ACTA MATHEMATICA VIETNAMICA
Volume 21, Number 1, 1996, pp. 111–123

ON REPRESENTATIONS OF ENTIRE FUNCTIONS BY DIRICHLET SERIES IN INFINITE DIMENSION

PHAN THIEN DANH

1. Introduction

For complex locally space E and F let $\mathcal{H}(E,F)$ denote the space of holomorphic functions on E with values in F. This space is equipped with the compact-open topology. Instead of $\mathcal{H}(E,\mathbf{C})$ we write $\mathcal{H}(E)$. In the present note we shall investigate the representations of an entire function f on E^*, the strongly dual space of E, in the exponential form

$$\text{(Exp)} \qquad f(x^*) = \sum_{j\geq 1} \xi_j \exp\langle x_j, x^*\rangle \quad \text{for} \quad x^* \in E^*$$

where $\xi_j \in \mathbf{C}$ and $x_j \in E$ for $j \geq 1$.

Such representations in the one complex variable case were first given by Leontiev [9], Korobeinik [7] and in the several complex variable case by L. H. Khoi, Mozakov, Napalkov, Chan Porn,... They have proved that for every convex domain D in $\mathbf{C}^n$ there exists a seqence $\{\lambda^j\} \subset \mathbf{C}^n$ such that

a) $\lim |\lambda^j| = +\infty$

b) every holomorphic function f on D can be written in the form

$$f(z_1,\dots,z_n) = \sum_{j\geq 1} \xi_j \exp(\lambda_1^j z_1 + \cdots + \lambda_n^j z_n).$$

In the case where E is a nuclear Frechet space the existence of a sequence $\{x_j\} \subset E$ for which (Exp) holds for entive functions on E^* was shown by Chan Porn [12]. However the growth of the sequence $\{x_j\}$ as in a) is not considered. Our aim here is to find a necessary and sufficient condition for E such that every entire function on E^* can be written in the form (Exp) in which the growth of the sequence $\{x_j\}$ is controlled. Recently, N. M. Ha and N. V. Khue ([4], [5]) and next L. M. Hai [6] have investigated the problem for entire functions on nuclear Frechet spaces in the interrelation with the linear topological invariants.

Received May 16, 1995; in revised form December 12, 1995.

2. Notions and results

We shall use notions from the theory of locally convex spaces as presented in the books of Pietsch [11] and Schaefer [13] and the theory of holomorphic functions as in the book of Colombeau [3]. All locally convex spaces are assumed to be complex vector spaces and Hausdorff.

Let E be a locally convex space. By $\mathcal{B}(E)$ we denote the family of all closed bounded balanced convex subsets of E. For each $B \in \mathcal{B}(E)$, write $E(B)$ for the normed space spanned by B and $H_b(E(B))$ the Frechet space of holomorphic functions of bounded type on $E(B)$. Here a holomorphic function on $E(B)$ is called of bounded type if it is bounded on every bounded set in $E(B)$.

Our main result is as follows.

Theorem 1. (1) *Let E be a nuclear Frechet space. Then for every $K \in \mathcal{B}(E)$, there exists an increasing sequence $\{K_n\} \subset \mathcal{B}(E)$ with $K_1 = K$ and sequences $\{x_j^n\} \subset E(K_n)$ such that*

(i)

$$\sum_{j\geq 1} \exp -\|x_j^n\|_{K_n} < \infty \quad \text{for} \quad n \geq 1 \tag{1}$$

(ii) *Every $f \in \bigcup_{n\geq 1} H_b([E(K_n)]^*)$ can be written in the form*

$$f(x^*) = \sum_{j\geq 1} \xi_j \exp\langle x_j^n, x^*\rangle \quad \text{for} \quad x^* \in [E(K_n)]^* \tag{2}$$

with some $n = n_f$ for which

$$\sum_{j\geq 1} |\xi_j| \exp \|x_j^n\|_{K_n} < \infty \quad \text{for all} \quad r > 0 \tag{3}$$

(iii) *If E is a Montel-Frechet space satisfying the conclusion of* (1), *then E is nuclear.*

Corollary 2. *Let E be a nuclear Frechet space. Then for every entire function f on E^* there exists a sequence $\{x_j\} \subset E$ and $L \in \mathcal{B}(E)$ such that*

$$\sum_{j\geq 1} \exp -\|x_j\|_L < \infty$$

and

$$f(x^*) = \sum_{j\geq 1} \xi_j \exp\langle x_j, x^* \rangle \quad \textit{for} \quad x^* \in E^*.$$

Moreover, the series is convergent in $H(E^*)$.

