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THE STRONG LAW OF LARGE NUMBERS FOR TWO-DIMENSIONAL ARRAYS OF ORTHOGONAL OPERATOR IN VON NEUMANN ALGEBRA

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Abstract. We investigate the strong law of large numbers for two-dimensional arrays of pairwise orthogonal operators in a von Neumann algebra A with faithful normal state. Some related results are considered.

1. Introduction and notations

The strong limit theorems for sequences of orthogonal operators in von Neumann algebra have been considered by some authors. In [3] the Randemacher-Menshov theorem has been proved. An other version of this theorem can be found in [1]. The non-commutative extension of the Weyl theorem was shown in [2]. In particular, the strong law of large numbers was given by R. Jajte in [4].

The aim of this paper is to give the strong law of large numbers for two-dimensional arrays of pairwise orthogonal operators in a von Neumann algebra with faithful normal state. Our results extend some results of [4], [5] to two-dimensional arrays and can be viewed as non-commutative extensions of some results of [6].

Note that in the special case when the state is tracial, some results on law of large numbers for sequences and for two-dimensional arrays of independent measurable operators have been considered in [5], [8], [10]. In particular, one can find the law of large numbers of Hsu-Robbins type for two-dimensional arrays of independent measurable operators in [11].

We start with some notations and definitions. Throughout of this paper, A denote a von Neumann algebra with faithful normal state Φ, and N is the set of all natural numbers.

For each self-adjoint operators X in A, we denote by $e_{\Delta}(X)$ the spectral projection of X corresponding to the Borel subset Δ of the real line R.

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Let X be a operator in A and X^* the adjoint of X. Then $X^*.X$ is a positive operator in A and there exists the positive operator $|X|$ in A such that

$$|X|.|X| = X^*X.$$

$|X|$ is called the positive square root of X^*X and is denoted by $(X^*X)^{1/2}$.

Two operators X and Y in A are said to be orthogonal if $\Phi(X^*.Y) = 0$. An array $(X_{mn},\ (m,n) \in N^2)$ is said to be the array of pairwise orthogonal operators in A if for all $(m,n) \in N^2$, $(p,q) \in N^2$, $(m,n) \neq (p,q)$, X_{mn} and X_{pq} are orthogonal.

Now let $(X_{mn}(m,n) \in N^2)$ be an array of operators in A. We say that X_{mn} is convergent almost uniformly $(a.u)$ to $X \in A$ as $(m.n) \to \infty$ if for each $\varepsilon > 0$ there exists a projection $p \in A$ such that $\Phi(p) > 1 - \varepsilon$ and $\|(X_{mn} - X)p\| \to 0$ as $\max(m,n) \to \infty$.

An array $(X_{mn}, (m,n) \in N^2) \subset A$ is said to be convergent almost completely $(a.c)$ to an operator $X \in A$ as $(m.n) \to \infty$, if for each $\varepsilon > 0$ there exists an array $(q_{mn}, (m,n) \in N^2)$ of projections in A such that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Phi(q_{mn}^{\perp}) < \infty \quad \text{and} \quad \|(X_{mn} - X)q_{mn}\| < \varepsilon$$

for all $(m,n) \in N^2$ (where $q_{mn}^{\perp} = E - q_{mn}$, E is the identity operator).

By the same method as for one-dimensional sequences we can prove that if the state Φ is tracial and $X_{mn} \to X(a,c)$, then $X_{mn} \to X(a,u)$ as $(m,n) \to \infty$ (see [4]).

For further information we refer to [7], [8], [10].

