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WEAK EXTENSION OF FRECHET-VALUED HOLOMORPHIC FUNCTIONS ON COMPACT SETS AND LINEAR TOPOLOGICAL INVARIANTS

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Abstract. It is shown that every holomorphic function on a nuclear Frechet space E with values in a Frechet space F is of uniform type if E has the linear topological invariant $(\widetilde{\Omega})$ and F has the linear topological invariant (DN) respectively. Based on the obtained result the equivalence of the holomorphicity and the weak holomorphicity of Frechet-valued functions on $\widetilde{L}$-regular compact subsets in a nuclear Frechet space is established.

1. Introdution

Let E be a Frechet space with a fundamental system of semi-norms $\{\, \|\cdot\|_k \,\}$. For each subset B of E, we define $\|\cdot\|^*_B\colon E^* \longrightarrow [0, +\infty]$ by

$$\|u\|^*_B = \sup\{|u(x)| : x \in B\}$$

where $u \in E^*$, the strongly dual space of E. Instead of $\|\cdot\|^*_{U_q}$ we write $\|\cdot\|^*_q$, where

$$U_q = \{x \in E \ : \ \|x\|_q \le 1\}.$$

Now we consider the following properties of E:

$$(DN)\ \exists p\ \exists d > 0\ \forall q\ \exists k,\ C > 0\ :\ \|\cdot\|_q^{1+d} \le C\|\cdot\|_k\ \|\cdot\|_p^d;$$

$$(\widetilde{\Omega})\ \forall p\ \exists q,\ d > 0\ \forall k\ \exists C > 0\ :\ \|\cdot\|_q^{*1+d} \le C\|\cdot\|_k^*\ \|\cdot\|_p^{*d}.$$

The above properties have been introduced and investigated by Vogt (see [14], [15], [16], [17], [18]). Note that the following equivalent form of the property (DN) has been formulated by Zaharjuta in [21]

$$(DN)_Z\ \exists p\ \forall q,\ d > 0\ \exists k,\ C > 0 : \|\cdot\|_q^{1+d} \le C\|\cdot\|_k\|\cdot\|_p^d.$$

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Let E and F be locally convex spaces and $f : E \longrightarrow F$ a holomorphic function. We say that f is of uniform type if there exists a continuous semi-norm ρ on E such that f can be factorized holomorphically through the canonical map $\omega_\rho : E \longrightarrow E_\rho$, where E_ρ is the Banach space associated with ρ. The uniformity of holomorphic functions on nuclear Frechet spaces $E \in (\overline{\overline{\Omega}})$ with values in the spaces of holomorphic functions on open subsets of $\mathbf{C}^n$ has been established by Meise-Vogt [6]. In the second section of the present paper we investigate the equality

$$H_u(E, F) = H(E, F), \tag{1}$$

where $H_u(E, F)$ denotes the set of Frechet-valued holomorphic functions of uniform type and $H(E, F)$ the set of Frechet-valued holomorphic functions on E, where E and F are Frechet spaces and, moreover, E and F have some linear topological invariants.

Applying the result of the second section, in the third section we investigate the relation between the weakly holomorphic extendability and the holomorphic extendability of Frechet-valued holomorphic functions on $\tilde{L}$-regular compact sets in nuclear Frechet spaces. The main aim of this section is to find some necessary and sufficient conditions for which

$$H(X, F) = H_w(X, F) \tag{2}$$

where $H(X, F)$ denotes the set of germs of Frechet-valued holomorphic functions on a $\tilde{L}$-regular compact set X in a nuclear Frechet space E and $H_w(X, F)$ is the set of Frechet-valued weakly holomorphic functions. In [9] N. V. Khue and B. D. Tac have shown that (2) holds in the case where X is compact, F^* is a Baire space and either E is a nuclear metric space or F is nuclear. Some more special cases for which (2) holds have been early established by Siciak [12], Waelbroeck [19]. In the case X is an open set and F^* is still Baire, (2) was first proved by Bogdanowicz [2], and then by Ligocka-Siciak [8], and in the more general case by Nguyen Thanh Van [10]. The Baireness of F^* plays a very important role in the works of the above authors. However, when F^* is not Baire, in particular, when F is a Frechet space which is not Banach, so far (2) has not been established.

2. Uniformity of holomorphic functions

In this section we shall prove the following

Theorem 2.1. *Let E be a nuclear Frechet space and F a Frechet space. Then*

$$H_u(E, F) = H(E, F)$$

if $E \in (\widetilde{\Omega})$ *and* $F \in (DN)$.

Proof. As in [6] it suffices to prove the following interpolation lemma. In the case of scalar holomorphic functions this lemma has been proved by Meise-Vogt [6]. Here by improving the proof of Meise-Vogt we extend the result to the case of Frechet-valued holomorphic functions.

Lemma 2.2 (Interpolation lemma). *Let* Y *and* Z *be Hilbert spaces,* X *a Banach space and* F *a Frechet space having the property* (DN). *Assume that the following hypotheses are satisfied:*

(i) $A \in \mathcal{L}(Y, Z)$ *is injective, of type* s *and* $A = v \circ u$, *where* $u \in \mathcal{L}(Y, X)$ *and* $v \in \mathcal{L}(X, Z)$. *Here a continuous linear map* A *between Banach spaces* Y *and* Z *is called to be of type* s *if it can be written in the form*

$$Ax = \sum_{j \geq 0} \lambda_j \langle x, a_j \rangle y_j$$

where $a_j \in E^*$, $\|a_j\| \leq 1$, $y_j \in F$, $\|y_j\| \leq 1$ *for every* $j \geq 0$ *and* $\{\lambda_j : j \geq 0\}$ *in* s, *i.e* $\sum\limits_{j \geq 0} |\lambda_j|^s < \infty$ *for every* $s > 0$.

