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BIFURCATION AND HOPF BIFURCATION AT MULTIPLE EIGENVALUES FOR EQUATIONS WITH LIPSCHITZ MAPPINGS

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Abstract. Some results on the existence of bifurcation and Hopf bifurcation at multiple eigenvalues for abstract equations concerning Lipschitz continuous mappings in Banach spaces are proved. The obtained results improve some well-known bifurcation results by Crandall and Rabinowitz, McLeod and Sattinger, Hopf etc. in the case involving Lipschitz continuous mappings. Some illustrative examples are given.

1. Introduction

Bifurcation problems and Hopf bifurcation problems play a very important role in different areas of applied mathematics and have been intensively studied in the literature. Several methods have been used, for instance, variational, topological, analytical, and numerical methods etc. In general, the bifurcation problem consists in determining bifurcation points of equations depending on a parameter in Banach spaces of the form

$$F(\lambda, v) = 0, \quad (\lambda, v) \in \Lambda \times \bar{D}, \tag{1.1}$$

where Λ is a subset of a normed space, D is a neighborhood of the origin in a Banach space X with the closure $\bar{D}$ and F is a nonlinear mapping from $\Lambda \times \bar{D}$ into another Banach space Y with $F(\lambda, 0) = 0$ for all $\lambda \in \Lambda$. $(\lambda, 0)$ is called a trivial solution. A point $(\bar{\lambda}, 0) \in \Lambda \times D$ is said to be a bifurcation point of the equation (1.1) if and only if

$$(\bar{\lambda}, 0) \in cl\{(\lambda, v) \in \Lambda \times \bar{D} | \quad F(\lambda, v) = 0 \text{ and } v \neq 0\},$$

where $clA = \bar{A}$, the closure of the set A.

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In the case the mapping F is differentiable, using the Implicit Function Theorem one can easily verify that $(\bar{\lambda}, 0)$ is a bifurcation point of (1.1) only if $\bar{\lambda}$ belongs to the spectral set of $F_x(\lambda, 0)$, i.e. $F_x(\bar{\lambda}, 0)u_0 = 0$ for some $u_0 \in X,\ u_0 \neq 0$.

The Hopf bifurcation problems study the bifurcation of periodic solutions of dynamic systems depending on a parameter

$$\frac{dv}{dt} = F(\lambda, v), \quad (\lambda, v) \in \Lambda \times \bar{D}. \tag{1.2}$$

We say that for $\lambda = \bar{\lambda} \in \Lambda$ a small periodic solution of the system (1.2) with periods close to T_0 bifurcates from the origin if for every $\varepsilon > 0$ there exists a point λ_ε in a neighborhood of $\bar{\lambda}(|\lambda - \bar{\lambda}|_\Lambda < \varepsilon)$ for which the system (1.2) has a nonzero T_ε – periodic solution $v_\varepsilon, (|T_\varepsilon - T_0| < \varepsilon)$ lying in the ε – neighborhood of the origin in X:

$$||v_\varepsilon(t)|| < \varepsilon \quad (-\infty < t < \infty).$$

In the case the mapping F is differentiable, such a bifurcation occurs when a complex conjugate pair of eigenvalues of $F_v(\lambda, 0)$ crosses the imaginary axis for a critical $\lambda = \bar{\lambda}$ while all other eigenvalues have negative real parts. There are several generalizations of the original result obtained by E. Hopf [4] which shows that the bifurcation of periodic solutions of the system (1.2) in a finite dimensional case occurs under the following assumptions:

a) the eigenvalue $\sigma(\lambda)$ of $F_v(\lambda, 0)$ crosses the imaginary axis for critical $\lambda = \bar{\lambda}$ with Re $\sigma'(\bar{\lambda}) \neq 0$, where σ' denotes the derivative of σ with respect to λ,

b) the purely imaginary eigenvalue $\sigma(\bar{\lambda}) = i\mu_0$ is simple,

c) $F_v(\bar{\lambda}, 0)$ has no eigenvalue of the form $ik\mu_0$, $k = 0, 2, 3, \ldots$, (see [1], [2], [7], [10], etc).

In [8] Kielhöfer studied the Hopf bifurcation of the evolution equation

$$\frac{du}{dt} + Au + B(\lambda)u = F(\lambda, u)$$

in a Hilbert space with mappings A, B, F depending analytically on λ and $B(0) = 0$, and F being a higher order term. The mapping A is assumed to have a purely imaginary eigenvalue $i\mu_0$ with multiplicity $r \geq 1$. Then, he investigated the Hopf bifurcation at $\lambda = 0$, using the method of Lyapunov–Schmidt for evolution equations, following Iudovich [6]. This reduces the above problem to the solution of a bifurcation equation in R^{2r} of the form

$$Dv + \tilde{B}(\lambda)v + G(\mu, \lambda, v) = 0.$$

The parameter μ corresponds to the unkwown period of the bifurcating solution. The vector v belongs to R^{2n-1} and the linear operators D and $\tilde{B}(\lambda)$ as well as the nonlinear operator $G(\mu, \lambda, .)$ map R^{2n-1} into R^{2n}. He found some necessary and sufficient conditions for the Hopf bifurcation of the above system, showing that the positive number of branches which bifurcate at $\lambda = 0$ depends only on the number of nontrivial solutions of four algebraic equations in R^{2n}. The purpose of this paper is to study the existence of bifurcation points and Hopf bifurcation points of the systems (1.1) and (1.2), respectively, with F of the form

$$F(\lambda, u) = -T(u) + L(\lambda, u) + H(\lambda, u) + K(\lambda, u), \qquad (\lambda, u) \in \Lambda \times \bar{D},$$