Proof. Given $f \in H(E^*)$. By Colombeau and Mujica (see [3]) we can find $K \in \mathcal{B}(E)$ such that f can be considered as a holomorphic function on $[E(K)]^*$ of bounded type. By applying Theorem 1 (1) there exists $L \in \mathcal{B}(E)$, $K \subset L$ and a sequence $\{x_j\} \subset E$ such that (1), (2) and (3) hold, where K_{n_f} and $\{x_j{}^n f\}$ are replaced by L and $\{x_j\}$, respectively.

Since for every continuous semi-norm $\|\cdot\|$ on E there exists $C > 0$ such that

$$\|x\| \leq C\big\|x\big\|_L \quad \text{for} \quad x \in E,$$

it follows that

$$\sum_{j\geq 1} |\xi_j| \exp\ r\|x_j\| < \infty \quad \text{for} \quad r > 0.$$

This yields the convergence of the series $\sum\limits_{j\geq 1} \xi_j \exp\langle x_j, x^* \rangle$ in $H(E^*)$. The corollary is proved.

3. Proof of Theorem 1

For the proof of Theorem 1 (1) we shall need the following

Lemma 3. *Let* T *be a nuclear map from a Banach space* X *to a Banach space* Y *and let* $\big\{f^\alpha\big\}_{\alpha\in I}$ *be a family of holomorphic functions on* Y. *Assume that there exists* C, $A > 0$ *such that*

$$|f^\alpha(y)| \leq C \exp A\|g\| \quad \textit{for all} \quad y \in Y \quad \textit{and} \quad \alpha \in I.$$

Then there exists an equicontinuous family $\big\{\mu_\alpha\big\}_{\alpha\in I} \subset [H_b(X)]^*$ *such that*

$$\langle \exp x, \mu_\alpha \rangle = f^\alpha(Tx) \quad \text{for all} \quad x \in X \quad \text{and all} \quad \alpha \in I.$$

Proof. By the hypothesis there exists $r,\ s > 0$ such that

$$\big\|\widehat{d^b f^\alpha}(0)\big\| \leq rs^k \quad \text{for all} \quad \alpha \in I \quad \text{and all} \quad b \geq 0,$$

where $\widehat{d^k f}(0)$ denotes the k-homogeneous polynomial associated to $d^k f^\alpha(0)$ [7].

We consider here a nuclear representation of T,

$$T(x) = \sum_{j \geq 1} \langle x, u_j \rangle e_j$$

with

$$a = \sum_{j \geq 1} \|u_j\| \ |e_j| < \infty.$$

For each $\alpha \in I$ and for $\sigma \in H_b(X^*)$, put

$$\langle \sigma, \mu_\alpha \rangle = \sum_{k \geq 0} \sum_{j_1, \ldots, j_{k \geq 1}} d^k f^\alpha(0)(e_{j_1}, \ldots, e_{j_k}) \frac{d^k \sigma(0)}{k!}(u_{j_1}, \ldots, u_{j_k}).$$

We have, by the Cauchy inequality

$$\begin{aligned}
&\sum_{k \geq 0} \sum_{j_1, \ldots, j_{k \geq 1}} \left| d^k f^\alpha(0)(e_{j_1}, \ldots, e_{j_k}) \right| \left| \frac{d^k \sigma(0)}{k!}(u_{j_1}, \ldots, u_{j_k}) \right| \\
&\leq \sum_{k \geq 0} \sum_{j_1, \ldots, j_{k \geq 1}} \frac{k^k}{k!} \|\widehat{d^k f^\alpha}(0)\| \ \|e_{j_1}\| \ldots \|e_{j_k}\| \ \frac{k^k}{k!} \ \left\| \frac{\widehat{d^k \sigma}(0)}{k!} \right\| \ \|u_{j_1}\| \ldots \|u_{j_n}\| \\
&\leq \sum_{k \geq 0} \sum_{j_1, \ldots, j_{k \geq 1}} \frac{k^{2k}}{(k!)^2} r s^k \big\|\sigma\big\|_{\rho/\rho^k} \|u_{j_1}\| \ \|e_{j_1}\| \ldots \|u_{j_n}\| \ \|e_{j_n}\| \\
&= r \big\|\sigma\big\|_\rho \sum_{k \geq 0} \frac{k^{2k}}{(k!)^2 \rho^k} \Big(\sum_{k \geq 1} \|u_j\| \ \|e_j\| \Big)^k \\
&= r \big\|\sigma\big\|_\rho \sum_{k \geq 0} \frac{a^k}{\rho^k} \ \frac{k^{2k}}{(k!)^2} = C(r, \rho) \big\|\sigma\big\|_\rho,
\end{aligned}$$