(ii) *There exist* $d > 0$, $C > 0$, *such that*

$$\|v^* y^*\|_X^{*1+d} \leq C \|A^* y^*\|_Y^* \|y^*\|_Z^{*d}, \tag{2}$$

for all $y^* \in Z^*$.

(iii) *There exist a neighbourhood* V *of* $O \in Z$ *and a holomorphic function* g *on* V *with values in* F_p, *where* F_p *is a Banach space associated with the semi-norm* p *in the definition of spaces having the property* (DN), *and a holomorphic function* $f : Y \longrightarrow F$, *such that*

$$\omega_p \circ f \Big|_{A^{-1}(V)} = g \circ A \Big|_{A^{-1}(V)},$$

where $\omega_p : F \longrightarrow F_p$ *is the canonical map.*

Then there exists holomorphic map $h : X \longrightarrow F$, *such that*

$$f = h \circ u.$$

Proof. By the spectral mapping theorem [11, 8.3] there exist a complete orthonormal system $\{e_j \big| j \in \mathbf{N}\}$ in Y, an orthonormal system $\{y_j \big| j \in \mathbf{N}\}$ in Z and a sequence $\lambda = \{\lambda_j \big|\ \lambda_j \geq 0 \text{ for } j \geq 1\}$ in s, such that

$$Ax = \sum_{j=1}^{\infty} \lambda_j (x \big| e_j) y_j.$$

Let χ_k denote the functional on Z given by

$$\chi_k(x) = (x\big|y_k)_Z, \quad k \geq 1.$$

Then $\|\chi_k\|_Z^* = 1$ and

$$\begin{aligned}\|A^*\chi_k\|_Z^* &= \sup\big\{|(Ax\big|y_k)_Z| : \|x\| \leq 1\big\}\\ &= \sup\big\{|\lambda_k(x\big|e_k)_Y| : \|x\| \leq 1\big\} = \lambda_k.\end{aligned}$$

Putting $\varphi_k = v^*\chi_k \in X^*$, we get

$$\|\varphi_k\|_X^* \leq C^{\frac{1}{1+d}}\ \lambda_k^{\frac{1}{1+d}},$$

for $k \geq 1$. Choose $0 < \delta < 1$, such that for $\mu = \Big\{\Big(\mu_j = \dfrac{\delta}{j}\Big)\Big\}$, the set

$$\Big\{y \in Z,\ y = \sum_{j\geq 1} \xi_j y_j,\ |\xi_j| \leq \mu_j = \frac{\delta}{j}\Big\} \subset V.$$

Without loss of generality, we may assume that

$$\sup\Big\{\|g(y)\|_p : y \in V\Big\} \leq 1.$$

For each $m \in M = \big\{(m_1, m_2, \dots, m_n, 0, 0, \dots)\big|m_i \in \mathbf{N}\big\}$ put

$$a_m = \Big(\frac{1}{2\pi i}\Big)^n \int\limits_{|\rho_1|=\mu_1} \cdots \int\limits_{|\rho_n|=\mu_n} \frac{g(\rho_1 y_1 + \cdots + \rho_n y_n)}{\rho_1^{m_1+1} \dots \rho_n^{m_n+1}} d\rho_1 \dots d\rho_n.$$

It follows that

$$\|a_m\|_{F_p} \leq \frac{1}{\mu^m} \quad \text{for } m \in M \quad (\text{where } \mu^m = \mu_1^{m_1} \dots \mu_n^{m_n}).$$

Since $Ae_j = \lambda_j y_j$ for $j \geq 1$, we have

$$\begin{aligned}a_m &= \Big(\frac{1}{2\pi i}\Big)^n \int\limits_{|\rho_1|=\mu_1} \cdots \int\limits_{|\rho_n|=\mu_n} \frac{g\Big(A(\frac{\rho_1}{\lambda_1}e_1 + \cdots + \frac{\rho_n}{\lambda_n}e_n)\Big)}{\rho_1^{m_1+1} \dots \rho_n^{m_n+1}} d\rho_1 \dots d\rho_n\\ &= \frac{1}{\lambda^m}\Big(\frac{1}{2\pi i}\Big)^n \int\limits_{|\omega_1|=r_1} \cdots \int\limits_{|\omega_n|=r_n} \frac{\omega_p.f(\omega_1 e_1 + \cdots + \omega_n e_n)}{\omega_1^{m_1+1} \dots \omega_n^{m_n+1}} d\omega_1 \dots d\omega_n\\ &= \omega_p\Big[\frac{1}{\lambda^m}\Big(\frac{1}{2\pi i}\Big)^n \int\limits_{|\omega_1|=r_1} \cdots \int\limits_{|\omega_n|=r_n} \frac{f(\omega_1 e_1 + \dots \omega_n e_n)}{\omega_1^{m_1+1} \dots \omega_n^{m_n+1}} d\omega_1 \dots d\omega_n\Big]\\ &= \omega_p(b_m),\end{aligned}$$

where

$$b_m = \frac{1}{\lambda^m}\Big(\frac{1}{2\pi i}\Big)^n \int\limits_{|\omega_1|=r_1} \cdots \int\limits_{|\omega_n|=r_n} \frac{f(\omega_1 e_1 + \cdots + \omega_n e_n)}{\omega_1^{m_1+1} \ldots \omega_n^{m_n+1}} d\omega_1 \ldots d\omega_n$$

with $r_i = \dfrac{\mu_i}{\lambda_i}$, $1 \leq i \leq n$.