where Λ is an open subset of a normed space Z with the norm defined by $|.|_\Lambda$, D is a neighborhood of the origin in a Banach space X. For any fixed $\lambda \in \Lambda$, T, $L(\lambda, .)$ are linear continuous mappings from X into another Banach space Y, $H(\lambda, .)$, $K(\lambda, .)$ are nonlinear Lipschitz continuous mappings of "higher order term" from $\bar{D}$ into Y with $H(\lambda, 0) = K(\lambda, 0) = 0$ and $H(\lambda, .)$ satisfies an a–homogeneous condition to be described later with $a > 1$. Let $\bar{\lambda} \in \Lambda$ be a characteristic value of the pair (T, L) (i.e. $T(v) + L(\bar{\lambda}, v) = 0$ for some $v \in X$, $v \neq 0$) such that the mapping $T + L(\bar{\lambda}, .)$ is Fredholm with nullity p and index zero the results of this paper are also valid for the case of positive index $s > 0$, $p > s \geq 0$). For the sake of simplicity we only investigate the case of index zero though. We shall prove, under some sufficient conditions, that $(\bar{\lambda}, 0)$ is a bifurcation point of the system (1.1) with F as above, using the Lyapunov–Schmidt procedure, the Banach Contraction Principle and the topological degree theory. Furthermore, we also describe parameter families of nontrivial solutions of the system (1.1) in a neighborhood of $(\bar{\lambda}, 0)$ in an analytical form. Our results in Section 2 generalize some well-known results obtained by Crandall and Rabinowitz [2], McLeod and Sattinger [9], Buchner, Marsden and Schecter [1]. They always need the differentiability conditions on those mappings. These also generalize the author's results in [11] where we assumed that the mappings H and K are Lipschitz continuous with respect to both λ and v. In Section 3, let $\bar{\lambda} \in \Lambda$ be such that the mapping $T + L(\bar{\lambda}, .)$ is Fredholm of index zero and has $\pm i\mu_0$ as eigenvalues with multiplicity p.

We assume that this mapping has no eigenvalue of the form $\pm i \; n\mu_0$ with $n = 0, 2, \ldots$. We shall give some sufficient conditions to show that $(\bar{\lambda}, 0)$ is a Hopf bifurcation point of the system (1.2) with F given as above, proving that the existence of Hopf bifurcation points can be reduced to the study of the nonvanishing of the topological degree of some mappings in a finite dimensional space (see Theorems 3.2, 3.6, 3.10, 3.14 below). In particular, this problem can be reduced to the existence of nontrivial regular solutions of algebraic equations in R^{2p} (see Corollaries 3.4, 3.8, 3.12, 3.16 below).

Section 4 is devoted to degenerate cases of Hopf bifurcation. We assume that the reduced algebraic equations of the system (1.2) have nontrivial nonregular solutions. We shall show that the existence of Hopf bifurcation points can also be reduced to the study of the nonvanishing of the topological degree of mappings by considering the first nonzero derivatives of the bifurcation equations at $(\bar{\lambda}, 0)$.

2. The main results for bifurcation

Throughout this paper, X and Y are always supposed to be real or complex Banach spaces with duals X^* and Y^*, respectively. Λ is an open subset of a normed space. The norms and the pairings between elements of X, X^* and Y, Y^* are denoted by the same symbols $\|.\|$ and $< ., . >$, respectively. The norm of the normed space containing Λ restricted to Λ is denoted by $|.|_\Lambda$. The symbol $R^n, n = 1, 2, \ldots$ stands for the n-dimensional Euclidean space whose norm will be denoted by the same symbol $|.|$ for all n. It is customary to simplify the notation for R^1 by dropping the superscript, $R^1 = R$. In this section we consider the existence of bifurcation points of the system (1.1) with F mentioned above. This means that we investigate the existence of bifurcation points of the equation

$$T(v) = L(\lambda, v) + H(\lambda, v) + K(\lambda, v), \qquad (\lambda, v) \in \Lambda \times \bar{D}, \tag{2.3}$$

where the mappings T, L, H and K are as in the introduction. Now let $\bar{\lambda} \in \Lambda$ be a characteristic value of the pair (T, L) with multiplicity p. It follows that the null space $Ker(T - L(\bar{\lambda}, .))$ of the mapping $T - L(\bar{\lambda}, .)$ is p-dimensional. We assume

$$Ker(T - L(\bar{\lambda}, .)) = [v^1, \ldots, v^p],$$

where the right side is the space spanned by $v^1, \ldots, v^p$.

By $(T - L(\bar{\lambda}, .))^*$ we denote the adjoint mapping of the mapping $T - L(\bar{\lambda}, .)$ and assume

$$Ker\ (T - L(\bar{\lambda}, .))^* = [\psi^1, \ldots, \psi^p].$$

By the Hahn - Banach Theorem one can find p functionals $\gamma^1, \ldots \gamma^p$ on X and p elements $z^1, \ldots, z^p$ in Y such that $< v^k, \gamma^j >= \delta_{kj}$ and $< z^m, \psi^n >= \delta_{m,n}$, $k, j, m, n = 1, 2, \ldots, p$, with δ_{kj}, δ_{mn} denoting the Kronecker δ. We set

$$\begin{aligned} X_0 &= Ker\ (T - L(\bar{\lambda}, .)) = [v^1, \ldots, v^p] \\ X_1 &= \{x \in X \mid < x, \gamma^j >= 0,\ j = 1, \ldots, p\} \\ Y_0 &= [z^1, \ldots, z^p] \\ Y_1 &= \{y \in Y \mid\ < y, \psi^j >= 0, j = 1, \ldots, p\}. \end{aligned}$$

It can be seen that $X = X_0 \oplus X_1$, $Y = Y_0 \oplus Y_1$ and the restriction of the mapping $T - L(\bar{\lambda}, .)$ on X_1 is a one-to-one linear continuous mapping from X_1 onto Y_1. The projections $P_X : X \to X_0, Q_X : X \to X_1$, $P_Y : Y \to Y_0$ and $Q_Y : Y \to Y_1$ are defined by

$$P_X(x) = \sum_{j=1}^{p} < x, \gamma^j > v^j, \quad Q_X(x) = x - P_X(x),\ x \in X,$$

$$P_Y(y) = \sum_{k=1}^{p} < y, \psi^k > z^k, \quad Q_Y(y) = y - P_Y(y),\ y \in Y.$$

Concerning the main results in this paper we impose the following hypotheses on the mappings L, H and K.

HYPOTHESIS 1. *There is a real number b such that $\alpha L(\bar{\lambda}, v) = L(\alpha^b \bar{\lambda}, v)$ holds for all $\alpha \in [0, 1]$ and $v \in \bar{D}$.*

HYPOTHESIS 2. *There exist a constant k_1 and a real number $a > 1$ and a real increasing function $\varrho : R \to R$ with $\lim_{\delta \to 0} \varrho(\delta) = 0$ such that*

i) $H(\lambda, tv) = t^a H(\lambda, tv)$ holds for all $t \in [0, 1]$, $\quad (\lambda, v) \in \Lambda \times \bar{D}$.

ii) $||Q_Y H\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|)^b}, u\right) - Q_Y H\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|)^b}, v\right)|| \leq k_1 ||u - v||$ holds for all $u, v \in \bar{D}$, uniformly in $\alpha \in [0, 1]$, where b is from Hypothesis 2.

iii) $||Q_Y K\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|)^b}, u\right) - Q_Y K\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|)^b}, v\right)|| \leq \varrho(||u - v||)||u - v||$ holds for all $u, v \in \bar{D}$ and uniformly in $\alpha \in [0, 1]$.

