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ON ITO STOCHASTIC INTEGRAL WITH RESPECT TO VECTOR STABLE RANDOM MEASURES

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Abstract. Let Z_p be a vector p-stable random measure with values in a q-smoothable Banach space, where $p > q$ if $p < 2$ and $q = 2$ if $p = 2$. It is shown that the stochastic integral $\int_0^1 udZ_p$ can be defined for processes u which are non-anticipating with respect to Z_p.

1. Introdution

The Ito stochastic integral with respect to the Brownian motion has been generalized in several directions. One of these directions is to define the Ito stochastic integral with respect to a larger class of integrators. A theory of stochastic integrals for semimartingale integrators has been developed by many authors (see [1] and the references therein). The Ito stochastic integral in which the integrator is a Levy process satisfying certain conditions was constructed by Gine and Marcus [5]. In [4] Dettweiler considered the Ito stochastic integral of vector-valued processes with respect to a real-valued Gaussian symmetric random measure. Mamporia [8] defined the Ito stochastic integral of operator-valued processes with respect to a vector-valued Wiener process.

The purpose of this paper is to construct the stochastic integral of non-anticipating processes with respect to a Banach space-valued p-stable random measure. Section 2 contains the definition and some properties of Banach space-valued p-stable random measures which will be used later. The main result of the paper is Theorem 3.5 (Section 3) which is based on Lemma 3.4. The inequality stated in Lemma 3.4 is an extension to the vector case of a similar inequality proved in [5, Lemma 3.3]. Theorem 3.6, which deals with the Gaussian case is an extension of the isometric property of the stochastic integral in the real case.

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2. Vector p-stable random measures

Let X be a separable Banach space and (T, Σ) be a measurable space. Denote by $L^0_X(\Omega)$ and $L^p_X(\Omega)$ the set of all X-valued r.v.'s and the set of X-valued r.v.'s having strong p-th moment, respectively. A mapping F from Σ into $L^0_X(\Omega)$ is called a X-valued random measure on (T, Σ) if for every sequence (A_n) of disjoint sets from Σ, the $F(A_n)$ are independent and

$$F(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} F(A_n) \quad \text{in } L^0_X(\Omega).$$

A X-valued random measure F is said to be p-stable $(0 < p \leq 2)$ if for each $A \in \Sigma$, $F(A)$ is a p-stable symmetric random variable. From now on, Z_p always denotes a X-valued p-stable random measure $(0 < p < 2)$ and Z denotes a X-valued symmetric Gaussian random measure.

Example. Let M be a real-valued p-stable random measure on (T, Σ). Given a M-integrable function $f\colon T \to X$, we can define a function $F\colon \to L^0_X(\Omega)$ by

$$F(A) = \int_A f dM.$$

By Rosinski's results [10], it is easy to show that F is a X-valued p-stable random measure.

Definition 2.1. A set function Q on Σ whose value on a set A is the covariance operator of $Z(A)$ is called a covariance measure of Z.

In order to study properties of the covariance measure Q it is useful to introduce an inner product on $L^2_X(\Omega)$. For $\xi, \eta \in L^2_X(\Omega)$ the inner product of ξ and η, denoted by $[\xi, \eta]$, is an operator from X' into X defined by

$$[\xi, \eta](a) = \int_\Omega \xi(\omega)\big(\eta(\omega), a\big)\, dP,$$

for each $a \in X'$.

By standard arguments, the inner products is seen to have the following properties

Proposition 2.2.

1) $[\xi, \eta]$ *is a nuclear operator.*

2) *If* $\lim \xi_n = \xi$ *and* $\lim \eta_n = \eta$ *in* $L^2_X(\Omega)$, *then* $\lim[\xi_n, \eta_n] = [\xi, \eta]$ *in the nuclear norm.*

3) $\|[\xi,\eta]\|_{\text{nuc.}} \leq \mathbf{E}\|\xi\|^2$. *Moreover, if* X *is of type 2, then there exists a constant* $C > 0$ *such that*

$$\mathbf{E}\|\xi\|^2 \leq C\|[\xi,\eta]\|_{\text{nuc.}} \quad \forall \xi \in L^2_X(\Omega).$$

We call $[\xi, \zeta]$ the covariance operator of ξ.