where

$$C(r, \rho) = r \sum_{k \geq 0} \frac{a^k k^{2k}}{\rho^k (k!)^2} < \infty \quad \text{for } \rho \text{ sufficiently large},$$

and

$$\big\|\sigma\big\|_\rho = \sup \Big\{ |\sigma(x^*)| : \|x^*\| < r \Big\}.$$

This inequality shows that the family $\big\{\mu_\alpha\big\}_\alpha$ is equicontinuous in $[H_b(X^*)]^*$.

Moreover, we also have

$$\begin{aligned}\langle exp\ x, \mu_\alpha \rangle &= \sum_{k\geq 0} \sum_{j_1,\dots,j_k\geq 1} d^k f^\alpha(0)(e_{j_1},\dots,e_{j_k})\Big(\frac{1}{k!}\Big)\langle x, u_{j_1}\rangle \dots \langle x, u_{j_k}\rangle \\ &= \sum_{k\geq 0} \frac{d^k f^\alpha(0)}{k!}\Big(\sum_{j\geq 1}\langle x, u_j\rangle e_j\Big) = \sum_{k\geq 0}\frac{d^k f^\alpha(0)}{k!}(Tx) \\ &= f^\alpha(Tx) \quad \text{for} \quad x \in X.\end{aligned}$$

The lemma is proved.

Proof of Theorem 1.

(1) Assume first that E is a nuclear Frechet space. Given $K \in \mathcal{B}(E)$. Put $K_1 = K$. Choose $K_2 \in \mathcal{B}(E)$, $K_1 \subset K_2$ such that $E(K_1)$ is dense in $E(K_2)$ and the identity map $E(K_1) \to E(K_2)$ can be written in the form

$$x = \sum_{k\geq 1} \lambda_k \langle x, u_k\rangle e_k$$

with

$$\sum_{k\geq 1} \|u_k\| + \sum_{k\geq 1}\|e_k\| \leq 1$$

and

$$\lambda_k \sim O(1/k^8).$$

For each $n \geq 1$, there exists a finite $1/\sqrt{n}$ - net A_n^1 of $nK_1 \setminus (n-1)K_1$ for the norm of $E(K_2)$ such that

$$\text{card } A_n^1 \leq (4Cn^2)^{\sqrt{Cn}}$$

where C is independent of n.

Indeed, choose $k_0 = \sqrt[4]{C}\ n^{3/8}$, where $C > 0$ such that

$$|\lambda_k| \leq C/k^8 \quad \text{for} \quad k \geq 1.$$

Then

$$\sum_{k\geq k_0} |\lambda_k|\ |\langle x, u_k\rangle|\ \|e_k\| \leq nC \sum_{k\geq k_0}\frac{1}{k^8} \leq n\frac{C}{k_0^4} \leq \frac{1}{6\sqrt{n}}$$

for all $x \in nK_1$.

Consider a finite $\dfrac{1}{2\sqrt{n}}$ - net A_n^1 of the set

$$W_n = \Big\{ \sum_{1 \le k \le k_0} \lambda_k \langle x, u_k \rangle e_k : x \in nK_1 \setminus (n-1)K_1 \Big\}$$

for the norm $\|.\|_{K_2}$ with

$$\operatorname{card} A_n \le \big(4Cn\sqrt{n}\big)^{\sqrt[4]{C}\ n^{3/8}} \le \big(4Cn^2\big)^{\sqrt{Cn}}.$$