In what follows, for the sake of simplicity, we set $M = H + K$. We can easily verify that the equation (1.1) is equivalent to the following $p+1$ equations

$$\begin{aligned} &Q_Y(T(v) - L(\lambda, v) - M(\lambda, v)) = 0, \\ &< T(v) - L(\lambda, v) - M(\lambda, v), \psi^j > = 0, \quad j = 1, \ldots, p. \end{aligned} \tag{2.4}$$

Since $v \in X$ can be written as $v = \sum_{j=1}^{p} \varepsilon_j v^j + w$ for some $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_p) \in R^p, w \in X_1$, we then conclude that to solve the system of equations (2.4) we need to find $\lambda \in \Lambda, \varepsilon = (\varepsilon_1, \ldots, \varepsilon_p) \in R^p$ and $w \in X_1$ satisfying

$$Q_Y\left(T\left(\sum_{j=1}^{p} \varepsilon_j v^j + w\right) - L\left(\lambda, \sum_{j=1}^{p} \varepsilon_j v^j + w\right) - M\left(\lambda, \sum_{j=1}^{p} \varepsilon_j v^j + w\right)\right) = 0$$

$$< T\left(\sum_{j=1}^{p} \varepsilon_j v^j + w\right) - L\left(\lambda, \sum_{j=1}^{p} \varepsilon_j v^j + w\right) - M\left(\lambda, \sum_{j=1}^{p} \varepsilon_j v^j + w\right), \psi^j > = 0,$$
$$j = 1, \ldots, p.$$

Next, let I_1 be a neighborhood of zero in $R, I_1 \subseteq (-1, 1)$, such that $\bar{\lambda}/(1 \pm |\alpha|)^b \in \Lambda$ holds for all $\alpha \in I_1$, where b is from Hypothesis 1, and let $U_1 = U(0, r)$ be an open ball with the center at the origin in R^p and the radius $r > 0$ such that $\sum_{j=1}^{p} \varepsilon_j v^j \in P_X(D)$ for all $(\varepsilon_1, \ldots \varepsilon_p) \in U_1$. Without loss of generality we may assume that $|\alpha|^{\alpha-1} \in I_1, |\alpha|\varepsilon \in U_1$ hold for all $\alpha \in I_1, \varepsilon \in U_1$. Setting $D_1 = Q_X(D)$ and by choosing D_1' smaller if necessary, we may assume $D_1 = D_1(0, r_1)$, the open ball with the center at the origin in X_1 and radius $r_1 > 0$. We define the mapping $G_\pm : I_1 \times U_1 \times \bar{D}_1 \to X_1$ by

$$G_\pm(\alpha, \varepsilon, w) = $$
$$- \Pi Q_Y\left(\pm|\alpha| T\left(\sum_{j=1}^{p} \varepsilon_j v^j + w\right) - (1 \pm |\alpha|) M\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|)^b} \varepsilon_j v^j + w\right)\right),$$

where $(\alpha, \varepsilon, w) \in I_1 \times U_1 \times D_1$, and Π is the inverse of the mapping $T - L(\bar{\lambda}, .)$ from Y_1 onto X_1.

The following lemma plays an important role in the proof of the main results of this paper.

LEMMA 2.1. *Let Hypotheses 1.2 be satisfied and let I_1, U_1, D_1 be as above. Then there exist neighborhoods I_2 of zero in $R, I_2 \subset I_1$, D_2 of the origin in $X_1, D_2 \subset D_1$ such that for any $(|\alpha|^{a-1}, |\alpha|x) \in I_2 \times U_1$ one can find a point*

$w_{\pm}(|\alpha|^{a-1}, |\alpha|x) \in D_2$ satisfying

1) $G_{\pm}(|\alpha|^{a-1}, |\alpha|x, w_{\pm}(|\alpha|^{a-1}, |\alpha|x)) = w_{\pm}(|\alpha|^{a-1}, |\alpha|x)$

2) there exists a constant $k_2 > 0$ such that for any $|\alpha|^{a-1} \in I_2, |\alpha|x^1, |\alpha|x^2 \in U_1$ we have

$$||w_{\pm}(|\alpha|^{a-1}, |\alpha|x^1 - w_{\pm}(|\alpha|^{a-1}, |\alpha|x^2)|| \leq k_2|x^1 - x^2|,$$

(consequently, for any fixed $\alpha \in I_2, w_{\pm}(|\alpha|^{a-1}, |\alpha|.)$ is a continuous mapping with respect to $x \in U_1$).

3) For any natural number n there exist constants E_n, F_n such that

$$\left|\left|\frac{w_{\pm}(|\alpha|^{a-1}, |\alpha|x)}{|\alpha|}\right|\right| \leq E_n(1 + |\bar{\lambda}| + |x|)|\alpha|^{a-1} + F_n|\alpha|^{(n+1)a-(n+2)}$$

holds for all $(|\alpha|^{a-1}, |\alpha|x) \in I_2 \times U_1, \alpha \neq 0$, (consequently, we conclude that $||w_{\pm}(|\alpha|^{a-1}, |\alpha|x)|| = o(|\alpha|)$ as $\alpha \to 0$, uniformly in $x \in U_1$).