From the above proposition we get

Proposition 2.3. *The covariance measure* Q *is a vector measure on* (T, Σ) *taking values in the Banach space* $N(X', X)$ *of nuclear operators from* X' *into* X.

Next, we consider the case $0 < p < 2$. Let S be the sphere of X and $\mathcal{M}(S)$ denote the set of real-valued measures of bounded variation on S. $\mathcal{M}(S)$ is a Banach space under the norm given by $\|\mu\| = |\mu|(S)$.

Definition 2.4. A set function Q_p on Σ, whose value on a set A is the spectral measure of $Z_p(A)$ is called a characteristic measure of Z_p.

By the properties of stable measures on Banach spaces [7], it is not difficult to show the following proposition.

Proposition 2.5.

1) *The characteristic measure* Q_p *of* Z_p *is a vector measure taking values in* $\mathcal{M}(S)$. *Moreover,* Q_p *is of bounded variation and the variation* $|Q_p|$ *is given by*

$$|Q_p|(A) = Q_p(A)(S).$$

2) *If* X *is of stable type p, then there exists a constant* $K > 0$ *such that*

$$P\{\|Z_p(A)\| > t\} \leq Kt^{-p}|Q_p|(A) \quad \forall A \in \Sigma, t \in \mathbf{R}.$$

3. Construction of the Ito vector p-stable stochacstic integral

Throughout this section, $T = [0,1]$, Σ is the σ-algebra of Borel sets of T. Let F be a X-valued random measure on (T, Σ) with the control measure μ (i.e. μ is a positive measure such that $F(A) = 0$ a.s. whenever $\mu(A) = 0$). We associate to F a family of increasing σ-algebra $\mathcal{F}_t \in \mathcal{F}$

as follows: $\mathcal{F}_t$ is the σ-algebra generated by the X-valued r.v.'s $F(A)$, $A \in \Sigma \cap [0,t]$ and by the sets of probability 0.

A process $u(t,\omega)$ on T is said to be non-anticipating (w.r.t. F) if it is jointly measurable and for each $t \in T$ the r.v. $u(t,\omega)$ is $\mathcal{F}_t$-measurable. A process $u(t,\omega)$ is said to be non-anticipating if there exists a finite partition $0 = t_1 < ... < t_{n+1} = 1$ and the r.v.'s $\alpha_i, i = 0,1,...,n$ such that α_0 is a.s.constant, α_i is $\mathcal{F}_{t_i}$-measurable for $i \geq 1$ and

$$u(t,\omega) = \sum_{i=0}^{n} \alpha_i(\omega) 1_{B_i}, \tag{3.1}$$

where for brevity we write B_0 for $\{0\}$ and B_i for $(t_i, t_{i+1}], i \geq 1$.

Definition 3.1. We say that a process $u(t,\omega)$ belongs to the class $\mathcal{V}_p(F,\mu)$ if it is non-anticipating in $L_p(\mu \otimes P)$ and there exists a sequence (u_n) of simple non-anticipating processes converging to u in $L_p(\mu \otimes P)$.

Remark 3.2. By the same argument as that given in Remark 3.2 on [5] we observe that when μ is continuous, $\mathcal{V}(F,\mu)$ is precisely the class of non-anticipating processes in $L_p(\mu \otimes P)$. If μ is any positive measure with $\mu\{0\} = 0$, then the class $\mathcal{V}_p(F,\mu)$ is still large enough to contain all the previsible processes in $L_p(\mu \otimes P)$.

