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ON A LINEAR DIFFERENTIAL EQUATION OF THE THIRD ORDER WITH TWO SINGULARITIES, BOTH IRREGULAR

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Abstract. The Euler transform is applied to a type of confluent hypergeometric linear differential equation of the third order so as to obtain convergent series solutions associated with irregular singularities. These results are expressed in the form of two different confluent hypergeometric functions of two variables.

1. Introduction

Recently, a linear differential of the second order with two singularities, both of which are irregular, was discussed; see Exton [3]. This equation occurs in the theory of the naked singularity in cosmology, and a complete solution was obtained by means of Laplace transforms. Any linear differential equation with all its singularities being irregular would, generally, be expected to be intractable because, even if a regular solution was possible, the Frobenius method would only yield a divergent series representation unless that series happened to terminate. A detailed discussion of linear differential equations with irregular singularities is to be found in Ince [5], Chapter 17, in particular Section 17.3, page 421.

In this study, we consider the third linear differential equation

$$x^2y''' - (ax^2 + bx + c)y'' + (px + q)y' + ry = 0, \quad a, b, c \neq 0. \tag{1.1}$$

The parameters p, q and r are to be determined in terms of a, b and c. It is clear that both of the singularities of (1.1) are irregular.

By means of the standard approach, namely, that of Frobenius, it would, in general, be expected that no convergent series solutions at all would be obtainable. The method used here to tackle this problem is that of the Euler transform as outlined by Ince [5], Section 18.4, page 454.

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Solutions are obtained in terms of the two following types of confluent double hypergeometric functions of Horn's [4] list; see Erdélyi [2], Section 5.7.1, page 224. These functions are given by

$$\Phi_3(a;b;x,y) = \sum \frac{(a,m)x^m y^n}{(b,m+n)m!n!} \tag{1.2}$$

and

$$\Gamma_2(b,b';x,y) = \sum \frac{(b,n-m)(b',m-n)x^m y^n}{m!n!}. \tag{1.3}$$

Both of these double series are convergent for all finite values of the variables x and y. As usual, the Pochhammer symbol (a,n) is defined by the expression

$$(a,n) = a(a+1)\dots(a+n-1) = \Gamma(a+n)/\Gamma(a); \quad (a,0) = 1. \tag{1.4}$$

In this study, the indices of summation run over all the non-negative integers, and any values of parameters leading to expressions which do not make sense are tacitly excluded.

2. Application of the Euler transform to (1.1)

The use of the Euler transform in an effective method of dealing with such equations as

(2.1)

$$Q(x)y^{(n)} - (\mu Q' + R(x))y^{(n-1)} + \big[\frac{1}{2}\mu(\mu+1)Q'' + (\mu+1)R'\big]y^{(n-2)} - \dots = 0,$$

where Q and R are polynomials such that one of Q and R is of degree n, while the degree of other is not greater than n.

The case when $n = 3$ and the degrees of Q and R are 3 and 2, respectively have been investigated to a considerable extent when the zeros of q are distinct. This leads to solutions related to the first Appell function F_1; see Appell et Kampé de Fériet [1], page 72. Other cases of the third order when the degree of two is less than 3 have not been worked out, however. The equation under consideration is of this type, and, as expected, solutions in terms of limiting cases of F_1 and related functions are obtained. Such limiting cases include the functions Φ_3 and Γ_2; see (1.2) and (1.3).

So as to bring (1.1) into the form (2.1) with $n = 3$, the parameters p, q and r are suitably expressed in terms of a, b and c and we have

$$x^2y''' - (ax^2+bx+c)y'' + (2ax+b-\mu)(\mu+1)y' - a(\mu+1)(\mu+2)y = 0. \tag{2.2}$$

Relative to the irregular singularity at the origin, we have an indicial function which is quadratic only, namely, $r(r-1)$, as expected. In keeping with the theory of irregular singularities, any associated series solutions are, in general, divergent. The equation obtained from (2.1) by replacing x by $1/x$ cannot be expressed in the form (2.1), and its solution is not attempted directly in the present context. Where (2.2) is concerned, we see that

$$Q = x^2 \quad \text{and} \quad R = ax^2 + (b-2\mu)x + c,$$

so that

$$R/Q = a + (b-2\mu)/x + cx^{-2}. \tag{2.3}$$

Using the method given by Ince [5], a solution of (2.2) may be written as the contour integral

$$y = \int_C (x-t)^{\mu+2} \exp{(at - c/t)} t^{b-2\mu-2} dt. \tag{2.4}$$

The contour of integration is any Pochhammer double loop in the t-plane, or any closed loop beginning and ending at a point at which the function

$$V = (x-t)^{\mu} \exp{(at - c/t)} t^{b-2\mu-2} \tag{2.5}$$

returns to its original value.

Taking the contour of integration to be a loop beginning and ending at $-\infty$, and encircling the origin once in the positive direction, we consider the expression

$$y = \int_{-\infty}^{(0+)} (x-t)^{\mu+2} t^{b-2\mu-2} \; \exp{(at - c/t)} dt, \quad \operatorname{Re}(a) < 0. \tag{2.6}$$

Apart from a constant multiplier, this integral may be written as

$$\begin{aligned} &\int_{-\infty}^{(0+)} (1 - x/t)^{\mu+2} t^{b-\mu} \exp{(at - c/t)} dt \\ &= \sum \frac{(-\mu-2, m) x^m (-c)^n}{m!n!} \int_{-\infty}^{(0+)} \exp{(at)} t^{b-\mu-m-n} dt, \end{aligned}$$

which is proportional to

$$\Phi_3(-\mu-2;\mu-b;ax,-ac)=\sum\frac{(-\mu-2,m)(ax)^m(-ac)^n}{(\mu-b,m+n)m!n!}\,. \tag{2.7}$$

Term-by-term integration is justified because the contour can be drawn so that the series converges upon it. The function (2.7) is a solution of (1.1) which converges for the whole x-plane, including the neighbourhoods of the irregular singularities at 0 to ∞. In the event that $\mathrm{Re}(a)>0$, we let the contour of integration begin and end at the point $+\infty$, and obtain the same result.

3. Other solutions of (1.1)

Similarly, if the contour of integration is taken to be a simple loop beginning and ending at the origin and encircling the point at infinity, a further solution in the form Φ_3 is obtained. This is

$$x^{2+\mu}\Phi_3(-\mu-2;b-2\mu;-c/x,-ac). \tag{3.1}$$

A solution which may be expressed as a function Γ_2, see (1.3), may be deduced by integrating around the Pochhammer double loop slung around the points 0 and x.

Let $t=xs$ in (2.4), when the solution is proportional to

$$x^{b-\mu}\int\exp\left(axs-c/(xs)\right)(-s)^{b-2\mu-2}(s-1)^{\mu+2}ds. \tag{3.2}$$

Readily justifiable integration term-by-term yields the solution

$$x^{b-\mu}\Gamma_2(\mu-b-1,b-2\mu-1;-ax,c/x). \tag{3.3}$$

4. Conclusion

Since the three solutions (2.7), (3.10) and (3.3) converge over the whole x-plane, a fundamental system of solutions of (1.1) has been obtained which is valid in the neighbourhoods of both of the irregular singularities A technique based upon a technique first investigated during the nineteenth century, and more fully worked out by Ince [5], page 454, has only now been successfully applied to discuss a linear differential equation, the only singularities of which are irregular. It is intended to extend this work subsequently.

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