PROOF. For fixed $t \in [0,1]$, we set $I(t) = tI_1, D(t) = tD_1$. Then, any $\alpha \in I(t), w \in D(t)$ can be written as $\alpha = t\alpha', w = tw'$ with $\alpha' \in I_1, w' \in D_1$. Now let $(|\alpha|^{a-1}, |\alpha|x) \in I(t) \times U_1$, $\alpha = t\alpha'$ and $w^1, w^2 \in D(t)$, $w^j = t{w'}^j$, $j = 1, 2$. We have

$$\begin{aligned}
&||G_{\pm}(|\alpha|^{a-1}, |\alpha|x, w^1) - G_{\pm}(|\alpha|^{a-1}, |\alpha|x, w^2)|| \leq \gamma\{|\alpha|^{a-1}||T||\ ||w^1 - w^2|| \\
&\quad + (1 + |\alpha|^{a-1})t^a \left|\left| H\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}, \sum_{j=1}^{p} |\alpha'|x_j v^j + {w'}^1\right) - \right.\right. \\
&\quad \left.\left. -H\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}, \sum_{j=1}^{p} |\alpha'|x_j v^j + {w'}^2\right)\right|\right| + \\
&(1 + |\alpha|^{a-1}) \left|\left| K\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}, \sum_{j=1}^{p} |\alpha|x_j v^j + w^1\right) - \right.\right. \\
&\quad \left.\left. -K\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}, \sum_{j=1}^{p} |\alpha|x_j v^j + w\right)\}\right|\right| \\
&\leq \gamma\{t^{a-1}(||T|| + 2k_1) + 2k_2(t))\}||w^1 - w^2||
\end{aligned}$$

where $\gamma = ||\Pi Q_Y||, k_1, a, b$ from Hypothesis 1, 2 and $k_2(t) = \varrho(2r_1t)$ satisfies

$\lim_{\delta \to u} k_2(t) = 0$ according to $\lim_{\delta \to u} \varrho(\delta) = 0$. Therefore, setting

$$C_1(t) = \gamma\{t^{a-1}(||T|| + 2k_1) + 2k_2(t)\},$$

we can see $\lim_{t \to 0} C_1(t) = 0$.

Further, for $(|\alpha|^{a-1}, |\alpha|x) \in I(t) \times U_1, \alpha = t\alpha', \alpha' \in I_1$, and $w \in D(t), w = tw', w' \in D_1$, we have

$$\begin{aligned}||G_{\pm}(|\alpha|^{a-1}, |\alpha|x, w)|| \leq &\{|\alpha|^{a-1}||T||(|\alpha x| + ||w||)) \\ &+ 2t^a k_1(|\alpha' x| + ||w'||) + 2k_2(t)||w||\} \\ &\leq \gamma\{t^a(|\alpha'|^{a-1}(||T|||\alpha' x| + ||w'|| + 2k_1(|\alpha' x| + ||w'||) \\ &+ 2k_2(t).t||w'|| \\ &\leq \gamma\{t^a(||T|| + 2k_1)(r + r_1) + 2k_2(t).tr_1\}.\end{aligned}$$

Setting $C_2(t) = \gamma\{t^a(||T|| + 2k_1)(r + r_1) + 2k_2(t)tr_1\}$ we can see $\lim_{t \to 0} \frac{C_2(t)}{t} = 0$. Consequently, we conclude that there exists $t_0 \in (0, 1)$ such that

$$C_1(t_0) < 1$$

and

$$C_2(t_0) \leq t_0 r_1.$$

Putting $I_2 = t_0 I_1$, $D_2 = t_0 D_1$, we deduce that for any $(|\alpha|^{a-1}, |\alpha|x) \in I_2 \times U_1$, the mapping $G_{\pm}(|\alpha|^{a-1}, |\alpha|x, .)$ is a contraction mapping and it maps $\bar{D}_2$ into itself. Applying the Banach Contraction Principle, we conclude that $G_{\pm}(|\alpha|^{a-1}, |\alpha|x, .)$ possesses a fixed point $w_{\pm}(|\alpha|^{a-1}, |\alpha|x)$ in D_2, i.e.,

$$G_{\pm}(|\alpha|^{a-1}, |\alpha|x, w_{\pm}(|\alpha|^{a-1}, |\alpha|x)) = w_{\pm}(|\alpha|^{a-1}, |\alpha|x).$$

This implies the first assertion of the lemma.

Now, let $(|\alpha|^{a-1}, |\alpha|x^1), (|\alpha|^{a-1}, |\alpha|x^2) \in I_2 \times U_1$, $\alpha = t_0\alpha', w_{\pm}(|\alpha|^{a-1}, |\alpha|x^j)$ $t_0\ w'_{\pm}(|\alpha|^{a-1}, |\alpha|x^j)$, $j = 1, 2$. We have

$$\begin{aligned}&||w_{\pm}(|\alpha|^{a-1}, |\alpha|x^1) - w_{\pm}(|\alpha|^{a-1}, |\alpha|x^2)|| \\ &= ||G_{\pm}(|\alpha|^{a-1}, |\alpha|x^1, w_{\pm}(|\alpha|^{a-1}, |\alpha|x^1)) - G_{\pm}(|\alpha|^{a-1}, |\alpha|x^2, w_{\pm}(|\alpha|^{a-1}, |\alpha|x^2))|| \\ &\leq \gamma\{t_0^{a-1}(||T|||x^1 - x^2| + ||w_{\pm}(|\alpha|^{a-1}, |\alpha|x^1) - w_{\pm}(|\alpha|^{a-1}, |\alpha|x^2)||) \\ &\quad + 2t_0^{a-1}k_1(|x^1 - x^2| + ||\bar{w}_{\pm}(|\alpha|^{a-1}, |\alpha|x^1) - w_{\pm}(|\alpha|^{a-1}, |\alpha|x^2)||) \\ &\quad + 2k_2(t_0)(|x^1 - x^2| + ||w_{\pm}(|\alpha|^{a-1}, |\alpha|x^1) - w_{\pm}(|\alpha|^{a-1}, |\alpha|x^2)||)\}.\end{aligned}$$

Hence

$$||w_{\pm}(|\alpha|^{a-1}, |\alpha|x^1) - w_{\pm}(|\alpha|^{a-1}, |\alpha|x^2)|| \leq \frac{\gamma t_0^{a-1}(||T|| + 2k_1 + 2k_2(t_0))}{1 - \gamma t_0^{a-1}(||T|| + 2k_1 + 2k_2(t_0))}|x^1 - x^2|$$

and we have proved assertion 2). The proof of assertion 3) proceeds exactly as the one of [11], Lemma 2.

This completes the proof of the lemma.