We note that $b_m \in F$ and

$$\|b_m\|_p = \|\omega_p(b_m)\|_{F_p} = \|a_m\|_{F_p} \leq \frac{1}{\mu^m} \cdot$$

Moreover, for $m \in M$, $q \geq 1$, $t > 0$

$$\|b_m\|_q \leq \frac{N(q,t)}{\lambda^m \mu^m t^{|m|}},$$

where

$$N(q,t) = \sup\Big\{\|f(x)\|_q \ : \ x = \sum_{j\geq 1} \xi_j e_j, \ |\xi_j| \leq t\mu_j, \ j \geq 1\Big\}$$

and $t^{|m|} = t^{m_1+\cdots+m_n}$.

For the positive number $d > 0$ in the condition (ii), we choose a positive number $0 < \delta_1 < \dfrac{1}{2}$ such that $\tilde{d}\delta_1 = d$ and put $\gamma = \dfrac{1}{2(1+d)}$ ·

Since $F \in (DN)$, for $q > 1$ we can take k, such that (DN) holds for $\tilde{d}$:

$$\|\cdot\|_q^{1+\tilde{d}} \leq \|\cdot\|_k \cdot \|\cdot\|_p^{\tilde{d}}$$

(we can consider $C = 1$). Then

$$\sum_{m\in M} r^{|m|}\|b_m\|_q \prod_{j=1}^{\infty}\big(\|\varphi_j\|_X^*\big)^{m_j} \le \sum_{m\in M} r^{|m|}\|b_m\|_q \prod_{j=1}^{\infty}\big(C^{\frac{1}{1+d}}\lambda_j^{\frac{1}{1+d}}\big)^{m_j}$$
$$= \sum_{m\in M} r^{|m|}(C\lambda)^{2m\gamma}\|b_m\|_q = \sum_{m\in M} r^{|m|}\big(\lambda^m\|b_m\|_q\big)^{\gamma}(C^2\lambda)^{m\gamma}\|b_m\|_q^{1-\gamma}$$
$$\le \sum_{m\in M} r^{|m|}\Big(\frac{N(q,t)}{\mu^m t^{|m|}}\Big)^{\gamma}\big(C^2\lambda\big)^{m\gamma}\big\|b_m\big\|_k^{\frac{1-\gamma}{1+\tilde d}}\big\|b_m\big\|_p^{\frac{(1-\gamma)\tilde d}{1+\tilde d}}$$
$$\le \sum_{m\in M} r^{|m|}\Big(\frac{N(q,t)}{\mu^m t^{|m|}}\Big)^{\gamma}(C^2\lambda)^{m\gamma}\Big(\frac{N(k,t)}{\lambda^m\mu^m t^{|m|}}\Big)^{\frac{1-\gamma}{1+\tilde d}}\Big(\frac{1}{\mu^m}\Big)^{\frac{(1-\gamma)\tilde d}{1+\tilde d}}$$
$$\le N(q,t)^{\gamma}\big(N(k,t)\big)^{\frac{1-\gamma}{1+\tilde d}}\sum_{m\in M}\Big(\frac{r}{t^{\gamma+\frac{1-\gamma}{1+\tilde d}}}\Big)^{|m|}\frac{\lambda^{m\left(\gamma-\frac{1-\gamma}{1+\tilde d}\right)}C^{2m\gamma}}{\mu^{m\left(\gamma+\frac{1-\gamma}{1+\tilde d}+\frac{1-\gamma}{1+\tilde d}\tilde d\right)}}$$
$$\le N(q,t)^{\gamma}N(k,t)^{\frac{1-\gamma}{1+\tilde d}}\sum_{m\in M}\Big(\frac{r}{t^{\gamma+\frac{1-\gamma}{1+\tilde d}}}\Big)^{|m|}\Big(\frac{\lambda^{\gamma-\frac{1-\gamma}{1+\tilde d}}C^{2\gamma}}{\mu}\Big)^{m}.$$

Put $\alpha = \gamma - \dfrac{1-\gamma}{1+\tilde d}$. Since $0 < \delta_1 < \dfrac{1}{2}$, it follows that $\alpha > 0$. Since $\lambda = \big(\lambda_j\big)_{j\in N}$ is in s, the sequence $\Big(\dfrac{\lambda_j^{\alpha}C^{2\gamma}}{\mu_j}\Big)$ is in ℓ^1 and, hence, for $R = \sum\limits_{j\ge 1}\dfrac{\lambda_j^{\alpha}C^{2\gamma}}{\mu_j} + 1$ we have $2R > R \ge \dfrac{\lambda_j^{\alpha}C^{2\gamma}}{\mu_j}$ for all j. This implies

$$0 < \sup\Big\{\frac{\lambda_j^{\alpha}C^{2\gamma}}{2R\mu_j} : j \ge 1\Big\} \le \frac{1}{2}\ .$$

