ACTA MATHEMATICA VIETNAMICA
Volume 20, Number 2, 1995, pp. 321-326

ON THE STABILITY OF THE CHARACTERIZATION OF DOUBLE-COMPOSED RANDOM VARIABLES

NGUYEN HUU BAO

Abstract. The random variable ζ is called a composed random variable of two random variables ξ and η if its characteristic function has the form

$$\psi(t) = a[\phi(t)]$$

where $a[z]$ is the generating function of η and $\phi(t)$ is the characteristic function of ξ. In our previous papers, we have considered a stability condition for characteristic functions of a class of composed random variables. In this paper, we consider another condition for the characteristic function of the composed random variable of ξ and η, if η is also a composed random variable.

Let us consider the random variable (r.v.) ξ with the characteristic function $\varphi(t)$. Let η be a r.v. with the generating function $a(z)$. It is known (see [1]) that the composed random variable of ξ and η has the characteristic function

$$\psi(t) = a[\varphi(t)].$$

In [1] we have dealt with the properties of those composed random variables and proved that it is a natural expansion of the class L containing the characteristic functions of the infinite divisible laws.

In [3] we considered some theorems on the stability for the composed random variables. The following theorem was given in [3].

THEOREM 1. *Suppose that $\psi_1(t)$ and $\psi_2(t)$ are two characteristic functions with the generating functions*

$$\psi_1(t) = a[\varphi_1(t)], \quad \psi_2(t) = a[\varphi_2(t)], \tag{1}$$

where $a[z]$ satisfies the following condition: If z_1, z_2 are two complex numbers such that $|z_1| \leq 1$, $|z_2| \leq 1$, then

$$|a(z_1) - a(z_2)| \leq K|z_2 - z_1| \tag{2}$$

Received December 22, 1994; in revised form March 30,1995.
This work was supported by the National Research Program for Theoretical Sciences, Vietnam.

where K is a constant. If for some sufficiently small ε $(0 < \varepsilon < 1)$ one can choose a number $T = T(\varepsilon)$ $(T(\varepsilon) \to +\infty$ when $\varepsilon \to 0)$ so that

$$\max_{|t| \leq T(\varepsilon)} |\varphi_1(t) - \varphi_2(t)| \leq \varepsilon, \tag{3}$$

then the following estimation holds:

$$\lambda(\Psi_1; \Psi_2) \leq \max \left\{ K\varepsilon,\ \frac{1}{T(\varepsilon)} \right\}.$$

For distribution functions $\Psi_1(x)$ and $\Psi_2(x)$ with the corresponding characteristic functions $\psi_1(t)$ and $\psi_2(t)$ the metric λ is defined by

$$\lambda(\Psi_1; \Psi_2) = \min_{T>0} \max \left\{ \max_{|t| \leq T} \frac{1}{2} |\psi_1(t) - \psi_2(t)|,\ \frac{1}{T} \right\}. \tag{4}$$

In this paper we consider the class of double-composed random variables. Namely, if ζ is a composed r.v. of the random variables ν_1 and η, where η is also a composed r.v. of two other random variables ν_2 and ξ, then ζ is called a double-composed random variable.

Next we give a stability theorem for the double-composed random variables. Let ζ be given by

$$\zeta = \sum_{k=1}^{\nu_1} \eta_k, \tag{5}$$

where ν_1 is a discrete r.v. independent of all $\eta_1, \eta_2, \cdots$ and has the negative binomial distribution, and $\eta_1, \eta_2, \cdots$ are independent identically distributed (i.i.d.) random variables with the same distribution function as the r.v. η. We assume that η is of the form

$$\eta = \sum_{k=1}^{\nu_2} \xi_k, \tag{6}$$

where ν_2 is a discrete r.v. independent of all $\xi_1, \xi_2, \cdots$ and has the Poisson distribution function with parameter λ, and $\xi_1, \xi_2, \cdots$ are i.i.d. random variables with the same distribution function as the r.v. ξ. We showed in [1] that ζ is a composed r.v. of ν_1 and η, η is a composed r.v. of ν_2 and ξ. Thus ζ is a double-composed random variable.