Further, let $\bar{\lambda}, \{v^1, \ldots, v^p\}$ and $\{\psi^1, \ldots, \psi^p\}$ be as above. We define the mapping $A : R^p \to R^p$, $A = (A_1, \ldots, A_p)$ by

$$A_k(x) = < T(\sum_{j=1}^{p} x_j v^j) - H(\bar{\lambda}, \sum_{j=1}^{p} x_j v^j), \psi^k >, x = (x_1, \ldots, x_p) \in R^p, \quad k = 1, \ldots, p, \quad (2.5)$$

and make the following hypothesis

HYPOTHESIS 3. *There is a point $\bar{x} \in R^p$ and an open neighborhood U^* of $\bar{x}$ not containing the origin in R^p such that the topological degree $\deg(A, U^*, 0)$ of the mapping A with respect to U^* and the origin is defined and different from zero.*

Then we have

THEOREM 2.2. *Under Hypotheses 1-3, $(\bar{\lambda}, 0)$ is a bifurcation point of the equation (2.3). More precisely, to any given $\delta > 0$ there exists a neighborhood I of zero in R such that for each $\alpha \in I, \alpha \neq 0$, we can find $x^{\pm}(\alpha) = (x_1^{\pm}(\alpha), \ldots, x_p^{\pm}(\alpha)) \in U^*$ and a nontrivial solution $(\lambda^{\pm}(\alpha), v^{\pm}(\alpha))$ of the equation (2.3) with*

$$\lambda^{\pm}(\alpha) = \frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})}b$$

and

$$v^{\pm}(\alpha) = \sum_{j=1}^{p} |\alpha| x_j^{\pm}(\alpha) v^j + w_{\pm}(|\alpha|^{a-1}, |\alpha| x^{\pm}(\alpha))$$

satisfying $|\lambda^{\pm}(\alpha) - \bar{\lambda})|_{\Lambda} < \delta$ and $0 < ||v^{\pm}(\alpha)|| < \delta$, where $w_{\pm}$ is from Lemma 2.1.

PROOF. Let I_1, U_1, D_1 be as above and let $\delta > 0$ be given. Using Lemma 2.1, we conclude that there is a neighborhood I_2 of zero in $R, I_2 \subset I_1$, such that for $\alpha \in I_2, \alpha \neq 0, |\alpha|x \in U_1$ we can find a fixed point $w_\pm(|\alpha|^{a-1}, |\alpha x)$ of the mapping $G_\pm(|\alpha|^{a-1}, |\alpha|x, .)$. It follows

$$w_\pm(|\alpha|^{a-1}, |\alpha|x) = \Pi Q_Y(\pm|\alpha|^{a-1}T(\sum_{j=1}^{p}|\alpha|x_jv^j + w_\pm(|\alpha|^{a-1}, |\alpha|x))$$
$$-(1 \pm |\alpha|^{a-1})M\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}, \sum_{j=1}^{p}|\alpha|x_jv^j + w_\pm(|\alpha|^{a-1}, |\alpha|x))\right).$$

Hence

$$Q_Y\Bigg\{T(w_\pm(|\alpha|^{a-1}, |\alpha|x)) - L(\bar{\lambda}, w_\pm(|\alpha|^{a-1}, |\alpha|x))$$
$$\pm|\alpha|^{a-1}T\left(\sum_{j=1}^{p}|\alpha|x_jv^j + w_\pm(|\alpha|^{a-1}, |\alpha|x)\right)$$
$$-(1 \pm |\alpha|^{a-1})M\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}, \sum_{j=1}^{p}|\alpha|x_jv^j + w_\pm|\alpha|^{a-1}, |\alpha|x))\right)\Bigg\} = 0.$$

Together with $T\left(\sum_{j=1}^{p}|\alpha|x_jv^j\right) - L\left(\bar{\lambda}, \sum_{j=1}^{p}|\alpha|x_jv^j\right) = 0$, we deduce

$$Q_Y\Bigg\{T\left(\sum_{j=1}^{p}|\alpha|x_jv^j + w_\pm(|\alpha|^{a-1}, |\alpha|x)\right)$$
$$-L\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}, \sum_{j=1}^{p}|\alpha|x_jv^j + +w_\pm(|\alpha|^{a-1}, |\alpha|x)\right)$$
$$-M\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}, \sum_{j=1}^{p}|\alpha|x_jv^j + w_\pm(|\alpha|^{a-1}, |\alpha|x)\right)\Bigg\} = 0. \tag{2.6}$$

Further, by choosing $I_2' \subset I_2$ if necessary we may assume $\bar{\lambda}/(1\pm|\alpha|^{a-1})^b \in \Lambda$ and $|\bar{\lambda} - \frac{\bar{\lambda}}{(1\pm|\alpha|^{a-1})^b}, | < \delta$ and $\alpha U^* \subset U_1$ for all $\alpha \in I_2, \sum_{j=1}^{p}|\alpha|x_jv^j \in P_XU(0,\delta)$ for all $|\alpha|x \in U_1$ and we also assume $D_1 \subset Q_XU(0,\delta)$. We proceed the proof exactly as the one of Theorem 1 in [11] to conclude that there exists a neighborhood I of zero in $R, I \subset I_2$, such that for any $\alpha \in I, \alpha \neq 0$, one can find $x^\pm(\alpha) \in$

$U^*, x^{\pm}(\alpha) = (x_1^{\pm}(\alpha), \ldots, x_p^{\pm}(\alpha)) \in U^*$ and

$$< T(v^{\pm}(\alpha)) - L\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}, v^{\pm}(\alpha)\right) - \\ - M\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}, v^{\pm}(\alpha)\right), \psi^k >= 0, \quad k = 1, \ldots, p \tag{2.7}$$

where

$$v^{\pm}(\alpha) = \sum_{j=1}^{p} |\alpha| x_j^{\pm}(\alpha) v^j + w_{\pm}(|\alpha|^{a-1}, |\alpha| x^{\pm}(\alpha)).$$

A combination of (2.6) and (2.7) gives

$$T(v^{\pm}(\alpha)) = L(\lambda^{\pm}(\alpha), v^{\pm}(\alpha)) + M(\lambda^{\pm}(\alpha, v^{\pm}(\alpha))$$

with

$$\lambda^{\pm}(\alpha) = \frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}.$$

It is clear that $|\lambda^{\pm}(\alpha) - \bar{\lambda}|_{\Lambda} < \delta$ and $0 < ||v^{\pm}(\alpha)|| < \delta$ for all $\alpha \in I,\ \alpha \neq 0$. This completes the proof of the theorem.

REMARK 2.3. Such a family $(\lambda^{\pm}(\alpha), v^{\pm}(\alpha)), \alpha \in I, \alpha \neq 0$ satisfying Theorem 2.2 is called a parameter family of nontrivial solution of the equation (2.3). Theorem 2.2 shows that we can always find at least two different parameter families of nontrivial solutions of (2.3).