Putting $\beta = \gamma + \frac{1-\gamma}{1+\tilde d} > 0$ and $t = \sqrt[\beta]{2Rr}$, we have

$$\begin{aligned}
&\sum_{m\in M} r^{|m|}\|b_m\|_q \prod_{j=1}^{\infty}\big(\|\varphi_j\|_X^*\big)^{m_j}\\
&\le N\big(q, \sqrt[\beta]{2Rr}\big)^{\gamma} N\big(k, \sqrt[\beta]{2Rr}\big)^{\frac{1-\gamma}{1+d}} \sum_{m\in M}\Big(\frac{\lambda^{\alpha}C^{2\gamma}}{2R\mu}\Big)^{m}\\
&= N\big(q, \sqrt[\beta]{2Rr}\big)^{\gamma} N\big(k, \sqrt[\beta]{2Rr}\big)^{\frac{1-\gamma}{1+d}} \prod_{j=1}^{\infty}\frac{1}{1-\Big(\dfrac{\lambda_j^{\alpha}C^{2\gamma}}{2R\mu_j}\Big)} < \infty,
\end{aligned}$$

for all $r > 0$.

This implies that the series $\sum\limits_{m\in M} b_m \prod\limits_{j=1}^{\infty}\big(\varphi_j(x)\big)^{m_j}$ defines a holomorphic function $h : X \longrightarrow F$. To complete the proof of the lemma we now show that $h\circ u = f$. For $x = \sum\limits_{j=1}^{n}\xi_j e_j \in Y$, consider $h\big(u(x)\big) = \sum\limits_{m\in M} b_m \prod\limits_{j=1}^{\infty}\varphi_j\big(u(x)\big)^{m_j}$. Now we have $\varphi_j\big(u(x)\big) = \chi_j\big(v(u(x))\big) = \lambda_j\xi_j$. Hence

$$h\big(u(x)\big) = \sum_{m\in M} b_m\lambda^m\xi^m = f(x)$$

This shows that $h\circ u = f$ on a dense subset of Y and, hence, $h\circ u = f$ on Y. The lemma is proved.

Now we continue the proof of Theorem 2.1.

Given $f : E \longrightarrow F$ a holomorphic function. Choose $p \ge 1$ such that (DN) holds for F. Consider the holomorphic function $\omega_p.f : E \longrightarrow F_p$. Then we can find q independent from p such that $\omega_p.f$ is bounded on a neighbourhood V of $0 \in E_q$ and, hence, induces a holomorphic function g on a neighbourhood V of $0 \in E_q$ with values in F_p. Since $E \in (\widetilde{\Omega})$, by [6] (Lemma 3.6) there exists a bounded balanced convex set B in E and r, C, $d > 0$ such that $E(B)$ is a Hilbert space which is dense in E and

$$(*)\qquad \big\|y\big\|_r^{*1+d} \le C\big\|y\big\|_B^{*d}\big\|y\big\|_q^{*} \quad \text{for} \quad y \in E^*.$$

Now consider the canonical map $\alpha_q : E(B) \longrightarrow E_q$. Write $E(B) = (E(B)\cap\ker\alpha_q)\oplus E_0$ and $\pi : E(B) \longrightarrow E_0$ for the orthogonal projection with $\ker\pi = E(B)\cap\ker\alpha_q$. Put $A = \alpha_q\big|_{E_0}$. Note that A is injective of type s because of the nuclearity of E. In virtue of lemma 2.2 to $X = E_r$, $Y = E_0$, $Z = E_q$, $u = \alpha_r\big|_{E_0}$, $v = \alpha_{rq} : E_r \longrightarrow E_q$ is the canonical map and $g : V \longrightarrow F_p$, we can find a holomorphic function $h : E_r \longrightarrow F$ such that

$$f\big|_{E_0} = h \circ \alpha_r\big|_{E_0}.$$

Now we prove that $f(x+\ker\alpha_q) = f(x)$, for $x \in E$. Since ω_p is injective it suffices to show that

$$\omega_p f(x + \ker\alpha_q) = \omega_p f(x),$$

for $x \in E$.

Let $x \in E$ and $y \in \ker\alpha_q$ be arbitrary. First, we assume that $x \in \alpha_q^{-1}(V)$. Consider the entier function

$$\varphi(\lambda) = \omega_p f(x + \lambda y).$$

Since $x + \lambda y \in \alpha_q^{-1}(V)$ for all $\lambda \in \mathbf{C}$ and by the boundedness of $\omega_p f$ on V it follows that $\varphi(\lambda)$ is bounded on $\mathbf{C}$. By Liouville's theorem $\varphi \equiv$ const. In particular,

$$\omega_p f(x) = \varphi(0) = \varphi(1) = \omega_p f(x + y).$$

Now let $x \in E$ be arbitrary. Consider the entier function on E with values in F_p, given by

$$\psi(z) = \omega_p f(z) - \omega_p f(z + y).$$

Since $z \in \alpha_q^{-1}(V)$, $\psi(z) = 0$ and, hence, by the identity theorem, we have $\psi(z) = 0$ for all $z \in E$. Hence,

$$\omega_p f(x) = \omega_p f(x + y).$$

For $x \in E(B)$ we can write $x = y + z$, $y \in E(B) \cap \ker \alpha_q$ and $z \in E_0$. Hence,

$$f(x) = f(y + z) = f(z) = h.\alpha_r(z) = h\alpha_r(z + y) = h\alpha_r(x).$$

From the density of $E(B)$ in E it follows that

$$f = h \circ \alpha_r$$

and Theorem 2.1 is proved.

3. Frechet-valued weakly holomorphic functions on compact subsets in nuclear Frechet spaces and the linear topological invariant (DN).