Recall that a Banach space X is said to be q-uniformly smooth ($1 \leq q \leq 2$) if the modulus of smoothness $\rho(t)$ satisfies $\rho(t) = O(t^q)$, where the modulus of smoothness is defined by

$$\rho(t) = \sup_{\substack{\|x\|=1 \\ \|y\|=t}} \left\{ \frac{\|x+y\| + \|x-y\| - 2}{2} \right\}.$$

A Banach space X is said to be q-smoothable if X is isomorphic to a q-uniformly smooth space. Assouad and Pisier [9] characterized q-smoothable Banach space by the following martingale inequality.

A Banach space X is q-smoothable if and only if there exists a constant $C > 0$ such that

$$\mathbf{E}\|\sum_{i=1}^{n} D_i\|^q \leq C \sum_{i=1}^{n} \mathbf{E}\|D_i\|^q,$$

for all X-valued martingale differences $(D_i)_1^n$ in $L_X^q(\Omega)$ $(n = 1,2,...)$. An immediate consequence of this characterization is that a q-smoothable Banach space is of type q.

The stochastic integral of a simple non-anticipating process u is defined by

$$\int_0^1 u\,dF = \sum_{i=0}^{n} \alpha_i F(B_i),$$

for u given by (3.1).

Lemma 3.3. *Let X be a 2-smoothable Banach space and Z be a X-valued Gaussian symmetric random measure with the covariance measure Q of finite variation. Then there exists a constant $C > 0$ such that for each simple non-anticipating process u in $L_2(|Q| \otimes P)$ we have*

$$\mathbf{E}\|\int_0^1 u\,dZ\|^2 \leq C\int_0^1 \mathbf{E}|u|^2\,d|Q|.$$

Proof. Let u be given by (3.1). For brevity we write Z_i for $Z(B_i)$. Since Z_i is symmetric, independent of $\mathcal{F}_{t_i}$ and α_i is $\mathcal{F}_{t_i}$-measurable with $\mathbf{E}|\alpha_i|^2 < \infty$, we have $\mathbf{E}\big[\alpha_i Z_i/\mathcal{F}_{t_i}\big] = 0$ and $\mathbf{E}\|\alpha_i Z_i\|^2 = \mathbf{E}|\alpha_i|^2\mathbf{E}\|Z_i\|^2$. Hence $(\alpha_i Z_i, \mathcal{F}_{t_{i+1}})_{i=0}^n$ constitutes a X-valued martingale differences sequence. In view of the Assouad-Pisier inequality there exists a constant $C_1 > 0$ such that

$$\mathbf{E}\|\sum_{i=0}^{n} \alpha_i Z_i\|^2 \leq C_1 \sum_{i=0}^{n} \mathbf{E}\|\alpha_i Z_i\|^2 = C_1 \sum_{i=0}^{n} \mathbf{E}|\alpha_i|^2 \mathbf{E}\|Z_i\|^2.$$

Noting that X is of type 2 and using Proposition 2.2 we get

$$\mathbf{E}\|Z_i\|^2 \leq C_2\|\big[Z_i, Z_i\big]\|_{nuc.} = C_2\|Q(B_i)\|_{nuc.} \leq C_2|Q|(B_i),$$

where C_2 is a constant. Consequently,

$$\mathbf{E}\|\int_0^1 u\,dZ\|^2 = \mathbf{E}\|\sum_{i=0}^{n} \alpha_i Z_i\|^2 \leq C_1C_2 \sum_{i=0}^{n} \mathbf{E}|\alpha_i|^2|Q|(B_i)$$

$$= C_1C_2\int_0^1 \mathbf{E}|u|^2\,d|Q|.$$

Lemma 3.4. *Let X be a q-smoothable Banach space and Z_p be a X-valued p-stable random measure with the characteristic measure Q_p. If $q > p$ then* *there exists a constant C such that for each simple non-anticipating process u in $L_p(|Q_p| \otimes P)$ and each $t > 0$, we have*

$$P\Big\{\|\int_0^1 u\,dZ\| > t\Big\} \le Ct^{-p}\int_0^1 \mathbf{E}|u|^p\,d|Q_p|.$$