REMARK 2.4. If there are two distinct points $\bar{x}^1, \bar{x}^2 \in R^p$ and two disjoint neighborhoods U_1^*, U_2^* in R^p of $\bar{x}^1, \bar{x}^2$, respectively, satisfying Hypothesis 3 and $I^1, I^2, (\lambda_j^{\pm}(\alpha), v_j^{\pm}(\alpha)),\ j = 1,2$ exist by Theorem 2.2 corresponding to $\bar{x}^1, U_1^*$ and $\bar{x}^2, U_2^*$, respectively, then $v_2^{\pm}(\alpha) \neq v_2^{\pm}(\alpha)$ for all $\alpha \in I^1 \cap I^2, \alpha \neq 0$.

REMARK 2.5. In Theorem 2.2 we need no continuity of the mapping $M(.,u)$ with respect to $\lambda \in \Lambda$ for any fixed $u \in D$ and no differentiability of the mappings L, M. Theorem 2.2 can be applied to consider bifurcation points of the equation

$$-\Delta u - \lambda u = h(\lambda)|u|^{\sigma} u \qquad \text{on } \Omega \subset R^n$$

together with the Dirichlet boundary condition, where h is not continuous with respect to $\lambda \in \Lambda$ and $\sigma > 0$. But, the other results in [1], [3], [9] and [11] cannot be applied.

Further, we obtain the following corollaries whose proofs are similar to the ones of Corollaries 2 and 3 in [1].

COROLLARY 2.6. *Let Hypotheses 1, 2 be satisfied and the mapping A defined in (2.5) be a potential operator with potential h, which possesses a local minimum and isolated critical point $\bar{x} \neq 0$. Then the conclusions of Theorem 2.2 continue to hold for some open neighborhood U^* of the point $\bar{x}$ in R^p.*

COROLLARY 2.7. *Let Hypotheses 1, 2 be satisfied. Let the mapping A defined in (2.5) be Gâtaux differentiable and let $\bar{x} \in R^p, \bar{x} \neq 0$, be such that*

$$A(\bar{x}) = 0 \tag{2.8}$$

and

$$\bar{\gamma} = \det\left(\frac{\partial A_k}{\partial x_j}(\bar{x})\right)_{j,k=1,\dots,p} \neq 0. \tag{2.9}$$

Then the conclusions of Theorem 2.2 continue to hold for some open neighborhood U^* of the point $\bar{x}$ in R^p.

REMARK 2.8. The point $\bar{x}$ satisfying (2.8) and (2.9) is called a regular solution of the equation $A(x) = 0$. In case $\bar{x}$ satisfies (2.8) but does not satisfy (2.9) we say that $\bar{x}$ is a degenerate solution. The bifurcation problem of the equation (2.3) in the degenerate case can also be investigated by the same ways as in [12]. We do not study it here.

Next, we consider the case when $\bar{\lambda}$ is a simple characteristic value of the pair (T, L) i.e. $Ker(T - L(\bar{\lambda}, .)) = [v^1]$ and $Ker(T - L(\bar{\lambda}, .))^* = [\psi^1]$. We assume that $< T(v^1), \psi^1 > \neq 0$ and make the hypothesis.

HYPOTHESIS 2'. *Hypothesis 2 is satisfied with $\frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}$ replaced by $\frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1}\beta)^b}$ for any fixed $\beta \in R$.*

We can prove the following theorem which generalizes the well-known theorem obtained by Crandall and Rabinowitz [3].

THEOREM 2.9. *Let $\bar{\lambda}, v^1, \psi^1$ be as above and let L, H and K satisfy Hypotheses 1, 2', respectively. Then $(\bar{\lambda}, 0)$ is a bifurcation point of the equation (2.3). More precisely, given $\delta > 0$ there exists a neighborhood I of zero in R such that for each $\gamma \in I, \gamma \neq 0$, one can find a point $\beta^{\pm}(\gamma) \in R$ and a nontrivial solution $(\lambda^{\pm}(\gamma), v^{\pm}(\gamma))$ of the equation (2.3) with*

$$\lambda^{\pm}(\gamma) = \frac{\bar{\lambda}}{(1 \pm |\gamma|^{a-1}\beta^{\pm}(\gamma))^b}$$

and

$$v^{\pm}(\gamma) = \pm|\gamma|v^1 + o(|\alpha|) \text{ as } \gamma \to 0$$

satisfying $|\lambda^{\pm}(\gamma)-\bar{\lambda}|_{\Lambda}<\delta$ *and* $0<||v^{\pm}(\gamma)||<\delta$.

PROOF. The proof of this theorem proceeds exactly as the one of Theorem 7 in [11], using the modification of Lemma 2.1.

3. The main results for Hopf bifurcation

Let X be a real or complex Banach space with the dual X^*. By $\mathcal{X} = C_{2\pi}(R,X), (\mathcal{X}^* = C_{2\pi}(R,X^*))$ we denote the spaces of $2\pi-$periodic continuous functions from R into X. By $\mathcal{Y} = C_0([0,2\pi],X), (\mathcal{Y}^* = C_0[0,2\pi],X^*))$ we denote the Banach spaces of continuous functions $h:[0,2\pi]\to X$ such that $h(0) = 0$. The topology in $\mathcal{X}$ and $\mathcal{Y}$ is the usual sup-norm topology. The pairing between elements in X and X^* will be denoted by (,) and the parings between elements in $\mathcal{X}$ and $\mathcal{X}^*$, also in $\mathcal{Y}$ and $\mathcal{Y}^*$ will be denoted by the same symbol $<,>$ defined by

$$<u,v>=\int_0^{2\pi}(u(t),v(t))dt, \qquad (u,v)\in\mathcal{X}\times\mathcal{X}^* \text{ or } (u,v)\in\mathcal{Y}\times\mathcal{Y}^*.$$

The space X can be considered as a subspace of $\mathcal{X}$. In this section we consider Hopf bifurcation of the equation

$$\frac{du}{dt}+T(u)+L(\lambda,u)+H(\lambda,u)+K(\lambda,u)=0, \quad (\lambda,u)\in\Lambda\times\bar{D} \tag{3.10}$$

where D is a neighborhood of the origin in $\mathcal{X}, \Lambda$ is an open subset of a normed space. For any $\lambda\in\Lambda, T, L(\lambda,.)$ are linear continuous mappings from a dense set of $\mathcal{X}$ into $\mathcal{Y}, H(\lambda,.), K(\lambda,.)$ are nonlinear mappings with $H(\lambda,0)=K(\lambda,0)=0$ for all $\lambda\in\Lambda$. Let i denote the complex number such that $i^2=-1$ and let $\bar{\lambda}\in\Lambda$ be such that linear mapping $T+L(\bar{\lambda},.)$ is Fredholm and has $\pm i\mu_0$ as eigenvalues with nullity p and index zero, $\mu_0\neq 0$. Without loss of generality we may assume $\mu_0=1$. In what follows we consider the existence of Hopf bifurcation points of the equation (3.10) in the case when $\pm i$ is an eigenvalue of the linear mapping $T+L(\bar{\lambda},.)$. We also assume that the mapping $T+L(\bar{\lambda},.)$ has no other eigenvalues of the form $\pm in$ with $n=0,2,\ldots$.