Let X be a compact subset in a locally convex space E and F a locally convex space. Here by standard notation $H(X, F)$ denotes the space of germs of holomorphic functions on X with values in F with the inductive limit topology. Recall that $f \in H(X, F)$ if there exists a neighbourhood V of X in E and a holomorphic function $\hat{f} : V \to F$, whose germ on X is f. A F-valued continuous function f on X is called weakly holomorphic on X if for every $x^* \in F^*$, the dual space of F, $x^* f$ can be holomorphically extended on a neighbourhood of X. By $H_w(X, F)$ we denote the space of F-valued weakly holomorphic functions on X.

The main result of this section is the following.

Theorem 3.1. *Let F be a reflexive Frechet space. Then $H(X, F) = H_w(X, F)$ for every $\tilde{L}$-regular compact subset X in any nuclear Frechet space E if and only if $F \in (DN)$.*

Here a compact subset X in a Frechet space E is called $\tilde{L}$-regular if $\big[H(X)\big]^* \in (\widetilde{\Omega})$.

We need the following.

Lemma 3.2. *Let F be a Frechet space with $F \in (DN)$. Then $\big[F^*_{bor}\big]^* \in (DN)$, where F^*_{bor} is the space F^*, equipped with the bornological topology associated to the topology of F^*.*

Proof. Let $\{U_n\}$ be a decreasing neighbourhood basis of zero in F. Since $F \in (DN)$ we have

$$\exists p\ \forall q\ \exists k, C > 0 : \|\cdot\|_q \le r\|\cdot\|_p + \frac{C}{r}\|\cdot\|_k \quad \text{for every } r > 0,$$

or in the equivalent form,

$$\exists p\ \forall q\ \exists k, C > 0 : U_q^0 \subseteq rU_p^0 + \frac{C}{r}U_k^0, \quad \text{for all } r > 0 \text{ [18]}.$$

For $u \in \big[F^*_{\mathrm{bor}}\big]^*$ and $r > 0$ we have

$$\begin{aligned}\|u\|_q^{**} &= \sup_{x^*\in U_q^0}|u(x^*)| \le \sup_{x^*\in rU_p^0+\frac{C}{r}U_k^0}|u(x^*)|\\ &\le r\sup_{x^*\in U_p^0}|u(x^*)| + \frac{C}{r}\sup_{x^*\in U_k^0}|u(x^*)|\\ &= r\|u\|_p^{**} + \frac{C}{r}\|u\|_k^{**}.\end{aligned}$$

Hence, $\left[F_{\mathrm{bor}}^*\right]^* \in (DN)$. The lemma is proved.

Lemma 3.3. *Let X be a $\tilde{L}$-regular compact set in a Frechet space E. Then X is a unique set, i.e. if $f \in H(X)$ and $f\big|_X = 0$, then $f = 0$ on a neighbourhood of X in E.*

Proof. Let $\{V_p\}$ be a decreasing neighbourhood basis of X in E. By the hypothesis

$$\forall p\ \exists q \ge 1,\ \exists d > 0\ \forall k \ge q\ \exists C > 0$$

$$\|f\|_q^{1+d} \le C\|f\|_k\|f\|_p^d, \quad \text{for} \quad f \in H^\infty(V_p).$$

Using the above inequality to f^n, $f \in H^\infty(V_p)$, we have

$$\begin{aligned}\|f\|_q^{1+d} &= \left(\|f\|_q^{n(1+d)}\right)^{\frac{1}{n}} = \left(\|f^n\|_q^{1+d}\right)^{\frac{1}{n}} \le C^{\frac{1}{n}}\left(\|f^n\|_k\|f^n\|_p^d\right)^{\frac{1}{n}}\\ &= C^{\frac{1}{n}}\left(\|f\|_k^n\|f\|_p^{nd}\right)^{\frac{1}{n}} \longrightarrow \|f\|_k\|f\|_p^d,\end{aligned}$$

as $n \to \infty$. Hence,

$$\|f\|_q^{1+d} \le \|f\|_k\|f\|_p^d \quad \text{for} \quad f \in H^\infty(V_p), \quad \forall k \ge q.$$

This inequality implies that as $k \to \infty$ we have

$$\forall p\ \exists q \ge p,\ d > 0 : \|f\|_q^{1+d} \le \|f\|_X\|f\|_p^d \quad \text{for} \quad f \in H^\infty(V_p),$$

which shows that X is a unique set. The lemma is proved.

Proof of Theorem 3.1

Sufficiency. Given $f \in H_w(X, F)$, where X is a $\tilde{L}$-regular compact set in a nuclear Frechet space E. Consider the linear map $\hat{f} : F^*_{bor} \to H(X)$ given by

$$\hat{f}(x^*) = \widehat{x^* f}$$

for $x^* \in F^*_{bor}$, where $\widehat{x^* f}$ is a holomorphic extension of $x^* f$ to some neighbourhood of X in E. By Lemma 3.3 X is an unique subset it follows that $\hat{f}$ has the closed graph and, hence, $\hat{f}$ is continuous. By lemma 3.2, $[F^*_{bor}]^* \in (DN)$, and by the hypothesis $[H(X)]^* \in (\widetilde{\Omega})$, Theorem 2.1 shows that there exists a neighbourhood W of $O \in F^*_{bor}$ such that $\hat{f}(W)$ is bounded in $H(X)$. Since $H(X)$ is regular there exists a neighbourhood U of X in E such that $\hat{f}(W)$ is contained and bounded in $H^\infty(U)$, the Banach space of bounded holomorphic functions on U. Then $\hat{f}(F^*) \subset H^\infty(U)$ and by the reflexivity of F, we can define a holomorphic function

$$g : U \longrightarrow F$$

by

$$g(z)(x^*) = \hat{f}(x^*)(z), \quad \text{for} \quad x^* \in F^*,\ z \in U.$$

We have

$$g(z)(x^*) = \hat{f}(x^*)(z) = f(z)(x^*) \text{ for every } z \in X \text{ and every } x^* \in F^*.$$

This yields $g\big|_X = f$ and, hence, $f \in H(X, F)$.