Suppose that $(X_1, X_2, \cdots, X_n)$ is a simple random sample from the set of values of η and there exist the absolute moments $E(|X_1|^k)$ for $k = 1, 2, 3, 4$. Put

$$\lambda_k = \sum_{k=1}^{n} x_i^k \quad (k = 1, 2, 3, 4) \tag{7}$$

and

$$T_1 = A\lambda_4 + 3B\lambda_2^2 + 2C\lambda_3\lambda_1 + 6\lambda_2\lambda_1^2 - \lambda_1^4. \tag{8}$$

Here

$$A = n(5-n); \quad B = n^2 - 5n + 7; \quad C = -(n^2 - 5n + 13). \tag{9}$$

The statistic T_1 is called ε-zero regression with respect to the statistic λ_1 if

$$E(T_1/\lambda_1) = \varepsilon. \tag{10}$$

The symbol $\Psi_\xi(x)$ denotes the distribution function of ξ, and $\Psi_2(x)$ denotes the distribution function with the corresponding characteristic function

$$g_2(t) = p\{1 - q[e^{\lambda(\frac{1}{1-it}-1)}]\}^{-1} \qquad (p + q = 1). \tag{11}$$

Let δ be a positive number satisfying the condition

$$\frac{1-\delta}{2} > \frac{\delta}{n}. \tag{12}$$

THEOREM 2. *If the statistic T_1 is an ε-zero regression with respect to the statistic λ_1, for some sufficiently small ε $(0 < \varepsilon < 1)$, then*

$$\lambda(\Psi_\zeta; \Psi_2) \le \mathcal{C} \cdot \varepsilon^{\frac{1-\delta}{2} - \frac{\delta}{n}} \tag{13}$$

where $\mathcal{C}$ is a constant independent of ε and $\lambda(\cdot;\cdot)$ is the metric defined by (4).

LEMMA. *Under the conditions of Theorem 2 we can choose a number $T = T(\varepsilon)$ $(T(\varepsilon) \to +\infty$ when $\varepsilon \to 0)$ such that for any t, $|t| \le T(\varepsilon)$, we have the estimation*

$$|g(t) - g_1(t)| \le \mathcal{C}\varepsilon^{\frac{1-\delta}{2} - \frac{\delta}{n}} \tag{14}$$

where

$$g_1(t) := e^{\lambda(\frac{1}{1-i\theta t}-1)} \tag{15}$$

PROOF. In the proof of the stability theorem in [2] with two characteristic functions $g(t)$ and $g_1(t)$ (as in (14) and (15)) we have the following estimation (for any t, $|t| \leq T$):

$$|g(t) - g_1(t)| \leq e^{4\lambda} \frac{C(\varepsilon)}{\varepsilon^{\delta/n}} |g(t)| \Big[\frac{2\alpha e^{C(\varepsilon)T}}{|b|} \mu_1 + e^{2C(\varepsilon)T} \mu_1 \frac{T}{2} \Big] T. \tag{16}$$

Here λ, μ_1, b, α are positive constants, independent of ε; δ is a positive number that satisfies condition (12), and

$$C(\varepsilon) := \frac{\varepsilon^{1-\delta}}{2n(n-1)(n-2)(n-3)}. \tag{17}$$

If we choose

$$T = T(\varepsilon) = C_1 \varepsilon^{\frac{\delta}{n} - \frac{1-\delta}{2}} \tag{18}$$

with any constant C_1 independent of ε, then $T(\varepsilon) \to +\infty$ when $\varepsilon \to 0$, and we also have

$$C(\varepsilon) \cdot T(\varepsilon) = C_2 \varepsilon^{1-\delta+\frac{\delta}{n}-\frac{1-\delta}{2}} = C_2 \varepsilon^{\frac{1-\delta}{2}+\frac{\delta}{n}} \tag{19}$$

where C_2 is a constant.

On the other hand, in [2] we have established the estimation

$$|g(t)| \leq e^{\ln \frac{1}{\varepsilon^{\delta/n}}} = \varepsilon^{-\frac{\delta}{n}} \quad (|t| \leq T(\varepsilon)), \tag{20}$$

where $T(\varepsilon)$ is chosen as in (18).

From (17), (19), (20) we can deduce the following estimation

$$\begin{aligned} |g(t) - g_1(t)| &\leq C_3 \varepsilon^{1-\delta-\frac{2\delta}{n}} \big[C_2 + C_4 \varepsilon^{\frac{\delta}{n} - \frac{1-\delta}{2}} \big] C_1 \varepsilon^{\frac{\delta}{n} - \frac{1-\delta}{2}} \\ &\leq C \cdot \varepsilon^{\frac{1-\delta}{2} - \frac{\delta}{n}} \end{aligned} \tag{21}$$

where C is a constant independent of ε.

