G_0,$$

such that $\tilde{v}(x) \not\equiv v(x)$. By applying the estimate (1.8) to $v(x) - \tilde{v}(x) \in H_{\Gamma_0}^{2m+s_0}(G_0)$ we have

$$\begin{aligned}
\|v - \tilde{v}\|_{H^{2m+s_0}(G_0)} &\leq \|A_0^{-1}\|_{s_0}.\|L(v - \tilde{v})\|_{H^{s_0}(G_0)} \\
&\leq \|A_0^{-1}\|_{s_0}.\|h(.,v + u_0) - h(.,\tilde{v} + u_0)\|_{H^{s_0}(G_0)} \\
&\leq M.\|A_0^{-1}\|_{s_0}.\|v - \tilde{v}\|_{H^{s_0}(G_0)} \\
&\leq M\|A_0^{-1}\|.\|v - \tilde{v}\|_{H^{2m+s_0}(G_0)}.
\end{aligned}$$

Since $\|v - \tilde{v}\|_{H^{2m+s_0}(G_0)} \neq 0$, it follows that

$$\mathcal{E} = \|A_0^{-1}\|_{s_0}.M \geq 1,$$

which contradicts the hypothese (3.4). The proof of the theorem 3.1 is completes.

Theorem 3.2. *For* $0 < a < 1$, $k_0 = [\dfrac{a}{1-a}] + 1$, *let*

$$f(x) \in H_a^s(G_0), \quad g_j(x) \in H^{2m+s-m_j}(G_0),$$

$j = 1, 2, \ldots, m$, $s \geq k_0 + 1$. *Let* $u_0(x)$ *be the unique solution of the problem* (2.1)-(2.2) *in* $H^{2m+s-(k_0+1)}(G_0)$, *and* $v(x)$ *the unique solution of the problem* (3.5)-(3.6) *in* $H^{2m+s-(k_0+1)}(G_0)$. *Then the function*

$$u(x) = v(x) + u_0(x) \tag{3.14}$$

is a solution of the problem (3.1)-(3.2) *in* $H^{2m+s-(k_0+1)}(G_0)$.

Proof. According to Theorems 2.1 and 3.1, the unique solution $u_0(x)$ of the problem (2.1)-(2.2) has been defined by (2.8) and the unique solution

$v(x)$ of the problem (3.5)-(3.6) has been defined by (3.13). Both of them belong to $H^{2m+s_0}(G_0)$ with $s_0 = s - (k_0 + 1) \geq 0$.

Let us consider the function $u(x) = v(x) + u_0(x)$. It is obvious that

$$\begin{aligned} Lu &= L(v + u_0) = Lv + Lu_0 = h(x, v + u_0) + Lu_0 \\ &= h(x, u) + Lu_0. \end{aligned}$$

Therefore, for $t \in (0, T]$ in the domain G_t we have

$$\begin{aligned} Lu - h(x, u) - f(x) &= h(x, u) + Lu_0 - h(x, u) - f(x) \\ &= Lu_0 - f(x). \end{aligned}$$

Then

$$\|Lu - h(., u) - f\|_{H^{s_0}(G_t)} = \|Lu_0 - f\|_{H^{s_0}(G_t)}.$$

On the other hand, since $u_0(x) \in H^{2m+s_0}(G_0)$ is the solution of the problem (2.1)-(2.2) in the sense of the definition 2.1, from (2.3) we get

$$\lim_{t \to 0} \|Lu_0 - f\|_{H^{s_0}(G_t)} = 0.$$

Therefore

$$\lim_{t \to 0} \|Lu - h(., u) - f\|_{H^{s_0}(G_t)} = 0.$$

We have thus proved that $u(x) = v(x) + u_0(x)$ satisfies the equation (3.1) (in the sense of the definition 3.1).

Moreover, it is clear that

$$B_j(x, D)u(x) = B_j(x, D)v(x) + B_j(x, D)u_0(x) = 0 + g_j(x) = g_j(x) \text{ on } \Gamma_0$$

for $j = 1, 2, \dots, m$. Thus, the function $u(x) = v(x) + u_0(x)$ satisfies the condition (3.2). This completes the proof of Theorem 3.2.

Remark. We can estimate the solution $u(x) = v(x) + u_0(x)$ of the problem (3.1) (3.2) as follows. Firstly, using (3.10), (3.11) for $v(x)$ we have

$$\begin{aligned} \|v(x)\|_{H^{2m+s_0}(G_0)} &\leq \|v - v_n\|_{H^{2m+s_0}(G_0)} + \|v_n - v_{n_1}\|_{H^{2m+s_0}(G_0)} \\ &\quad + \cdots + \|v_2 - v_1\|_{H^{2m+s_0}(G_0)} + \|v_1\|_{H^{2m+s_0}(G_0)} \\ &\leq \|v - v_n\|_{H^{2m+s_0}(G_0)} + (\mathcal{E}^{n-1} + \mathcal{E}^{n-2} + \cdots + \mathcal{E} + 1)\|v_1\|_{H^{2m+s_0}(G_0)} \\ &\leq \|v - v_n\|_{H^{2m+s_0}(G_0)} + \frac{\|A_0^{-1}\|_{s_0}}{1 - \mathcal{E}} \cdot \|h(., u_0)\|_{H^{s_0}(G_0)}. \end{aligned}$$