Proof. Let u be given by (3.1). For brevity we write M_i for $Z_p(B_i)$. We have

$$\begin{aligned}
P\Big\{\|\int_0^1 u\,dZ_p\| > t\Big\} &= P\Big\{\|\sum_{i=0}^n \alpha_i M_i\| > t\Big\}\\
&= P\Big\{\|\sum_{i=0}^n \alpha_i M_i\| > t, \max_i \|\alpha_i M_i\| > t\Big\}\\
&+ P\Big\{\|\sum_{i=0}^n \alpha_i M_i\| > t, \max_i \|\alpha_i M_i\| \le t\Big\}\\
&\le \sum_{i=0}^n P\big\{\|\alpha_i M_i\| > t\big\}\\
&+ P\Big\{\|\sum_{i=0}^n \alpha_i M_i 1_{\{\|\alpha_i M_i\|\le t\}}\| > t\Big\}. \qquad (3.2)
\end{aligned}$$

Since X is of type $q > p$, by Proposition V.5.1 of [14] X is of stable type p. From Proposition 2.5 and the independence of α_i and M_i we obtain

$$P\big\{\|\alpha_i M_i\| > t\big\} = \int_0^\infty P\big\{\|M_i\| > t/x\big\}\,dP\{|\alpha_i| \le x\} \le$$

$$Kt^{-p}|Q_p|(B_i)\int_0^\infty x^p\,dP\{|\alpha_i| \le x\} = Kt^{-p}|Q_p|(B_i)\mathbf{E}|\alpha_i|^p,$$

where K is a constant. Hence

$$\begin{aligned}
\sum_{i=0}^n P\{\|\alpha_i M_i\| > t\} &\le Kt^{-p}\sum_{i=0}^n \mathbf{E}|\alpha_i|^p|Q_p|(B_i)\\
&= Kt^{-p}\int_0^1 \mathbf{E}|u|^p\,d|Q_p|, \qquad (3.3)
\end{aligned}$$

Next, for brevity we denote the set $\{\|\alpha_i M_i\| \le t\}$ by C_i. Since M_i is symmetric, independent of $\mathcal{F}_{t_i}$ and α_i is $\mathcal{F}_{t_i}$-measurable , we have

$$\mathbf{E}\big[\alpha_i M_i 1_{C_i}/\mathcal{F}_{t_i}\big] = \alpha_i \mathbf{E}\big[M_i 1_{\{\|M_i\| \le t/|\alpha_i|\}}/\mathcal{F}_{t_i}\big] = 0,$$

i.e. $\big(\alpha_i M_i 1_{C_i}, \mathcal{F}_{t_{i+1}}\big)_{i=0}^{n}$ is a X-valued bounded martingale differences sequence. In view of the Assouad-Pisier inequality there exists a constant $C_1 > 0$ such that

$$P\Big\{\|\sum_{i=o}^{n} \alpha_i M_i 1_{C_i}\| > t\Big\} \le t^{-q}\mathbf{E}\|\sum_{i=o}^{n} \alpha_i M_i 1_{C_i}\|^q$$

$$\le C_1 t^{-q} \sum_{i=0}^{n} \mathbf{E}\|\alpha_i M_i 1_{C_i}\|^q. \tag{3.4}$$

By the independence of M_i and α_i we get

$$\mathbf{E}\|\alpha_i M_i 1_{C_i}\|^q = \int_0^\infty y^q \mathbf{E}\big[\|M_i\|^q 1_{\{\|M_i\|<t/y\}}\big]\, dP\{|\alpha_i| \le y\}. \tag{3.5}$$