Necessity. By Vogt [17] it suffices to show that every continuous linear map T from $H(\Delta)$ to F is bounded on a neighbourhood of $0 \in H(\Delta)$. Consider $T^* : F^* \longrightarrow [H(\Delta)]^* \cong H(\overline{\Delta})$. Since $T^*(x^*) \in H(\overline{\Delta})$ for every $x^* \in F^*$, we can define a map $f : \overline{\Delta} \longrightarrow F^{**}$ given by

$$f(z)(x^*) = \delta_z(T^*(x^*)),$$

for $x^* \in F^*$, $z \in \overline{\Delta}$ and δ_z being the Dirac functional defined by z,

$$\delta_z(\sigma) = \sigma(z) \quad \text{for} \quad \sigma \in H(\overline{\Delta}).$$

By the hypothesis $f \in H_w(\overline{\Delta}, F)$. Since $\overline{\Delta}$ is $\widetilde{L}$-regular it implies that $f \in H(\overline{\Delta}, F)$. Thus there exists a neighbourhood V of $\overline{\Delta}$ such that $f \in H(V, F)$. Without loss of generality we may assume that $B = f(V)$ is bounded in F. It follows that T^* is bounded on B^o. Put $T^*(B^o) = C \subset [H(\Delta)]^*$. Thus $U = C^0$ is a neighbourhood of $0 \in H(\Delta)$ and $T(U) \subset B^{oo}$ is bounded in F. Theorem 3.1 is proved.

Now we consider Theorem 3.1 in the case X is a compact determining polydisc in a dual space of a Frechet nuclear space.

Theorem 3.4. *Let E be a Frechet nuclear space with a basis and having a continuous norm. Then $H_w(X, F) = H(X, F)$ for every compact determining polydisc X in E^* such that $H(X)$ is semi-reflexive and every Banach space F if and only if $E \in (DN)$.*

Proof. Necessity. Let $\{e_n\}_{n=1}^{\infty}$ be a basis of E. Write $E = Ce_1 \oplus F$ where F is the closed subspace of E generated by $\{e_n\}_{n=2}^{\infty}$. Take an open polydisc U of E of the form

$$U = \big\{z = \sum_{i=1}^{\infty} z_n e_n = (z_n)_{n=1}^{\infty} \in E : \sup_{n\geq 1} |z_n| < 1\big\}.$$

Hence,

$$X = U^M = \big\{w = \sum_{i=1}^{\infty} w_n e_n^* = (w_n)_{n=1}^{\infty} \in E^* : \sup_{n\geq 1} |w_n| \leq 1\big\} = \overline{\Delta} \times Y\,,$$

$$\overline{\Delta} = \{w_1 \in \mathbf{C} : |w_1| \leq 1\} \text{ and } Y = \big\{(w_n)_{n=2}^{\infty} : \sup |w_n| \leq 1\big\},$$

where $\{e_n^*\}_{n=1}^{\infty}$ is the dual basis of $\{e_n\}_{n=1}^{\infty}$. Note that $X \supset \operatorname{conv}\{e_n^*\}_{n=1}^{\infty}$ and, hence, X is a compact determining polydisc.

Indeed, given $f \in H(X)$ such that $f|_X = 0$. Take a balanced convex open neighbourhood $W \supset X$ such that $f \in H(W)$. For each $m \geq 1$, put $L = \operatorname{span}\{e_1^*, \dots, e_m^*\}$ and consider

$$V = \big\{\sum_{i=1}^{m} \lambda_i e_i^* : \sum_{i=1}^{m} |\lambda_i| \leq 1\big\}.$$

Note that V is a neighbourhood of $0 \in L$ and by the hypothesis $f|_V = 0$. Hence, $f|_{W\cap L} = 0$. Thus

$$f|_{W\cap \operatorname{span}\{e_i^*: i\geq 1\}} = 0.$$

From the density of $W \cap \operatorname{span}\{e_i^* : i \geq 1\}$ in W and the continuity of f it implies that $f|_W = 0$.

First we show that $H(X)$ is regular.

Indeed, given a balanced convex bounded subset A in $H(X)$. Consider the normed space $\mathbf{E}_1 = H(X)(A)$ spanned by A and the function $f : X \to \mathbf{E}_1^*$ given by

$$f(x)(\sigma) = \sigma(x)$$

for $x \in X$ and $\sigma \in \mathbf{E}_1$.

Since $H(X) = [H(X)]^{**}$ we infer that f is weakly holomorphic. By the hypothesis, f can be extended to a bounded holomorphic function $\widehat{f}$ on a neighbourhood V_1 of X in E^*. From the relation

$$\sigma(x) = f(x)(\sigma) = \widehat{f}(x)(\sigma),$$

for every $x \in X$ and $\sigma \in A$ and from the uniqueness of X, it follows that A is contained and bounded in $H^\infty(V_1)$. Therefore $H(X)$ is regular and by [4] $H(U)$ is bornological.

