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A PARAMETRIC SIMPLEX METHOD FOR OPTIMIZING A LINEAR FUNCTION OVER THE EFFICIENT SET OF A BICRITERIA LINEAR PROBLEM

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Abstract. The problem of optimizing a linear function over the efficient set of a multiple objective problem has many applications in multiple criteria decision making. The main difficulty of this problem is that its feasible region, in general, is a nonconvex set. In this paper we develop a fast algorithm for optimizing a linear function over the efficient set of a bicriteria linear programming problem. The method is a parametric simplex procedure using one parameter in the objective function.

1. Introduction

Let X be a nonempty bounded polyhedral convex set in R^n given by a system of linear inequalities and/or equalities. Let C denote a $(p \times n)$-matrix. The multiple objective linear programming problem (MP) given by

$$\text{(MP)} \qquad V_{\max}\{Cx, x \in X\}$$

is the problem of finding all efficient point of Cx over X. This problem is called bicriteria linear programming problem if C has two rows.

We recall that a point x^0 is said to be an *efficient point* of Cx over X if there no is vector $y \in X$ such that $Cy \geq Cx^0$. An efficient point is often called *Pareto* or *nondominated point.*

Throughout this paper, for two vectors of k-dimensions $a = (a_1, \dots, a_k)$ and $b = (b_1, \dots, b_k)$ we write $a \geq b$ (resp. $a > b$) if and only if $a_i \geq b_i$ (resp. $a_i \geq b_i, a \neq b$) for every i.

Let X_E denote the efficient set of Problem (MP). The linear optimization problem over the efficient set of (MP) can be stated as

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$$\text{(P)} \qquad \max\{d^T x, x \in X_E\}.$$

Recently, interest in this problem has been intensifying since it has many applications in multiple criteria decision making [11, 12]. Problem (P) can be classified as a global optimization problem since its feasible domain, in general, is a nonconvex set.

A few algorithms have been proposed for finding globally optimal solution of (P). Philip [10] first proposed Problem (P) and outlined a cutting plane method for finding an optimal solution of this problem. In [1, 2], using the fact that one can find a simplex S in R^p for which Problem (P) is reduced to the infinitely-constrained Problem (Q) formulated as

$$\text{(Q)} \qquad \max d^T x$$

subject to

$$\lambda^T Cx \geq \lambda^T Cy \quad \forall y \in X,$$
$$x \in X, \quad \lambda \in S,$$

Benson proposed two algorithms for finding a global solution to (P). In both these methods, at each iteration k, Problem (Q) is relaxed by Problem (P_k) given by

$$(P_k) \qquad \max d^T x$$

subject to

$$\lambda^T Cx \geq \lambda^T Cx^i \quad (i = 1, \ldots, k)$$
$$x \in X, \quad \lambda \in S.$$

In the first algorithm this relaxed problem is solved by a branch-and-bound procedure using the convex envelope of the bilinear term $\lambda^T Cx$. In the second algorithm it is weakened by finding a feasible point (λ^k, x^k) such that $d^T x^k > LB_k$, where LB_k is the best known lower bound found at iteration k.

By taking $h(\lambda, x) := \max_{y\in X} \lambda^T Cy - \lambda^T Cx$, Problem (Q) is reduced to the problem

$$\max d^T x$$

subject to

$$h(\lambda, x) \leq 0, \quad x \in X, \quad \lambda \in S.$$

Since $h(\lambda, x)$ is a convex-linear function, this problem is a special case of the problem considered in [7]. In [8] an algorithm is proposed which solves (Q) directly. The main computational effort of this algorithm involves searching the vertices of every generated simplex in R^p. The algorithm therefore is resonably efficient when p is small while n, the number of the variables, may be fairly large. Perhaps due to its importance and inherent difficulty several algorithmic ideas have also been suggested for solving Problem (P) (see e.g. [3]).