2. Preliminaries

In the sequel we'll need the following lemmas

Lemma 2.1. *Let (Y_{mn}) be a two-dimensional array of pairwise orthogonal operators in A. Put*

$$t_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} Y_{i,j}.$$

Then there exists an array (B_{mn}) of positive operators in A such that

$$|t_{mn}|^2 \leq (m+1)(n+1)B_{mn} \quad \textit{for} \quad 1 \leq i \leq 2^m \quad 1 \leq j \leq 2^n$$

and

$$\Phi(B_{mn}) \leq (m+1)(n+1)\sum_{i=1}^{2^m}\sum_{j=1}^{2^n}\Phi(|Y_{ij}|^2)$$

Proof. We start with the dyadic representations of i and j. Divide the interval $I = (,2^m]$ into intervals $(0, 2^{m-1}]$ and $(2^{m-2}, 2^m]$, each of these intervals into halves, and so on, we obtain in this way a sequence of partitions of I. For a positive integer $i \leq 2^m$, we take its dyadic representation. Then the interval $(0, i]$ can be written as the sum of at most m disjoint intervals $I^{(i)}$, each of which belongs to different partition; that is

$$(0, i] = \sum_{k=0}^{m} I_k^{(i)}, \tag{2.1}$$

where $I_k^{(i)}$ is empty or element of k^{th} partition. Analogously, we have

$$(0, j] = \sum_{\ell=1}^{m} J_\ell^{(i)} \tag{2.2}$$

where $j_\ell^{(i)}$ is empty or a element of ℓ^{th} partition of $[0, 2^n]$.

Using (2.1) and (2.2) we obtain

$$(0, i] \times (0, j] = \sum_{k=0}^{m}\sum_{\ell=0}^{n} I_k^{(i)} \times J_\ell^{(j)} = \sum_{k=0}^{m}\sum_{\ell=0}^{n}\Delta_{k\ell}^{(ij)},$$

and therefore

$$t_{ij} = \sum_{u=1}^{i}\sum_{v=1}^{j} Y_{uv} = \sum_{k=0}^{m}\sum_{\ell=0}^{n}\Big(\sum_{u,v\in\Delta_{k\ell}^{i,j}} Y_{uv}\Big)$$

$$B_{mn} = \sum_{\Delta_{k\ell}}\Big|\sum_{(u,v)\in\Delta_{k\ell}} Y_{uv}\Big|^2,$$

where $\Delta_{k\ell} = I \times J$, I (and J) runs over all intervals with appear as the elements of the partitions of $(0, 2^m]$ (and of $(0, 2^n]$, respectively). By Schwarz inequality,

$$\Big|\sum_{k=1}^{n} Z_k\Big|^2 \leq n\sum |Z_k|^2.$$

Thus,

$$|t_{ij}|^2 \leq (m+1)(n+1)\sum_{k=1}^{m}\sum_{\ell=1}^{n}\left(\Big|\sum_{(u,v)\in\Delta^{ij}} Y_{uv}\Big|^2\right) \leq (m+1)(n+1)B_{mn}.$$

Moreover, B_{mn} does not depend on $i \in (0, 2^m]$, $j \in (0, 2^n]$ and

$$\begin{aligned}\Phi(B_{mn}) &\leq \sum_{\Delta_{k\ell}} \Phi\left(\Big|\sum_{(u,v)\in\Delta_{k\ell}} Y_{uv}\Big|^2\right) = \sum_{\Delta_{k\ell}}\sum_{(u,v)\in\Delta_{k\ell}} \Phi(|Y_{uv}|^2)\\ &= (m+1)(n+1)\sum_{i=1}^{2^m}\sum_{j=1}^{2^n}\Phi(|Y_{ij}|^2)\end{aligned}$$

which completes the proof.

Lemma 2.2. *Let (X_{mn}) be an array of positive operators in A and (ε_{mn}) an array of positive numbers. If*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \varepsilon_{mn}^{-1}\Phi(X_{mn}) < \frac{1}{2}\ ,$$

then there exists a projection $p \in A$ such that for all m, $n \in N$

$$\Phi(p) \geq 1 - 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\varepsilon_{mn}^{-1}\Phi(X_{mn}) \quad \textit{and} \quad \|pX_{mn}p\| \leq 2\varepsilon_{mn}.$$

For sequences this lemma was proved in [5], (2.2.13). It also holds for arrays because all X_{mn} are positive operators and all ε_{mn} are positive numbers.