Since $L_b(F, H(\Delta)) \cong H(\Delta)\widehat{\otimes}_\pi F^*$ is contained as a complemented subspace of $H(\Delta)\widehat{\otimes}_\pi H(W_1) \cong H(\Delta)\widehat{\otimes}_\varepsilon H(W_1) \cong H(\Delta \times W_1) = H(U)$ with $U = \Delta \times W_1$, it follows that $H(\Delta)\widehat{\otimes}_\pi F^*$ is bornological. By [19, Theorem 4.9] $F \in (DN)$ and, hence, $E \in (DN)$.

Sufficiency. Assume that $E \in (DN)$ and given $f \in H_w(X, F)$, where X is a compact determining polydisc of the form

$$X = \big\{w = (w_n) \in E^* : \sup_n \big|\frac{w_n}{\alpha_n}\big| \le 1\big\}$$

in E^*.

From the determination of X it follows that $\alpha_n \neq 0$ for every n. Moreover, by [4] $H(X)$ is regular. By the determination of X we can define the linear map

$$\widehat{f} : F^* \to H(X),$$

given by

$$\widehat{f}(x^*) = x^* f \quad \text{for } x^* \in F^*,$$

where $\widehat{x^* f}$ is a holomorphic extension of $x^* f$ to some neighbourhood of X. By [4] on a neighbourhood (depending on x^*) of X, we have

$$\widehat{f}(x^*)(w) = \sum_{m \in N^{(N)}} b_m(x^*)\, w^m,$$

where $b_m(x^*)$ are the linear functionals on F^* defined by

$$b_m(x^*) =$$

$$\Big(\frac{1}{2\pi_i}\Big)^n \int\limits_{|\lambda_1| = |\alpha_1|} \cdots \int\limits_{|\lambda_n| = |\alpha_n|} \frac{\widehat{f}(x^*)(\lambda_1 e_1^* + \cdots + \lambda_n e_n^*)}{\lambda_1^{m_1+1} \cdots \lambda_n^{m_n+1}}\, d\lambda_1 \cdots d\lambda_n.$$

We prove that $b_m(x^*)$ is continuous on F^* for every $m \in N^{(N)}$. Fix $m \in N^{(N)}$, $m = (m_1, m_2, \dots, m_n, 0, 0, \dots)$ and put

$$X_m = X \cap \operatorname{span}\{e_1^*, \dots, e_n^*\} = \{(w_1, \dots, w_n) : |w_i| \le |\alpha_i|,\ i = \overline{1, n}\}.$$

Consider the function $f_m = f|_{X_m}$. By the hypothesis $f_m \in H_w(X_m, F)$ and since X_m is compact and F is a Banach space it implies that $f_m \in H(X_m, F)$ [12]. Thus there exists a neighbourhood V of X_m in span $\{e_1^*, \dots, e_n^*\}$ such that $f_m : V \to F$ is holomorphic. Hence,

$$\begin{aligned} &b_m(x^*) = \\ &\Big(\frac{1}{2\pi i}\Big)^n \int\limits_{|\lambda_1|=|\alpha_1|} \cdots \int\limits_{|\lambda_n|=|\alpha_n|} \frac{\widehat{f}(x^*)(\lambda_1 e_1^* + \cdots + \lambda_n e_n^*)}{\lambda_1^{m_1+1} \cdots \lambda_n^{m_n+1}} d\lambda_1 \cdots d\lambda_n = \\ &\Big(\frac{1}{2\pi i}\Big)^n \int\limits_{|\lambda_1|=|\alpha_1|} \cdots \int\limits_{|\lambda_n|=|\alpha_n|} \frac{\widehat{f}_m(x^*)(\lambda_1 e_1^* + \cdots + \lambda_n e_n^*)}{\lambda_1^{m_1+1} \cdots \lambda_n^{m_n+1}} d\lambda_1 \cdots d\lambda_n \end{aligned}$$

is continuous on F^*.

Now we prove that $\widehat{f} : F^* \to H(X)$ is continuous. Take $\mu \in [H(X)]^*_\beta \cong H(U)$ [4], where $U = \{z = (z_n) \in E : \sup\limits_n |z_n \alpha_n| < 1\}$. By [4] we can write

$$\mu(z) = \sum_{m \in N^{(N)}} a_m(\mu)\, z^m\,, \quad \text{for } z \in U$$

and

$$\langle \widehat{f}(x^*),\, \mu\rangle = \sum_{m \in N^{(N)}} b_m(x^*)\, a_m(\mu).$$

From

$$\sup\Big\{\Big|\sum_{m \in J} b_m(x^*)\, a_m(\mu)\Big| : J \subset N^{(N)},\ J \text{ finite}\Big\} < \infty$$

for $x^* \in F^*$, we infer that

$$\sup\Big\{\Big|\sum_{m \in J} b_m(x^*)\, a_m(\mu)\Big| : J \subset N^{(N)},\ J \text{ finite},\ \|x^*\| \le 1\Big\} < \infty.$$

Hence,

$$\sup\big\{|\langle \widehat{f}(x^*),\, \mu\rangle| : \|x^*\| \le 1\big\} < \infty,$$

for every $\mu \in [H(X)]^*_\beta$. This yields that $\widehat{f}$ is continuous on F^*. Thus there exists a neighbourhood Ω of X in E^* such that $\widehat{f} : F^* \to H^\infty(\Omega)$. The function $g : \Omega \to F$ given by

$$g(z)(x^*) = \widehat{f}(x^*)(z)$$

is holomorphic on Ω and $g|_X = f$. Theorem 3.4 is proved.