In this paper we propose a parametric simplex algorithm for finding a global solution of Problem (P). In the case when X_E is the efficient set of a bicriteria linear problem we show that solving (P) amounts to performing a parametric simplex tableau with one parameter in the objective function. In an important special case when $d = \alpha c^1 + \beta c^2$ with $\alpha \leq 0$, $\beta \geq 0$, in particular $d = -c^1$ we specialized a result of [4] by showing that a globally optimal solution of Problem (P) can be obtained by solving at most two linear programs. Thus, in this case there exist polynomial time algorithms for solving (P).

2. Preliminaries

The algorithm we are going to describe in the next section is based upon the following theorem whose proof can be found, for example in [10].

Theorem 2.1. *A vector x^0 is an efficient point of Cx over X if and only if there exists a $\lambda > 0$ such that x^0 is an optimal solution for the scalarized problem*

$$(\mathrm{L}^0_\lambda) \qquad \max\{\lambda^T Cx,\ x \in X\}.$$

By dividing to $\sum \lambda_i$ one can always assume that $\sum \lambda_i = 1$. Thus, in the case $p = 2$ Problem (L^0_λ) can be written as

$$(\mathrm{L}_\lambda) \qquad \max\langle \lambda c^1 + (1-\lambda)c^2, x\rangle, \quad \text{subject to} \quad x \in X,$$

where c^1 and c^2 are the rows of the matrix C.

From Theorem 2.1 and the fact that the set of optimal solutions of a linear program is a face of its feasible domain it follows that there exists a finite set I of real numbers such that $X_E = \cup_{\lambda \in I} X_\lambda$, where X_λ denotes the solution set of the linear program (L^0_λ).

Let $\xi(\lambda)$ denotes the optimal value of (L^0_λ). Then solving Problem (P) amounts to solving the following linear programs, one for each $\lambda \in I$,

$$\max d^T x$$

subject to

$$(\mathrm{P}_\lambda) \qquad x \in X, \lambda^T C x = \xi(\lambda).$$

Let x^λ be an optimal solution and $\eta(\lambda)$ be the optimal value of this problem. Then $\eta(\lambda_*) := \max\{\eta(\lambda) : \lambda \in I\}$ is the globally optimal value and x^{λ_*} is an optimal solution of Problem (P).

3. A parametric simplex algorithm for linear optimization over the efficient set of a bicriteria linear problem

The results described in the previous section suggest applying the parametric simplex method for solving Problem (P). For the case of bicriteria linear problem the algorithm runs as follows.

Let $\lambda_1, \ldots, \lambda_K$ be the critical values of the parametric linear program (L_λ) in the interval $[0, 1]$ (this means that the optimal basic of the linear program (L_λ) is constant in each interval $(\lambda_i, \lambda_{i+1})$ and changes when λ passes through one λ_i). For each λ_i $(i = 1, \ldots, K)$ we solve linear program (L_{λ_i}) by the simplex method. From the obtained optimal simplex tableau of (L_{λ_i}) we use simplex pivots to obtain the maximal value α_i of the linear function $d^T x$ over the solution set of (L_{λ_i}). Then it is clear that the maximal value of α_i $(i = 1, \ldots, K)$ gives the optimal value of the original problem (P).

For each λ let $c^\lambda := \lambda c^1 + (1 - \lambda)c^2$, and denote by α^* the optimal value of (P). Assume that $X := \{x \in R^n : ax = b, x \geq 0\}$, where A is a $(m \times n)$-matrix and $b \in R^m$. Then the algorithm can be described in detail as follows:

ALGORITHM

Assume that we are given the critical values $\lambda_1, \ldots, \lambda_K$ of the parametric linear program (L_λ).

Let α_0 be a lower bound for α^* and $x^0 \in X_E$ such that $d^T x^0 = \alpha_0$. Set $i = 1$.