Lemma 2.3. *Let (X_{mn}) be an array of operators in A. If*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \Phi(|X_{mn}|^2) < \infty, \tag{2.3}$$

then X_{mn} converge almost uniformly to zero as $(m,n) \to \infty$.

Proof. Let $\varepsilon > 0$ be given. By (2.3) we can find an array (ε_{mn}) of positive numbers such that $\varepsilon_{mn} \to \infty$ as $(m,n) \to \infty$ and

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \varepsilon_{mn} \Phi(|X_{mn}|^2) < \varepsilon/2.$$

By Lemma 2.2 there exists a projection $p \in A$ such that

$$\Phi(p) \geq 1 - 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \varepsilon_{mn}^{-1} \Phi(|X_{mn}|^2) > 1 - \varepsilon$$

$$\|p|X_{mn}|^2 p\| \leq 2\varepsilon_{mn}.$$

Thus $0 \leq \|X_{mn}p\| = \|p|X_{mn}|^2 p\|^{1/2} \leq \sqrt{2\varepsilon_{mn}} \to 0$ as $(m,n) \to \infty$. This means $X_{mn} \to 0$ $(a.u)$ as $(m,n) \to \infty$, and the proof is completed.

3. Strong law of large numbers

The main result of this section is following theorem

Theorem 3.1. *Let A be a von Neumann algebra with a faithful normal state Φ and let (X_{mn}) be a two-dimensional array of pairwise orthogonal operators in A. If*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Big(\frac{lg m \; lg n}{mn}\Big)^2 \Phi(|X_{mn}|^2) < \infty$$

then

$$S_{mn} = \frac{1}{m.n} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}$$

converge almost uniformly to zero as $(m,n) \to \infty$.

Proof. Put

$$S_{mn} = \frac{1}{m.n} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}.$$

Let $2^k < m \leq 2^{k+1}$, $2^\ell < n \leq 2^{\ell+1}$. Then

(3.1)

$$|S_{mn} - S_{2^k 2^\ell}|^2 = \Big[\Big(\frac{1}{m.n} - \frac{1}{2^k 2^\ell}\Big)\sum_{i=1}^{2^k}\sum_{j=1}^{2^\ell} X_{ij}$$

$$+\frac{1}{m.n}\Big(\sum_{i=2^k+1}^{m}\sum_{j=2^\ell+1}^{n} X_{ij}\Big) + \sum_{i=1}^{2^k}\sum_{j=2^\ell+1}^{n} X_{ij} + \sum_{i=2^k+1}^{m}\sum_{j=1}^{n} X_{ij}\Big]^2$$

$$\leq 4\left[\Big(\frac{1}{m.n} - \frac{1}{2^{k+1}}\Big)^2\Big|\sum_{i=1}^{2^k}\sum_{j=1}^{2^\ell} X_{ij}\Big|^2 + \frac{1}{(m,n)^2}\Big|\sum_{i=2^k+1}^{m}\sum_{j=2^\ell+1}^{n} X_{ij}\Big|^2\right]$$

$$+\frac{4}{(m,n)^2}\left[\Big|\sum_{i=1}^{2^k}\sum_{j=2^\ell+1}^{n} X_{ij}\Big|^2 + \Big|\sum_{i=2^k+1}^{m}\sum_{j=1}^{n} X_{ij}\Big|^2\right]$$