4. Examples

In this section we give examples of $\tilde{L}$-regular compact sets.

Let $\alpha = \{\alpha_n\}$ be an exponent sequence such that

$$\Lambda_1(\alpha) = \{(\xi_1, \dots, \xi_n, \dots) : \xi_i \in \mathbf{C},\ \sum_{i=1}^{\infty} |\xi_i| r^{\alpha_i} < \infty \text{ for } 0 < r < 1\}$$

is nuclear and $a = \{a_j\} \in \Lambda_1(\alpha)$, $a_j \geq 0$ for $j \geq 1$. Then the set

$$D_a = \{x' \in \Lambda_1^*(\alpha) : \sup_{j \geq 1} |x'_j| a_j < 1\}$$

is open in $\Lambda_1^*(\alpha)$, and is called an open polydisc. It is known [6] that $(H(D_a), \tau_0) \in (\widetilde{\Omega})$ if and only if $a > 0$ and $\liminf_{j\to\infty} a_j^{\frac{1}{\alpha_j}} = 0$. On the other hand, by [3]

$$\big(H(D_a)\big)^* \cong H(D_a^0),$$

where

$$D_a^0 = \Big\{x \in \Lambda_1(\alpha) : \sup_{x' \in D_a} |x'_j x_j| \leq 1\Big\}.$$

Thus, D_a^0 is $\tilde{L}$-regular in $\Lambda_1(\alpha)$ if and only if $a > 0$ and $\liminf_{j\to\infty} a_j^{\frac{1}{\alpha_j}} = 0$. However, as in [6] for $0 < r < 1$ set $a := \big(r^{\alpha_j}\big)_{j \in N}$. Then $\big(H(D_a), \tau_0\big) \in (\widetilde{\Omega})$ but $\big(H(D_a), \tau_0\big) \overline{\in} (\overline{\Omega})$. Hence D_a^0 is $\tilde{L}$-regular but not L-regular.

References

1. B. J. Boland and S. Dineen, *Holomorphic functions on fully nuclear spaces*, Bull. Soc. Math. France **106** (1978), 311-336.
2. W. M. Bogdanowicz, *Analytic continuation of holomorphic functions with values in a locally convex space*, Proc. Amer. Math. Soc. **22** (1969), 660-666.

3. S. Dineen, *Analytic functionals on fully nuclear spaces*, Studia. Math. **53** (1982), 11-32.

4. ______ , *Complex analysis in locally convex spaces*, Math. Stud. **57**, 1981, North-Holland.

5. R. Meise and D. Vogt, *Structure of spaces of holomorphic functions on infinite dimensional polydisc*, Studia. Math. **LXXV** (1983), 235-252.

6. ______ , ______ , *Holomorphic functions of uniformly bounded type on nuclear Frechet spaces*, Studia Math. **83** (1986), 117-166.

7. L. M. Hai, N. V. Khue, and N. T. Nga, *Weak meromorphic extension*, Colloq. Math. **LXIV**, Fasc. 1 (1993), 65-70.

8. E. Ligocka and J. Siciak, *Weak analytic continuations*, Bull. Acad. Polon. Sci. Math. **20** (1972), 461-466.

9. N. V. Khue and B. D. Tac, *Extending holomorphic maps from compact sets in infinite dimensions*, Studia Math. **95** (1990), 263-272.

10. N. T. Van, *Fonctions séparément analytiques et prolongement analytique faible en dimension infinie*, Ann. Polon. Math. **33** (1976), 71-83.

11. A. Pietsch, *Nuclear locally convex spaces*, 2nd edition, Berlin, Springer-Verlag, 1972.

12. J. Siciak, *Weak analytic continuation from compact subsets of* $\mathbf{C}^n$, in: Lecture Notes in Math. **364**, Springer (1974), 92-96.

13. R. L. Soraggi, *Holomorphic Germs on certain locally convex spaces*, Ann. Math. Pura ed App., **144**, No 1 (1986), 1-22.

14. D. Vogt, *On two classes of (F)-spaces*, Arch. Math. **45** (1985), 255-266.

15. ______ , *Characterisierung der Unterraume eines nuklearen stabilen Potenzreihenraumes von endlichem Typ*, Studia. Math. **71** (1982), 251-270.

16. ______ , *Eine Characterisierung der Potenzreiheräume von endlichem Typ und ihre Folgerungen*, Manuscripta. Math. **37** (1982), 269-301.

17. ______ , *Frechetraume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J. Reine Angew. Math. **345** (1983), 182-200.

18. ______ , *Subspaces and quotient spaces of (s)*, in Functional Analysis: Surveys and Recent Results, K.-D. Bierstedt, B. Fuchssteiner (Eds.), North-Holland Math. Studies **27** (1977), 167-187.

19. ______ , *Some results on continuous linear maps between Frechet spaces*, in Functional Analysis: Surveys and Recent Results III', K. D. Bierstedt, B. Fuchssteiner (Eds.), North-Holland. Math. Studies **90** (1984), 349-381.

20. L. Waelbroeck, *Weak analytic functions and the closed graph theorem*, Proc. Conf. on Infinite Dimensional Holomorphy, Lecture Notes in Math. **364**, Springer-Verlag 1974, 97-100.

21. V. P. Zaharjuta, *Isomorphism of spaces of analytic functions*, Sov. Math. Dokl. **22** (1980), 631-634.

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