Integration by parts and Proposition 2.5 yield

$$\mathbf{E}\big[\|M_i\|^q 1_{\{\|M_i\|<t/y\}}\big] = \int_0^{t/y} x^q\, dP\{\|M_i\| \le x\}$$

$$\le q \int_0^{t/y} x^{q-1} P\{\|M_i\| > x\}\, dx$$

$$\le Kq|Q_p|(B_i) \int_0^{t/y} x^{q-1}x^{-p}\, dx = C_2|Q_p|(B_i)(t/y)^{q-p}, \tag{3.6}$$

where $C_2 = \frac{Kq}{q-p}$

From (3.5) and (3.6) we get

$$\mathbf{E}\|\alpha_i M_i 1_{C_i}\|^q \le C_2|Q_p|(B_i)t^{q-p} \int_0^\infty y^p\, dP\{|\alpha_i| \le y\}$$

$$= C_2|Q_p|(B_i)t^{q-p}\mathbf{E}|\alpha_i|^p.$$

Using this and (3.4) we get

$$
\begin{aligned}
P\Big\{\|\sum_{i=o}^{n}\alpha_i M_i 1_{C_i}\| > t\Big\} &\le C_1C_2t^{-p}\sum_{i=o}^{n}\mathbf{E}|\alpha_i|^p|Q_p|(B_i) \\
&= Ct^{-p}\int_0^1 \mathbf{E}|u|^p\, d|Q_p|, \qquad (3.7)
\end{aligned}
$$

where $C = C_1C_2$.

Finally, we get the inequality stated in Lemma 3.4 by using (3.2), (3.3) and (3.7).

Having established Lemma 3.3 and Lemma 3.4, we obtain the following result by a standard extension procedure

Theorem 3.5

1) *Under the assumption stated in Lemma 3.3 there exists a unique linear continuous mapping* $u \to \int_0^1 u\, dZ$ *from* $\mathcal{V}_2(Z, |Q|)$ *into* $L^2_X(\Omega)$ *such that for each simple non-anticipating* u *given by* (3.1) *we have*

$$\int_0^1 u\, dZ = \sum_{i=0}^{n}\alpha_i Z(B_i).$$

2) *Under the assumption stated in Lemma 3.4 there exists a unique linear continuous mapping* $u \to \int_0^1 u\, dZ$ *from* $\mathcal{V}_p(Z_p, |Q_p|)$ *into* $L^0_X(\Omega)$ *such that for each simple non-anticipating* u *given by* (3.1) *we have*

$$\int_0^1 u\, dZ_p = \sum_{i=0}^{n}\alpha_i Z_p(B_i).$$

The following theorem gives an extension of the isometric property of the stochastic integral w.r.t. a real-valued Gausian symmetric random measure to the vector case.

Theorem 3.6. *Let* u, v *be processes in* $\mathcal{V}_2(Z, |Q|)$. *Under the assumption stated in Lemma 3.3, the function* $t \to \mathbf{E}u(t)v(t)$ *is* Q*-integrable and we* *have*

$$\left[\int_0^1 u\,dZ, \int_0^1 v\,dZ\right] = \int_0^1 \mathbf{E}u(t)v(t)\,dQ. \tag{3.8}$$

In particular, the covariance operator of $\int_0^1 u\,dZ$ *is* $\int_0^1 \mathbf{E}|u(t)|^2\,dQ$.

Proof. At first, suppose that u and v are simple non- anticipating.

If $u = \sum_{i=0}^{n} \alpha_i 1_{B_i}$, $v = \sum_{i=0}^{n} \beta_i 1_{B_i}$, then

$$\left[\int_0^1 u\,dZ, \int_0^1 v\,dZ\right] = \sum_{i=0}^{n} [Z_i, Z_i]\mathbf{E}\alpha_i\beta_i =$$
$$\sum_{i=0}^{n} Q(B_i)\mathbf{E}\alpha_i\beta_i = \int_0^1 \mathbf{E}u(t)v(t)\,dQ,$$

where $Z_i = Z(B_i)$ and the first equality follows from the fact that Z_i, α_i, β_i are independent of Z_j for $i < j$.