We have

$$S^{(1)} = 4\left[\Big(\frac{1}{m.n} - \frac{1}{2^{k+1}}\Big)^2\Big|\sum_{i=1}^{2^k}\sum_{j=1}^{2^\ell} X_{ij}\Big|^2 + \frac{1}{(m,n)^2}\Big|\sum_{i=2^k+1}^{m}\sum_{j=1}^{n} X_{ij}\Big|^2\right]$$

$$\leq 4\left[\Big(\frac{1}{2^{k+\ell+2}} - \frac{1}{2^{k+\ell}}\Big)^2\Big|\sum_{i=1}^{2^k}\sum_{j=1}^{2^\ell} X_{ij}\Big|^2 + \frac{1}{2^{2(k+1)}}\Big|\sum_{i=2^k+1}^{m}\sum_{j=2^\ell+1}^{n} X_{ij}\Big|^2\right]$$

$$\leq 4\left[9.2^{-2(k+\ell+2)}\Big|\sum_{i=1}^{2^k}\sum_{j=1}^{2^\ell} X_{ij}\Big|^2 + \frac{1}{2^{-2(k+1)}}\Big|\sum_{i=2^k+1}^{m}\sum_{j=2^\ell+1}^{n} X_{ij}\Big|^2\right]$$

$$\leq 2^{2-2(k+1)}\left(\Big|\sum_{i=1}^{2^k}\sum_{j=1}^{2^\ell} X_{ij}\Big|^2 + \Big|\sum_{i=2^k+1}^{m}\sum_{j=2^\ell+1}^{n} X_{ij}\Big|^2\right).$$

Applying Lemma 2.1 we obtain:

$$\Big|\sum_{i=2^k+1}^{m}\sum_{j=2^\ell+1}^{n} X_{ij}\Big|^2 \leq (k+2)(\ell+2)B_{k\ell}$$

where $B_{k\ell}$ is a positive operator independent of $m \in (2^k, 2^{k+1}]$, $n \in (2^\ell, 2^{\ell+1})$ and

$$\Phi(B_{k\ell}) \leq (k+1)(\ell+2)\sum_{i=2^k+1}^{2^{k+1}}\sum_{j=2^\ell+1}^{2^{\ell+1}} \Phi(|X_{ij}|^2).$$

Thus $S^{(1)} \leq D_{k,\ell}$, where $D_{k\ell} \in A_+$ and

$$\Phi(D_{k\ell}) \leq 2^{2-2(k+1)} \left(\sum_{i=1}^{2^k} \sum_{j=1}^{2^\ell} \Phi(|X_{ij}|^2) \right) + (k+2)(\ell+2) \sum_{i=2^k+1}^{2^{k+1}} \sum_{j=2^\ell+1}^{2^{\ell+1}} \Phi(|X_{ij}|^2).$$

By the assumption

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Big(\frac{\lg m \ \lg n}{m.n} \Big)^2 \Phi(|X_{ij}|^2) < \infty.$$

We have

$$\begin{aligned}
\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \Phi(D_{k\ell}) &\leq \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} 2^{2-2(k+1)} \frac{2^{k+\ell}}{k^2\ell^2} \sum_{i=1}^{2^k} \sum_{j=1}^{2^\ell} \Phi(|X_{ij}|^2) \Big(\frac{\lg i \ \lg j}{ij} \Big)^2 \\
&\quad + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} 2^{2-2(k+\ell)} \frac{2^{2(k+\ell+2)}(k+2)^2(\ell+2)^2}{(k+1)^2(\ell+2)^2} \cdot \\
&\quad \sum_{i=2^k+1}^{2^{k+1}} \sum_{j=2^\ell+1}^{2^{\ell+1}} \Phi(|X_{ij}|^2) \Big(\frac{\lg i \ \lg j}{ij} \Big)^2 \\
&\leq \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{4}{k^2\ell^2} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \Phi(|X_{ij}|^2) \Big(\frac{\lg i \ \lg j}{ij} \Big)^2 \qquad (3.2) \\
&\quad + \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} C_2 \sum_{i=2^k+1}^{\infty} \sum_{j=2^k+1}^{\infty} \Phi(|X_{ij}|^2) \Big(\frac{\lg i \ \lg j}{ij} \Big)^2 \\
&\quad (C_1 + C_2) \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \Phi(|X_{ij}|^2) \Big(\frac{\lg i \ \lg j}{ij} \Big)^2 < \infty
\end{aligned}$$