Iteration i $(i = 1, \ldots, K)$

Step 1. Solve the following linear program by the simplex method

$$\max < \lambda_i c^1 + (1 - \lambda_i)c^2, x >$$

$$subject\ \ to\ \ x \in X.$$

Let w^i be the obtained optimal basic solution, $B_1 = (z_{jk})$ the inverse corresponding basic matrix, and J_i be the set of the basic indices.

Step 2 (The case when w^i is also an optimal solution of (P_{λ_i}).

If either

$$\Delta_{ik} := c_k^{\lambda_i} - \sum_{j \in J_i} z_{jk} c_j^{\lambda_i} < 0 \quad \forall k \notin J_i$$

or

$$d_k - \sum_{j \in J_i} z_{jkd_j} \leq 0 \quad \forall k \notin J_i,$$

then set

$$x^i := w^i \ if \ \ d^T w^i > \alpha_{i-1} \quad \text{and} \quad x^i = x^{i-1} \quad \text{otherwise},$$

and $\alpha_i = d^T x^i$.

If $i = K$, then terminate: $x^* := x^i$ is an optimal solution of Problem (P).

If $i < K$, then increase i by 1 and go to iteration i.

Step 3 (The case when w^i is not an optimal solution of (P_{λ_i})).

If

$$d_k - \sum_{j \in j_i} z_{jk} d_j > 0 \quad \text{for some } k \notin J_i,$$

then let

$$J_i^+ = \{k \notin J_i : c_k^{\lambda_i} - \sum_{j \in J_i} z_{jk} c_j^{\lambda_i} < 0\}$$

and solve the linear program (M_{λ_i}) given as

$$(\mathrm{M}_{\lambda_i}) \qquad \max\{d^T x : x \in X, x_k = 0 \ \ \forall k \in J_i^+\}.$$

Let y^i be the obtained optimal solution of this linear program, and set

$$x^i := y^i \ if \ d^T y^i > \alpha_{i-1} \quad \text{and} \quad x^i = x^{i-1} \quad \text{otherwise},$$

and $\alpha_i = d^T x^i$. Increase i by 1 and go to iteration i.

Remarks 1. To solve the linear program (M_{λ_i}) it is expedient to start from the obtained solution w^i of Program (L_{λ_i}). Program (M_{λ_i}) can be rewritten in the form

$$\max\{d^T x : x \in X, \lambda_i c^1 + (1 - \lambda_i) c^2 = \xi_i\}$$

where ξ_i is the optimal value of (L_{λ_i}).

2. It is evident that the above algorithm terminates after K iteration yielding a global optimal solution of Problem (P).

3. Sometimes the critical values of the parametric program (L_λ) are not given. In this case, at the beginning of each iteration i we need to compute the critical value λ_i by using the parametric simplex method with one parameter in the objective function (see [5]).

4. A Special Case

An important special case of Problem (P) occurs when $d = -c^i$ for some $i \in \{1, \ldots, p\}$. This case have been considered in [1, 4, 7]. For the case when $p = 2$ and d is a linear combination of the two rows of C, Benson in [4] has shown that the maximal value of $d^T x$ over X_E attains at a vertex of X which is also an optimal solution of at least one of the following three linear programs:

$$(\mathrm{L}_i) \qquad \max\{c^i x : x \in X,\} \quad (i = 1, 2)$$

$$(\mathrm{L}) \qquad \max\{c^T x : x \in X,\}$$

Using this result Benson [4] give a procedure for maximizing $d^T x$ over X_E by generating all basic solutions of these linear problems.

In this section we specialize this result by observing that an optimal solution of Problem (P) with $d = \alpha c^1 + \beta c^2$ and $\alpha \le 0, \beta \ge 0$ can be obtained by solving at most two linear programs. Namely, we have the following lemma.