Letting $n \to +\infty$ we get

$$\|v\|_{H^{2m+s_0}(G_0)} \leq \frac{\|A_0^{-1}\|_{s_0}}{1-\mathcal{E}} \cdot \|h(.,u_0)\|_{H^{s_0}(G_0)}.$$

Finally, for $u(x) = v(x) + u_0(x)$ we obtain

$$\|u\|_{H^{2m+s_0}(G_0)} \leq \|u_0\|_{H^{2m+s_0}(G_0)} + \frac{\|A_0^{-1}\|_{s_0}}{1-\mathcal{E}} \|h(.,u_0)\|_{H^{s_0}(G_0)},$$

where $\mathcal{E} = M.\|A_0^{-1}\|_{s_0} < 1$.

Theorem 3.3. *Under the hypotheses of Theorem 3.2 the function $u(x)$ which has been defined by* (3.14) *is the unique solution of the problem* (3.1)-(3.2).

Proof. We only have to prove the uniqueness of the solution $u(x) = v(x) + u_0(x)$. Suppose that there is another any solution $\tilde{u}(x) \in H^{2m+s_0}(G_0)$ of the problem (3.1)-(3.2), such that $\tilde{u}(x) \not\equiv u(x)$. Then

$$\begin{aligned} &\lim_{t\to 0} \|L\tilde{u} - h(.,\tilde{u}) - f\|_{H^{s_0}(G_t)} = 0, \\ &\|B_j\tilde{u} - g_j\|_{H^{2m+s_0-m_j-\frac{1}{2}}(\Gamma_0)} = 0 \quad (j = 1, 2, \ldots, m), \end{aligned}$$

we have

$$u(x) - \tilde{u}(x) \in H^{2m+s_0}_{\Gamma_0}(G_0).$$

By applying (1.8) to the function $u(x) - \tilde{u}(x)$ we get

$$\|u - \tilde{u}\|_{H^{2m+s_0}(G_0)} \leq \|A_0^{-1}\|_{s_0}.\|L(u-\tilde{u})\|_{H^{s_0}(G_0)}. \tag{3.15}$$

On the other hand, we have

(3.16)
$$\begin{aligned} &\|L(u-\tilde{u})\|_{H^{s_0}(G_0)} \leq \|L(u-\tilde{u})\|_{H^{s_0}(G_0-G_t)} + \|L(u-\tilde{u})\|_{H^{s_0}(G_t)} \\ &\leq \|L(u-\tilde{u})\|_{H^{s_0}(G_0-G_t)} \\ &\quad + \|Lu - h(.,u) - f + [h(.,u) - h(.,\tilde{u})] - [L\tilde{u} - h(.,\tilde{u}) - f]\|_{H^{s_0}(G_t)} \\ &\leq \|L(u-\tilde{u})\|_{H^{s_0}(G_0-G_t)} + \|Lu - h(.,u) - f\|_{H^{s_0}(G_t)} \\ &\quad + \|h(.,u) - h(.,\tilde{u})\|_{H^{s_0}(G_t)} + \|L\tilde{u} - h(.,\tilde{u}) - f\|_{H^{s_0}(G_t)}. \end{aligned}$$

Recall that under the definition 3.1,

$$\lim_{t\to 0}\big\|Lu - h(.,u) - f\big\|_{H^{s_0}(G_t)} = 0,$$
$$\lim_{t\to 0}\big\|L\tilde{u} - h(.,\tilde{u}) - f\big\|_{H^{s_0}(G_t)} = 0.$$

Moreover, $\lim\limits_{t\to 0}\big\|L(u-\tilde{u})\big\|_{H^{s_0}(G_0-G_t)} = 0$ because $L(u-\tilde{u}) \in H^{s_0}(G_0)$. Therefore, from (3.16) we obtain, as $n \to +\infty$,

$$\big\|L(u-\tilde{u})\big\|_{H^{s_0}(G_0)} \le \big\|h(.,u) - h(.,\tilde{u})\big\|_{H^{s_0}(G_0)}. \tag{3.17}$$

Combining (3.15) and (3.17) gives

$$\big\|u-\tilde{u}\big\|_{H^{2m+s_0}(G_0)} \le \big\|A_0^{-1}\big\|_{s_0}\big\|h(.,u) - h(.,\tilde{u})\big\|_{H^{s_0}(G_0)}.$$

Applying (3.3) to the right hand side of the last estimate we have

$$\begin{aligned}\big\|u-\tilde{u}\big\|_{H^{2m+s_0}(G_0)} &\le M\big\|A_0^{-1}\big\|_{s_0}\big\|u-\tilde{u}\big\|_{H^{s_0}(G_0)}\\ &\le M\big\|A_0^{-1}\big\|_{s_0}\big\|u-\tilde{u}\big\|_{H^{2m+s_0}(G_0)}.\end{aligned}$$

It follows that

$$\mathcal{E} = M\big\|A_0^{-1}\big\|_{s_0} \ge 1,$$

which contradicts the hypothesis (3.4). The proof of the theorem 3.3 is complete.

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