Let $v^1,\ldots v^p$ be independent eigenvectors corresponding to eigenvalue i, i.e.,

$$T(v^j)+L(\bar{\lambda},v^j)=iv^j, j=1,\ldots,p.$$

It then follows

$$Ker(T+L(\bar{\lambda},.)-iI)=[v^1,\ldots v^p],$$

where I denotes the identity mapping. A simple calculation shows

$$Ker\ \left(\frac{d}{dt}+T+L(\bar{\lambda},.)\right)=[\phi^1,\dots,\phi^{2p}]$$

with $\phi^{2k-1}(t)=Re\ (e^{-it}v^k)$ and $\phi^{2k}(t)=Im(e^{-it}v^k), k=1,2,\dots,p$.

Let

$$Ker\ (T+L(\bar{\lambda},.)-iI)^*=[\gamma^1,\dots,\gamma^p].$$

Without loss of generality we may assume $< v^k,\gamma^j > = \delta_{jk}, \quad j,k=1,\dots,p$. If then follows

$$Ker\ \left(\frac{d}{dt}+T+L(\bar{\lambda},.)\right)^*=[\psi^1,\dots,\psi^{2p}]$$

with $\psi^{2k-1}(t)=Re\ (e^{-it}\gamma^k), \psi^{2k}(t)=Im(e^{-it}\gamma^k), k=1,\dots,p$. We put

$$\mathcal{X}_0=[\psi^1,\dots,\psi^{2p}].$$

Then $\mathcal{X}_0$ has a complementary orthogonal subspace $\mathcal{X}_1$, i.e. $\mathcal{X}=\mathcal{X}_0\oplus\mathcal{X}_1$. By the Hahn-Banach Theorem we can find $2p$ functionals $f^1,\dots,f^{2p}$ on $\mathcal{X}$ and $2p$ elements $z^1,\dots,z^{2p}$ in $\mathcal{Y}$ such that $<\phi^k,f^j>=\delta_{kj}$ and $<z^m,\psi^n>=\delta_{mn}, k,j,m,n=1,\dots,2p$. We set

$$\begin{aligned}\mathcal{Y}_0 &=[z^1,\dots,z^{2p}]\\ \mathcal{Y}_1 &=\{y\in\mathcal{Y}| < y,\psi^k >=0, \quad k=1,\dots,2p\}\end{aligned}$$

If then follows $\mathcal{Y}=\mathcal{Y}_0\oplus\mathcal{Y}_1$. Let $P_{\mathcal{X}},Q_{\mathcal{X}},P_{\mathcal{Y}},Q_{\mathcal{Y}}$ be defined as in Section 2 with X,Y replaced by $\mathcal{X},\mathcal{Y}$ and p replaced by $2p$ respectively. It is clear that $\frac{d}{dt}+T+L(\bar{\lambda},.)$ is a one to one mapping from $\mathcal{X}_1$, onto $\mathcal{Y}_1$.

The problem of finding $(1\pm\varrho)2\pi-$ periodic solutions of the equation (3.10) will be done by finding 2π- periodic solutions of the equation

$$\frac{du}{dt}+(1\pm\varrho)\{T(u)+L(\lambda,u)+H(\lambda,u)+K(\lambda,u)\}=0 \quad (\varrho,\lambda,u)\in R\times\Lambda\times\hat{D}, \tag{3.11}$$

which has ϱ close to zero and letting $t=(1\pm\varrho)\tau$. As in Section 2, the equation

(3.11) is equivalent to the following $2p+1$ equations

$$Q_{\mathcal{Y}}\{\frac{du}{dt}+(1\pm\varrho)(T+L(\lambda,u)+H(\lambda,u)+K(\lambda,u)\}=0,$$

$$<\frac{du}{dt}+(1\pm\varrho)(T(u)+L(\lambda,u)+K(\lambda,u)+K(\lambda,u),\psi^j>=0,$$
$$j=1,\ldots,2p.$$

We assume that the mappings L, H and K satisfy Hypotheses 1 and 2, respectively. In this section we shall give some sufficient conditions such that $(\bar{\lambda},0)$ is a Hopf bifurcation point of the equation (3.10).

As before, since any $u\in\mathcal{X}$ can be writen as $u=\sum_{j=1}^{2p}\epsilon_j\phi^j+w$ for some $\epsilon=(\epsilon_1,\ldots,\epsilon_{2p})\in R^{2p}, w\in\mathcal{X}_1$, we then conclude that to solve the system of equations (3.12) we need to find $\lambda\in\Lambda, \epsilon=(\epsilon_1,\ldots,\epsilon_{2p})\in R^{2p}$ and $w\in\mathcal{X}_1$ satisfying

$$Q_{\mathcal{Y}}\{\frac{d(\sum_{j=1}^{2p}\epsilon_j\phi^j+w)}{dt}+(1\pm\varrho)(T\sum_{j=1}^{2p}\epsilon_j\phi^j+w)+L(\sum_{j=1}^{2p}\epsilon_j\phi^j+w)$$
$$+H(\lambda,\sum_{j=1}^{2p}\epsilon_{j=1}\phi^j+w)+K(\lambda,\sum_{j=1}^{2p}\epsilon_j\phi^j+w))\}=0,$$
$$<\frac{d(\sum_{j=1}^{2p}\epsilon_j\phi^j+w)}{dt}+(1\pm\varrho)(T(\sum_{j=1}^{2p}\epsilon_j\phi^j+L(\lambda,\sum_{j=1}^{2p}\epsilon_j\phi^j+w)$$
$$+H(\lambda,\sum_{j=1}^{2p}\epsilon_j\phi^j+w)+K(\lambda,\sum_{j=1}^{2p}\epsilon_j\phi^j+w)),\psi^j>=0,\quad k=1,\ldots,2p.$$