Lemma 4.1. *Let X_2 denote the solution set of the linear program (L_2). Then any solution of the program*

$$(\mathrm{L}_{12}) \qquad \max\{c^1 x : x \in X_2\}$$

is also an optimal solution of the problem

$$(\mathrm{P}_1) \qquad \max\{(\alpha c^1 + \beta c^2) x : x \in X_E\}.$$

Proof. Let x^1 be any solution of (L_{12}). We first show that x^1 is efficient. Indeed, otherwise there exists a point $x \in X$ such that $c^i x \ge c^i x^1 \quad (i =$

$1,2)$ and $Cx \neq Cx^1$. Then, since $x^1 \in X_2$, it follows that $x \in X_2$ and that $c^1x > c^1x^1$ which contradicts the fact that x^1 is an optimal solution of (L_{12}). Hence $x^1 \in X_E$. Now let x^* be a global optimal solution of (P_1). Then

$$(\alpha c^1 + \beta c^2)x^* \geq (\alpha c^1 + \beta c^2)x^1.$$

From $x^1 \in X_2$ it follows that $c^2x^1 \geq c^2x^*$, and therefore $c^1x^1 \geq c^1x^*$. This together with $x^* \in X_E$ implies that $c^ix^1 = c^ix^*$, $(i = 1, 2)$. Hence x^1 is a global optimal solution of (P_1). The lemma is proved.

By Lemma 4.1, solving Problem (P_1) amounts to solving the two linear programs (L_2) and (L_{12}). From the linear programming [5] we know that if B is an optimal basic matrix, J_B is the set of basic indices and Δ_k $(k \in J_B)$ is the entries of the first row in the simplex tableau corresponding to an optimal solution of (L_2), then the solution set of X_2 of Problem (L_2), which is the feasible set of Problem (L_{12}), is given by

$$\{x \in X,\ x_k = 0,\ k \in J_B^+\}$$

where

$$J_B^+ = \{k \notin J_B : \Delta_k < 0\}.$$

Note that if $\Delta_k < 0$ for every $k \notin J_B$, then solving (L_{12}) is avoided, since (L_2) has a unique optimal solution.

We close the paper by elucidating how to calculate the critical values of the parameters. For simplicity assume that the set

$$X := \{x \in R^n : Ax = b,\ x \geq 0\}$$

is nondegenerate and that A is a full rank matrix.

Let x be a given optimal solution of the linear program

$$\max\langle \lambda c^1 + (1 - \lambda c^2, x\rangle : x \in X\}.$$

Denote by B the basic matrix corresponding to x, and by J the set of basic indices. If $B = (z_{jk})$, then from the linear programming [5] we get

$$\Delta_k = \lambda c_k^1 - (1-\lambda)c_k^2 - \sum_{j \in J} z_{jk}(\lambda c_j^1 + (1-\lambda)c_j^2) \leq 0 \quad \forall k \notin J$$

which implies that

$$\lambda(c_k^1 - c_k^2 - \sum_{j \in J}(c_j^1 - c_j^2)) \leq \sum_{j \in J} z_{jk}c_j^2 - c_k^2, \forall k \notin J. \tag{3.1}$$

(c_k^i stands for the kth component of the vector c^i). Let

$$\begin{aligned}
m_k &:= c_k^1 - c_k^2 - \sum_{j\in J} z_{jk}(c_j^1 - c_j^2),\\
t_k &:= \sum_{j\in J} z_{jk}c_j^2 - c_k^2,\\
J^- &:= \{k \notin J : m_k < 0\},\\
J^+ &:= \{k \notin J : m_k > 0\},\\
\alpha^- &:= \max\{t_k/m_k : k \in J^-\},\\
\alpha^+ &:= \max\{t_k/m_k : k \in J^+\}.
\end{aligned}$$

Thus, by virtue of (3.1) we have $\alpha^- \le \alpha^+$. Moreover, B is also an optimal basic solution of Problem (L_λ) for every λ which belongs to the interval $[\alpha^-, \alpha^+]$.

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