where $C_1 = \sum\limits_{k=1}^{\infty} \sum\limits_{\ell=1}^{\infty} \dfrac{4}{k^2\ell^2}$ and C_2 is the constant satisfying

$$\frac{2^5(k+1)^2(\ell+2)^2}{(k+1)^2(\ell+2)^2} < C_2 \quad (\forall k, \ \forall \ell).$$

We now estimate the second term of (3.1). We have

$$\Big|\sum_{i=1}^{2^k}\sum_{j=2^\ell}^{n} X_{ij}\Big|^2 = \sum_{i=1}^{2^k}\Big|\sum_{j=2^\ell+1}^{n} X_{ij}\Big|.$$

Applying proposition 4.4.2 of [5] for sequences (X_{ij}), $j \in N$, $i = 1, \ldots, 2^k$, we obtain

$$\Big|\sum_{j=2^\ell+1} X_{ij}\Big|^2 \leq (\ell+2)\overline{B}_{i\ell}$$

where $\overline{B}_{i\ell}$ is a positive operator independent of $n \in (2^\ell, 2^{\ell+1})$ and

$$\begin{aligned}\sum_{i=1}^{\infty} n^{-2}(\ell+2)B_{i\ell} &\leq \sum_{j=1}^{\infty}\Big(\frac{\lg j}{j}\Big)^2\Phi(|X_{ij}|^2)\\ &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\Big(\frac{\lg i\ \lg j}{ij}\Big)^2\Phi(|X_{ij}|^2) < \infty.\end{aligned}$$

Put $\overline{B}_{k\ell} = (\ell+1)\sum\limits_{i=1}^{2^k} B_{i\ell}$. Then $\overline{B}_{k\ell}$ depends only k, ℓ, $\sum\limits_{k=1}^{\infty}\sum\limits_{\ell=1}^{\infty}\Phi((mn)^2 B_{k\ell}) < \infty$ and

$$\Big|\sum_{i=1}^{2^k}\sum_{j=2^i+1}^{n} X_{ij}\Big| \leq B_{k\ell}.$$

Analogously, we can find a positive operator $\overline{\overline{B}}_{k\ell}$ such that $\overline{\overline{B}}_{k\ell}$ depends only on k, ℓ, $\sum\limits_{k=1}^{\infty}\sum\limits_{\ell=1}^{\infty}\Phi((mn)^{-2}\overline{\overline{B}}_{k\ell}) < \infty$ and

$$\Big|\sum_{i=2^k+1}^{m}\sum_{j=1}^{2^\ell} X_{ij}\Big|^2 = \sum_{j=1}^{2^\ell}\Big|\sum_{j=1}^{m} X_{ij}\Big|^2 \leq \overline{\overline{B}}_{k\ell}$$

Thus

$$\begin{aligned} S^{(2)} &= \frac{4}{(mn)^2}\left[\Big|\sum_{i=1}^{2^k}\sum_{j=2^\ell+1}^{n} X_{ij}\Big|^2 + \Big|\sum_{i=2^k+1}^{m}\sum_{j=1}^{2^k} X_{ij}\Big|^2\right]\\ &\leq \frac{4}{(mn)^2}((B_{k,\ell}) + (B_{k\ell})) = \overline{D}_{k\ell} \qquad (3.3)\end{aligned}$$

where $\overline{D}_{k\ell} \in A$ and $\sum\limits_{k=1}^{\infty}\sum\limits_{\ell=1}^{\infty} \Phi(\overline{D}_{k\ell}) < \infty$. Combining (3.1), (3.2). (3.3), we get