Such a net exists, because W_n is contained in the image of

$$\Big\{ \big\{\xi_k\big\}_{1 \le k \le k_0} \in \mathbf{C}^{k_0} : |\xi_k| \le Cn \Big\}$$

under the map

$$S : \mathrm{C}^{k_0} \longrightarrow E(K_2) : S\big(\big\{\xi_k\big\}_{1 \le k \le k_0}\big) = \sum_{1 \le k \le k_0} \xi_k e_k.$$

We have for $A_1 = \bigcup\limits_{n \ge 1} A_n^1 = \{x_j^1\} \subset E(K_1)$,

$$(4)_1 \qquad \sum_{x \in A_1} \exp\ -\big\|x\big\|_{K_1} = \sum_{n \ge 1} \sum_{x \in A_n^1} \exp\ -\big\|x\big\|_{K_1}$$
$$\le \sum_{n \ge 1} \big(4Cn^2\big)^{\sqrt{Cn}} e^{-(n-1)} < \infty$$

because

$$e^{-1} \frac{\big(4C(n+1)^2\big)^{\sqrt{C(n+1)}}}{(4Cn)^{2\sqrt{6Cn}}} =$$
$$e^{-1} \Big[\Big(\frac{4\big((n+1)^2\big)}{4Cn^2} \Big)^n \Big]^{\frac{\sqrt{C(n+1)}}{n}} \Big(4Cn^2\Big)^{\sqrt{C(n+1)} - \sqrt{Cn}} \to e^{-1} \text{ as } n \to \infty.$$

Put

$$\operatorname{Exp}_r(K_2) = \Big\{ f \in H(E) : |||f|||_r = \sup \Big\{ \frac{|f(x)|}{\exp\ r\big\|x\big\|_{K_2}} : x \in E(K_2) \Big\} < \infty \Big\}.$$

For each $\varepsilon > 0$ choose n_0 and $C_1^r \geq 1$ such that

$$e^{\frac{6r}{\sqrt{n_0}} - n_0}/2 < \varepsilon, \quad e^{6r/\sqrt{n_0}} < 3/2 \quad \text{and}$$

$$|||f|||_{2r} \leq C_1^n \sup\Big\{\frac{|f(x)|}{\exp 2r\|x\|_{K_2}} : \|x\|_{K_2} > n_0^{+1}\Big\} \quad \text{for} \quad f \in \mathrm{Exp}_r(K_2).$$

Given $f \in \mathrm{Exp}_r(K_2)$ with $|||f|||_r \leq 1$. For each $x \in (n+1)K_1 \setminus nK_1$, $n \geq n_0$, take $y_x \in A_1$ such that $\|x - y_x\|_{K_2} < \dfrac{1}{\sqrt{n}}$. Since

$$\begin{aligned}
\big|f(x) - f(y_x)\big| &\leq \int_0^1 \big|f'(x + t(x - y_x))(x - y_x)\big| dt \\
&= \int_0^1 \Big|\frac{1}{2\pi i} \int_{|\lambda|=2} \frac{f(x + (t+\lambda)(x - y_x))}{\lambda^2} dx\Big| dt \\
&\leq \frac{1}{2} \sup \Big\{\big|f(x + \lambda(x - y_x))\big| : |\lambda| \leq 3\Big\},
\end{aligned}$$

we have for $x \in (n+1)K_1 \setminus nK_1$, $n \geq n_0$,

$$\begin{aligned}
\frac{|f(x)|}{\exp 2r\|x\|_{K_2}} &\leq \frac{|f(y_x)|}{\exp 2r\|y_x\|_{K_2}} \exp 2r\|x - y_x\|_{K_2} + \frac{|f(x) - f(y_x)|}{\exp 2r\|x\|_{K_2}} \\
&\leq \frac{2|f(y_x)|}{\exp 2r\|y_x\|_{K_2}} + \frac{1}{2} \sup \Big\{\frac{|f(x + \lambda(x - y_x))|}{\exp 2r\|x + \lambda(x - y_x)\|_{K_2}} \times \\
&\quad \times \exp 2r|\lambda| \ \|x - y_x\|_{K_2} : |\lambda| \leq 3\Big\} \leq \frac{2|f(y_x)|}{\exp 2r\|y_x\|_{K_2}} \\
&\quad + \frac{1}{2} \sup \Big\{\frac{|f(z)|}{\exp r\|z\|_{K_2}} \exp -r\|z\|_{K_2} : n \leq \|z\|_{K_2} \leq n+1\Big\} \\
&\quad + \frac{3}{4} \sup \Big\{\frac{|f(z)|}{\exp 2r\|z\|_{K_2}} : \|z\|_{K_2} \geq n+1\Big\} \\
&\leq \frac{2|f(y_x)|}{\exp 2r\|y_x\|_{K_2}} + \frac{3}{4} \sup\Big\{\frac{|f(z)|}{\exp 2r\|z\|_{K_2}} : \|z\|_{K_2} \geq n+1\Big\} + \varepsilon
\end{aligned}$$