Next, let u and v be bounded, $|u(t,\omega)| \le M$ and $|v(t,\omega)| \le M$. Suppose that (u_n) and (v_n) are simple non-anticipating processes converging to u and v in $L_2(|Q| \otimes P)$, respectively. Using Proposition 2.2 and formula (3.8) for (u_n), (v_n) we obtain

$$\begin{aligned}\left[\int_0^1 u\,dZ, \int_0^1 v\,dZ\right] &= \lim_n \left[\int_0^1 u_n\,dZ, \int_0^1 v_n\,dZ\right] \\ &= \lim_n \int_0^1 \mathbf{E}u_n(t)v_n(t)\,dQ\end{aligned} \tag{3.9}$$

in $N(X', X)$.

On the other hand, since $u_n \to u$ and $v_n \to v$ in $L_2(|Q| \otimes P)$, there exists a subsequence (n_k) such that $\lim_k \mathbf{E}|u_{n_k}(t) - u(t)|^2 = 0$ $|Q|$-a.e. and $\lim_k \mathbf{E}|v_{n_k}(t) - v(t)|^2 = 0$ $|Q|$-a.e.. This implies that $\lim_{k\to\infty} \mathbf{E}u_{n_k}(t)v_{n_k}(t) =$

$\mathbf{E}u(t)v(t)$ $|Q|$-a.e.. Using this and the dominated convergence theorem for the vector measure Q we get

$$\lim_{k\to\infty}\int_0^1 \mathbf{E}u_{n_k}(t)v_{n_k}(t)\,dQ = \int_0^1 \mathbf{E}u(t)v(t)\,dQ \quad \text{in } N(X', X). \tag{3.10}$$

From (3.9) and (3.10) we get (3.8) for bounded processes in $\mathcal{V}_2(Z, |Q|)$.

Finally, let u and v be arbitrary. Put $u_n(t,\omega) = u(t,\omega)$ if $|u(t,\omega)| \leq n, v_n(t,\omega) = v(t,\omega)$ if $|v(t,\omega)| \leq n$ and $u_n(t,\omega) = v_n(t,\omega) = 0$ otherwise. Clearly, $u_n \to u$ and $v_n \to v$ in $L_2(|Q| \otimes P)$. Again, by using Proposition 2.2 and formula (3.8) for (u_n) and (v_n) we have

$$\Big[\int_0^1 u\,dZ, \int_0^1 v\,dZ\Big] = \lim_{n\to\infty}\int_0^1 \mathbf{E}u_n(t)v_n(t)\,dQ \quad \text{in } N(X', X). \tag{3.11}$$

Reasoning as in the second part of the proof, we can choose a subsequence (n_k) such that

$$\lim_{k\to\infty} \mathbf{E}u_{n_k}(t)v_{n_k}(t) = \mathbf{E}u(t)v(t) \quad |Q|\text{-a.e..}$$

Since $u, v \in L_2(|Q| \otimes P)$, it follows that $\int_0^1 \mathbf{E}|u(t)v(t)|\,d|Q| < \infty$ which implies that the function $t \to \mathbf{E}|u(t)v(t)|$ is Q-integrable. Because $|\mathbf{E}u_{n_k}(t)v_{n_k}(t)| \leq \mathbf{E}|u(t)v(t)|$ for all t, we get

$$\lim_{k\to\infty}\int_0^1 \mathbf{E}u_{n_k}(t)v_{n_k}(t)\,dQ = \int_0^1 \mathbf{E}u(t)v(t)\,dQ \quad \text{in } N(X', X) \tag{3.12}$$

by using the dominated convergence theorem. From (3.11) and (3.12) we get the formula (3.8) as desired.

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