PROOF OF THEOREM 2. Let us consider the double-composed random variable

$$\zeta = \sum_{k=1}^{\nu_1} \eta_k$$

(see (5)). If the r.v. η has the negative binomial distribution function and its generating function has the form:

$$a(z) = p[1 - qz]^{-1} \qquad (p + q = 1), \tag{22}$$

then the second condition of Theorem 1 is satisfied. Indeed, for any complex numbers z_1 z_2 with $|z_1| \leq 1$ and $|z_2| \leq 1$, we have

$$|a(z_1) - a(z_2)| = |\frac{p}{1 - qz_1} - \frac{p}{1 - qz_2}| \leq \frac{pq|z_1 - z_2|}{|1 - qz_1| \cdot |1 - qz_2|}.$$

Since $q \leq 1$, then

$$|1 - qz_1| \geq |1 - q|z_1|| \geq 1 - q,$$
$$|1 - qz_2| \geq |1 - q|z_2|| \geq 1 - q,$$

and

$$|a(z_1) - a(z_2)| \leq \frac{pq|z_1 - z_2|}{(1 - q)^2} = \frac{q}{p}|z_1 - z_2|. \tag{23}$$

Note that, for $|t| \leq T(\varepsilon) = C_1 \varepsilon^{\frac{\delta}{n} - \frac{1-\delta}{2}}$,

$$|g(t) - g_1(t)| \leq C\varepsilon^{\frac{1-\delta}{2} - \frac{\delta}{n}}, \tag{24}$$

where C is a constant independent of all ε.

Applying Theorem 1 we get

$$\lambda(\Psi_\varepsilon(x); \Psi_2(x)) \leq C_1 \varepsilon^{(\frac{1-\delta}{2} - \frac{\delta}{n})} \quad (C_1 := \frac{p}{q}C).$$

This completes the proof of the theorem.

THEOREM 3. *Assume that the random variable ξ has the $\varepsilon-$exponential distribution function. This means that there exists $T = T(\varepsilon)$, $T(\varepsilon) \to \infty$ as $\varepsilon \to 0$, such that for $|t| \leq T(\varepsilon)$ we have $|\varphi(t) - \frac{\theta}{1 - i\theta t}| \leq \varepsilon$. Then the following estimation holds*

$$\lambda(\Psi_\zeta; \Psi_2) \leq \max\{\frac{q}{p}e^{4\lambda}\varepsilon, \ \frac{1}{T(\varepsilon)}\}.$$

PROOF. Let $g(t)$ be the characteristic function of η. According to Theorem 1 there exists $T = T(\varepsilon)$ such that, for any t satisfying $|t| \leq T(\varepsilon)$, we have

$$|g(t) - g_1(t)| \leq e^{4\lambda}|\varphi(t) - \frac{\theta}{1 - i\theta t}| \leq e^{4\lambda}\varepsilon.$$

On the other hand, if $\varphi_\zeta(t)$ is the characteristic function of ζ, then

$$|\varphi_\zeta(t) - g_2(t)| \leq \frac{q}{p}|g(t) - g_1(t)|.$$

This implies that

$$|\varphi_\zeta^{(t)} - g_2(t)| \leq \frac{q}{p} e^{4\lambda} \varepsilon, \quad (|t| \leq T(\varepsilon)).$$

According to the definition of the metric $\lambda(\cdot;\cdot)$, we have

$$\lambda(\Psi_\zeta; \Psi_2) \leq \max\{\frac{q}{p} e^{4\lambda} \varepsilon;\ \frac{1}{T(\varepsilon)}\},$$

which completes the proof.

REFERENCES

[1] Nguyen Huu Bao, *On the Stability of the Characterization of the Distribution Functions"*, Doctor thesis, Hanoi, 1989.

[2] Nguyen Huu Bao, *On the stability of the characterization of a random sum of i.i.d. variables by constant regression*, Reports of the Faculty of Mathematics, Mechanics, and Informatics, Hanoi University, December, 1989.

[3] Nguyen Huu Bao, *On the stability of the characterization of the composed random variables*, Vietnam Journal of Mathematics, 1995 (to appear).

HYDROLOGICAL FACULTY
HANOI WATER RESOURCES COLLEGE