These inequalities imply (as $\varepsilon \to 0$)

$$\frac{3}{4} \sup \Big\{\frac{|f(x)|}{\exp 2r\|x\|_{K_2}} : \|x\|_{K_2} \geq n+1\Big\} \leq 2 \sup \Big\{\frac{|f(x)|}{\exp 2r\|x\|_{K_2}} : x \in A_1\Big\}.$$

Hence

$$(5)_1 \ \left|\left|\left|f\right|\right|\right|_{2r} \leq M_1^r \sup \left\{\frac{|f(x)|}{\exp\, 2r\left\|x\right\|_{K_2}} : x \in A_1\right\} \quad \text{for} \quad f \in \mathrm{Exp}_r(K_2)$$

where

$$M_1^r = 8C_1^r$$

Since $E(K_1)$ is dense in $E(K_2)$, from (1) we get

$$(6)_1 \ \sup\left\{\frac{|f(x)|}{\exp\, 2r\left\|x\right\|_{K_2}} : x \in f(K_2)\right\} \leq M_1^r \sup \left\{\frac{|f(x)|}{\exp\, 2r\left\|x\right\|_{K_2}} : x \in A_1\right\}$$

for $f \in \mathrm{Exp}_r(K_2)$.

Repeating the above argument for $K = K_2$ we can find $K_3 \in \mathcal{B}(E)$, $K_2 \subset K_3$ with $E(K_2)$ is dense in $E(K_3)$ and a sequence $A_2 = \{x_j^2\} \subset E(K_2)$ satisfying $(1)_2$, $(2)_2$ and $(3)_2$.

Continuing this process we get an increasing sequence $\{K_n\} \subset \mathcal{B}(E)$ with $K_1 = K$ and sequences $A_n = \{x_j^n\} \subset E(K_n)$ such that

$$(4)_n \qquad \sum_{j\geq 1} \exp\, -\left\|x_j^n\right\|_{K_n} < \infty,$$

$$(5)_n \qquad \left|\left|\left|f\right|\right|\right|_{2r} \leq M_n^r \sup \left\{\frac{|f(x)|}{\exp\, 2r\left\|x\right\|_{K_{n+1}}} : x \in A_n\right\}$$

for all $f \in \mathrm{Exp}_r(K_{n+1})$, all $n \geq 1$, $r \geq 0$, and

$$(6)_n \qquad \sup \left\{\frac{|f(x)|}{\exp\, 2r\left\|x\right\|_{K_{n+1}}} : x \in E_{n+1}\right\} \leq M_n^r \sup \left\{\frac{|f(x)|}{\exp\, 2r\left\|x\right\|_{K_{n+1}}} : x \in A_n\right\}$$

for all $f \in \mathrm{Exp}_r(K_{n+1})$, $n \geq 1$, $r > 0$. Moreover the canonical maps $E_n \longrightarrow E_{n+1}$ are nuclear.

For each $n \geq 1$, put

$$L_n = \left\{(\xi_j) \subset \mathbf{C} : \sum_{j\geq 1} |\xi_j| \exp\, r\left\|x_j^n\right\|_{K_n} < \infty \quad \text{for all } r \geq 0\right\}.$$

By $(4)_n$, L_n are nuclear Frechet spaces. Define

$$R_p : \sum_{1\le n\le p} L_n \longrightarrow H_b(E^*_{p+1})$$

by

$$R_p\big((\xi^1_j),\dots,(\xi^p_j)\big) = \sum_{n=1}^{p}\sum_{j\ge 1}\xi^n_j \exp\langle x^n, x^n_j\rangle \quad \text{for } x^*\in E^*_{p+1}$$

and

$$R = \lim R_p : \sum_{n\ge 1} L_n \longrightarrow \bigcup_{n\ge 1} H_b(E^*_n).$$