$$|S_{mn} - S_{2^k 2^\ell}|^2 \leq S^{(1)} + S^{(2)} \leq D_{k\ell} + \overline{D}_{k\ell} = \overline{\overline{D}}_{k\ell}$$

where $\overline{\overline{D}}_{k\ell} \in A$ and $\sum\limits_{k=1}^{\infty}\sum\limits_{\ell=1}^{\infty} \Phi(\overline{\overline{D}}_{k\ell}) < \infty$. Moreover

$$\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \Phi(|S_2^k.2^\ell|^2) = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{1}{2^{2(k+\ell)}} \sum_{i=1}^{2^k}\sum_{j=1}^{2^\ell} \Phi(|X_i|^2)$$
$$\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{1}{2^{2(k+\ell)}} \frac{2^{2(k+\ell)}}{k^2\ell^2} \sum_{i=1}^{2^k}\sum_{j=1}^{2^\ell} \Phi(|X_i|^2)\Big(\frac{\lg i \lg j}{ij}\Big)^2 < \infty.$$

Using the Lemma 2.3 for $(S_{mn} - S_2^k.2^\ell)$ and $(S_2^k.2^\ell)$ we obtain

$$S_{mn} = (S_{mn} - S_2^k.2^\ell) + S_2^k.2^\ell \to 0 + 0 \quad (a.u) \text{ as}$$
$$2^k 2^\ell < m.n < 2^{k+1}.2^{\ell+1} \to 0.$$

This completes the proof.

Under some conditions stronger than those of Theorem 3.1 we can obtain the almost completely convergence of the averages. Namely, it is easy to prove the following theorem.

Theorem 3.2. *Let (X_{mn}) be an array of pairwise orthogonal operators in A. If there exists an array (a_{mn}) of positive numbers such that $a_{mn} \downarrow 0$ as $(m,n) \to 0$ and*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\Phi(|X_{ij}|^2) < \infty \tag{3.4}$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{(ij)^2 a_{ij}} < \infty \tag{3.5}$$

Then

$$\frac{1}{mn}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} X_{ij} \to 0 \quad (a.c) \quad \text{as} \quad (m,n) \to \infty.$$

Proof. Put

$$S_{mn} = \frac{1}{mn}\sum_{k=1}^{m}\sum_{j=1}^{n} X_{ij}$$

Then

$$\begin{aligned}\Phi(|S_{mn}|^2) &= \frac{1}{(m,n)^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\Phi(|X_{ij}|^2)\\ &\leq \frac{1}{(m,n)^2.a_{mn}}\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\Phi(|X_{ij}|^2)\end{aligned}$$

Using (3.4) and (3.5) we obtain

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Phi(|S_{mn}|^2) &\leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m,n)^2.a_{mn}}\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\Phi(|X_{ij}|^2)\\ &\leq \Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m,n)^2 a_{mn}}\Big)\Big(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\Phi(|X_{ij}|^2)\Big) < \infty.\end{aligned}$$

Now, let $\varepsilon > 0$ be given. Put $q_{mn} = e_{[0,\varepsilon^2]}(|S_{mn}|^2)$. By Lemma 2.2 of [8], we have

$$\||S_{mn}|^2 q_{mn}\| < \varepsilon^2.$$

Thus

$$\begin{aligned}\||S_{mn}|q_{mn}\| &= (\|S_{mn}q_{mn}\|^2)^{1/2} \leq \|q_{mn}|S_{mn}|^2 q_{mn}\|^{1/2}\\ &\leq \||S_{mn}|^2 q_{mn}\|)^{1/2} < \varepsilon \quad \forall m \in N, \quad \forall n \in N.\end{aligned}$$

Moreover

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Phi(q_{mn}^{\perp}) \leq \varepsilon^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Phi(|S_{mn}|^2) < \infty.$$

This means that

$$S_{mn} = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}$$

converges almost completely to zero as $(m,n) \to \infty$, which completes the proof.

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