To complete the necessary part of the proof it suffices to show that the map

$$R = R^{**} : \Big[\prod_{n\ge 1} L^*_n\Big]^* \longrightarrow \Big[\lim\operatorname{proj}\big[H_b(E^*_n)\big]^*\Big]^*$$

is surjective because

$$\Big[\prod_{n\ge 1} L^*_n\Big]^* \cong \sum_{n\ge 1} L^{**}_n \cong \sum_{n\ge 1} L_n$$

and

$$\bigcup_{n\ge 1} H_b(E^*_n) \hookrightarrow \Big[\lim\operatorname{proj}\,\big[H_b(E^*_n)\big]^*\Big]^*.$$

Given $g \in \Big[\lim\ \operatorname{proj}\ \big[H_b(E^*_n)\big]^*\Big]^*$. Choose $p \ge 3$ such that $g \in \big[H_b(E^*_{p-1})\big]^{**}$. Consider the commutative diagram

$$\begin{tikzcd}
\big[H_b(E^*_{p+3})\big]^* \arrow[rr, "\omega^{p+1}_{p+3}"] \arrow[dd, "R^*_{p+2}"'] & & \big[H_b(E^*_{p+1})\big]^* \arrow[rr, "\omega^{p}_{p+1}"] \arrow[dd, "R^*_{p}"'] & & \big[H_b(E^*_{p})\big]^* \arrow[rr, "\omega^{p-1}_{p}"] \arrow[dd, "R^*_{p-1}"'] & & \big[H_b(E^*_{p-1})\big]^* \arrow[dd, "R^*_{p-2}"'] \arrow[dl, "g"'] \\
 & & & & & \mathbf{C} & \\
\prod\limits_{1\le n\le p+2} L^*_n \arrow[rr, "\Pi^{p}_{p+2}"'] & & \prod\limits_{1\le n\le p} L^*_n \arrow[rr, "\Pi^{p-1}_{p}"'] & & \prod\limits_{1\le n\le p-1} L^*_n \arrow[rr, "\Pi^{p+1}_{p-1}"'] & & \prod\limits_{1\le n\le p-2} L^*_n
\end{tikzcd}$$

where ω^{p+1}_{p+3}, ω^{p}_{p+1}, ω^{p-1}_{p}, Π^{p}_{p-2}, Π^{p-1}_{p} and Π^{p-2}_{p-1} are canonical maps. It is easy to see that $(6)_{p-1}$ together with Lemma 3 imply

$$\ker R^*_{p-\Lambda} \subseteq \ker\omega^{p-1}_{p}.$$

Thus g can be considered as a linear functional $\overline{g} : \operatorname{Im} R^*_{p-\Lambda} \longrightarrow \mathbf{C}$.

Let us check

$$\Pi^{p-1}_{p+2}\big(\text{cl Im}\, R^*_{p+2}\big) \subseteq \operatorname{Im} R^*_{p-1}.$$

Let $\{W_k\}$ be a decreasing neighbourhood basis of $0 \in \sum\limits_{1\le n\le p+2} L_n$ and

$$M = \bigcup_{k\ge 1} \text{cl}\,\big(\operatorname{Im} R^*_{p+2} \cap W^0_k\big).$$

Since $\sum\limits_{1\le n\le p+2} L_n$ is nuclear Frechet and M is sequentially closed, it follows that M is closed and hence

$$\text{cl Im}\, R^*_{p+2} = M.$$

Assume that

$$\Big\{\eta^\alpha = R^*_{p+2}(\mu^\alpha)\Big\}_{\alpha\in I} \longrightarrow \eta \in \prod_{1\le n\le p+2} L^*_n$$

with $\{\eta^\alpha\} \subseteq W^0_k$, the polar of W_k in $\Big(\sum\limits_{1\le n\le p+2} L_n\Big)^*$.

Choose $r > 0$ such that

$$\sup\Big\{\frac{|f^\alpha(x)|}{\exp r\big\|x\big\|_{K_{n+1}}} : x \in A_n, \quad \alpha\in I, \quad n = 1,\dots,p+2\Big\} < \infty,$$

where

$$f^\alpha(x) = \langle \exp\langle x^*, x\rangle, \mu^\alpha\rangle \quad \text{for} \quad x \in E_{p+3}.$$

Such a $r > 0$ exists because

$$\langle \exp\langle x^*, x^n_j\rangle, \mu^\alpha\rangle = \eta^\alpha_j \quad \text{for} \quad \alpha\in I, \quad j\ge 1 \quad \text{and } n = 1,\dots,p+2.$$

By Lemma 3 from $(4)_{p+2}$ and $(5)_{p+2}$ it follows that $\big\{\omega^{p+1}_{p+3}(\mu^\alpha)\big\}$ is equicontinuous in $\big[H_b(E^*_{p+1})\big]^*$, and hence without loss of generality we may assume that $\omega^p_{p+3}(\mu_\alpha) \longrightarrow \mu$. Obviously,

$$\Pi^{p-1}_{p+2}\eta = R^*_{p-1}\mu.$$

It remains to show that $\overline{g}\ \Pi^{p-1}_{p+2}$ is continuous on Cl $\operatorname{Im} R^*_{p+2}$. Since Cl $\operatorname{Im} R^*_{p+2}$ is a (DFN)-space it suffices to check that

$$\overline{g}\ \Pi^{p-1}_{p+2}(\eta^k) \longrightarrow 0 \quad \text{for every sequence} \quad \{\eta^k\} \subset \text{cl } \operatorname{Im} R^*_{p+2}, \quad \eta^k \to 0.$$

By $(5)_p$ and Lemma 3 applying the inclusion

$$\Pi^p_{p+2}\big(\text{cl } \operatorname{Im} R^*_p\big) \subseteq \operatorname{Im} R^*_p$$

we can find an equicontinuous family $\big\{\mu_k\big\} \subset \big[H_b(E^*_{p+1})\big]^*$ such that

$$R^*_p(\mu_k) = \Pi^p_{p+2}(\eta^k) \quad \text{for} \quad k \geq 1.$$

Then

$$\omega^p_{p+1}(\mu_k) \longrightarrow 0$$

and hence

$$\lim \overline{g}\, \Pi^{p-1}_{p+2}(\eta^k) = \lim g\omega^{p-1}_{p+1}(\mu_k) = 0.$$

(2) It suffices to prove that every continuous linear map T from E^* into $\ell^\infty(S)$ is nuclear for every set S. Choose $K \in \mathcal{B}(E)$ such that T can be considered as a continuous linear map from $\big[E(K)\big]^*$ into $\ell^\infty(S)$. Let $\{K_n\}$ and $\{x^n_j\} \subset E(K_n) := E_n$ satisfy (i), (ii) with $K = K_1$ of the theorem. Since E is a Frechet-Montel space, without loss of generality we may assume that the canonical maps $E_n \longrightarrow E_{n+1}$ are compact. As in (1) consider the maps

$$R_p : \bigoplus_{\eta=n\leq p} L_n \longrightarrow H_b(E^*_{p+1})$$

and

$$R = \lim\operatorname{ind} R_p : \bigoplus_{p\geq 1} L_p \longrightarrow \bigcup_{p\geq 1} H_b(E^*_p).$$

By the hypothesis we have

$$H_b(E^*_1) \subseteq \bigcup_{p\geq 1} H_b(E^*_p) = \bigcup_{p\geq 1} \operatorname{Im} R_p.$$

By a result of Leiterer [8] we can find $p \geq 1$ such that

$$H_b(E^*_1) \subseteq \operatorname{Im} R_p.$$

Moreover, the identity map $H_b(E_1^*) \longrightarrow \operatorname{Im} R_p$ is continuous. The closed graph theorem implies that this map is also continuous for the quotient topology $\operatorname{Im} R_p = \bigoplus L_n/\operatorname{kern} R_p$. Consider the map $\hat{T} : \big[\ell^\infty(S)\big]^* \longrightarrow H_b(E_1^*) \subset \operatorname{Im} R_p$,

$$\hat{T}(\mu)(x^*) = \mu(Tx^*) \quad \text{for} \quad x^* \in E_1^*.$$

It follows that $\hat{T}$ is continuous linear. Since $\{x_j^n\}$ satisfies (i) for $n \geq 1$, the space $\bigoplus\limits_{1 \leq n \leq p} L_n$ is nuclear Frechet. Hence $\hat{T}$ can be lifted to a continuous linear map

$$\tilde{T} : \big[\ell^\infty(S)\big]^* \longrightarrow \prod_{1 \leq n \leq p} L_n.$$

This means that

$$\begin{aligned}\mu(Tx^*) = \hat{T}(\mu)(x^*) = R_p\tilde{T}(\mu)(x^*) &= R_p\Big(\sum_{n=1}^{p}\sum_{j\geq 1}\xi_j^n\big(\tilde{T}(\mu)\big)e_j^n\Big)\\ &= \sum_{n=1}^{p}\sum_{j\geq 1}\xi_j^n\big(\tilde{T}(\mu)\big)\exp\langle x_j^n, x^*\rangle \quad \text{for } x^* \in E_p^* \text{ and } \mu \in \big[\ell^\infty(S)\big]^*,\end{aligned}$$

in which

$$\sum_{n=1}^{p}\sum_{j\geq 1}\big|\xi_j^n\big(\tilde{T}(\mu)\big)\big| \exp r\big\|x_j^n\big\|_{K_n} < \infty \quad \text{for all } r \geq 0,$$

where $\{e_j^n\}$ is the canonical basis of L_n for $n \geq 1$.

This inequality yields

$$\begin{aligned}&\sum_{n=1}^{p}\sum_{j\geq 1}\big\|\xi_j^n\tilde{T}\big\| \exp\,\big\|x_j^n\big\|_{K_n} \leq \\ &\leq \Big\{\sup\sum_{n=1}^{p}\big\|\xi_j^n\tilde{T}\big\| \exp\,2\big\|x_j^n\big\|_{K_n}\Big\}\sum_{n=1}^{p}\sum_{j\geq 1}\exp\,-\big\|x_j^n\big\|_{K_n} < \infty.\end{aligned}$$

Hence

$$T(x^*) = \sum_{n=1}^{p}\sum_{j\geq 1}\xi_j\tilde{T}\exp\langle x_j^n, x^*\rangle \quad \text{for } x^* \in E^*$$

with

$$\sum_{n=1}^{p} \sum_{j \geq 1} \left\| \xi_j^n \tilde{T} \right\| \, \left\| x_j^n \right\|_{K_n} < \infty,$$

which means that T is nuclear. The theorem is proved.

Acknowledgement

The author would like to thank Prof. Nguyen Van Khue for suggestions that let to improvements of the presentation of the paper.

References

1. P. J. Boland, *Holomorphic functions in nuclear spaces*, Trans. Amer. Math. Soc. **209** (1975), 275-281.
2. P. J. Boland, *Malgrange theorem for entire functions in nuclear spaces*, Lecture Notes in Math. **364** (1974), 135-144.
3. J. F. Colombeau, *Differential Calculus and Holomorphy*, North-Holland Math. Stud. 65, North-Holland 1982.
4. Nguyen Minh Ha and Nguyen Van Khue, *The property* (H_u) *and* $(\widetilde{\Omega})$ *with the exponential representation of holomorphic functions*, Publ. Mat. **38** (1) (1994), 37-41.
5. Nguyen Minh Ha and Nguyen Van Khue, *The property* (DN) *and the exponential representation of holomorphic functions*, to appear in Mon. Math.
6. Le Mau Hai, *Linear topological invariants and Dirichlet representation of holomorphic functions*, to appear in Rendi. Math.
7. Yu. Korobeinik, *Representation systems*, Uspekhi of Math. Nauk. **31** (1) (1981), 73-126 (in Russian).
8. J. Leiterer, *Banach coherent Frechet sheaves*, Math. Narch. **85** (1978), 91-100.
9. A. F. Leontiev, *Series of Exponents*, Nauka, Moscow 1976 (in Russian).
10. M. C. Matos, *On convolution operators in spaces of entire functions of a given type and order*, Complex Analysis, Functional Analysis and Approximation Theory, J. Mujica (Editor), Elsevier Science Publishers, North-Holland 1986.
11. A. Pietsch, *Nuclear Locally Convex Spaces*, Akademie Verlag, Berlin 1972.
12. Chan Porn, *Sur la représentation exponentiel des entiers fonctions dans infini dimension*, Acta Math. Vietnamica **17** (1992), 1-10.
13. H. Schaefer, *Topological Vector Spaces*, Macmillan, New-York 1966.

Department of Mathematics, Ho Chi Minh City University of Education