We take I_1, U_1 and $D_1\subset\mathcal{X}_1$ as in Section 2 and define the mapping $G_\pm$: $I_1\times I_1\times U_1\times D_1\to\mathcal{X}_1$ by

$$G_\pm(\varrho,\alpha,\varepsilon,w)=$$
$$-\Pi Q_{\mathcal{Y}}\left\{\pm\frac{\alpha d\left(\sum_{j=1}^{2p}\varepsilon_j\phi^j+w\right)}{dt}+(\pm\alpha\pm\varrho+\alpha\varrho)T\left(\sum_{j=1}^{2p}\varepsilon_j\phi^j+w\right)\right.$$
$$\left.\pm\varrho L\left(\bar{\lambda},\sum_{j=1}^{2p}\varepsilon_j\phi^j+w\right)+(1\pm\alpha)(1\pm\varrho)M\left(\frac{\bar{\lambda}}{(1\pm\alpha)^b},\sum_{j=1}^{2p}\varepsilon_j\phi^j+w\right)\right\},$$
$$\text{for } (\varrho,\alpha,\varepsilon,w)\in I_1\times I_1\times U_1\times D_1,$$

where Π is the inverse of the mapping $\frac{d}{t} + T + L(\bar{\lambda}, .)$ from $\mathcal{Y}_1$ onto $\mathcal{X}_1$ and $M = H + K$. We have

LEMMA 3.1. *Under Hypotheses 1, 2 there exists a neighborhoods I_0 of zero in $R, I_2 \subset I_1, D_2$ of the origin in $\mathcal{X}_1, D_2 \subset D_1$, such that for any fixed $\alpha \in I_2, \alpha \neq 0, (|\alpha|^a, |\alpha|^a, |\alpha|x) \in I_2 \times I_2 \times U_1$, $x = (x_1, \ldots, x_{2p})$, one can find a point $w_\pm(|\alpha|^a, |\alpha|^a, |\alpha|x) \in D_2$ satisfying:*

1) $w_\pm(|\alpha|^a, |\alpha|^a, |\alpha|x) = G_\pm(|\alpha|^a, |\alpha|^a, |\alpha|x, w_\pm(|\alpha|^a, |\alpha|^a, |\alpha|x))$

2) There exists a constant $k_0 > 0$ depending on α such that for $\alpha \in I_2, \alpha \neq 0, |\alpha|x^1, |\alpha|x^2 \in U_1$ we have

$$||w_\pm(|\alpha|^a, |\alpha|^a, |\alpha|x^1) - w_\pm(|\alpha|^a, |\alpha|^a, |\alpha|x^2)|| \leq k_0|x^1 - x^2|.$$

It follows that for any fixed $\alpha \in I_0, w_\pm(|\alpha|^a, |\alpha|^a, |\alpha|.)$ is a continuous mapping with respect to $x \in U_1$.

3) Let $\alpha \in I_2, \alpha \neq 0$, be fixed. For any natural number n there exists constants E_n, F_n such that

$$\left|\left|\frac{w_\pm(|\alpha|^a, |\alpha|^a, |\alpha|x)}{\alpha}\right|\right| \leq E_n(1 + |\bar{\lambda}| + |x|)|\alpha|^{a-1} + F_n|\alpha|^{(n+1)a-(n+2)}$$

holds for all $(|\alpha|^a, |\alpha|^a, |\alpha|x) \in I_2 \times I_2 \times U_1$. As a consequence, $||w_\pm(|\alpha|^a, |\alpha|^a, |\alpha|x)|| = o(|\alpha|)$ as $\alpha \to 0$.

PROOF. The proof of this lemma proceeds exactly as the one of Lemma 2.1.

Now, let $\bar{\lambda}, \phi^1, \ldots, \phi^{2p}, \psi^1, \ldots, \psi^{2p}$ be as above. We define the mapping $\mathcal{A} : R^{2p} \to R^{2p}, \mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_{2p})$ by

$$\mathcal{A}_k(x) = < H\left(\bar{\lambda}, \sum_{j=1}^{2p} x_j \phi^j\right), \psi^k >, x = (x_1, \ldots, x_{2p}) \in R^{2p},$$
$$k = 1, \ldots, 2p, \tag{3.12}$$

and make

HYPOTHESIS 4. *There is a point $\bar{x} \in R^{2p}$ and an open neighborhood U^* of $\bar{x}$ not containing the origin in R^{2p} such that the topological degree, deg $(\mathcal{A}, U^*, 0)$, of the mapping $\mathcal{A}$ with respect to U^* and the origin is defined and different from zero.*

We have

THEOREM 3.2. *Under Hypotheses 1, 2, 4, $(\bar{\lambda}, 0)$ is a Hopf bifurcation point of periodic solutions of the equation (3.10). More precisely, there exists a neighborhood I_0 of zero in R. For any $\alpha \in I_0, \alpha \neq 0$, one can find a point $x^{\pm}(\alpha) = (x_1^{\pm}(\alpha), \dots, x_{2p}^{\pm}(\alpha)) \in U^*$ such that $(\varrho_{\pm}(\alpha), \lambda_{\pm}(\alpha), v_{\pm}(\alpha)), \alpha \in I_0, \alpha \neq 0$, with*

$$\varrho_{\pm}(\alpha) = \pm\, |\alpha|^a$$
$$\lambda_{\pm}(\alpha) = \frac{\bar{\lambda}}{(1 \pm |\alpha|^a)^b}$$

$$v_{\pm}(\alpha) = \sum_{j=1}^{2p} |\alpha| x_j^{\pm}(\alpha)\phi^j + w_{\pm}(|\alpha|^a, |\alpha|^a, |\alpha| x^{\pm}(\alpha))$$

satisfies the equation (3.10), $\varrho_{\pm}(\alpha) \to 0, \lambda_{\pm}(\alpha) \to \bar{\lambda}, v_{\pm}(\alpha \to 0$ as $\alpha \to 0, v_{\pm}(\alpha) \neq 0$ as $\alpha \neq 0$, and $u_{\pm}(\alpha(t)) = v_{\pm}(\alpha)\left(\frac{t}{1+\varrho_{\pm}(\alpha)}\right)$ is $(1 + \varrho_{\pm}(\alpha))2\pi$ - periodic , where $w_{\pm}$ is from Lemma 3.1.

PROOF. Let $I_2, U_1, w_{\pm}$ be from Lemma 3.1 and U^* be from Hypothesis 4. Without loss of generality we may assume $\alpha U^* \subset U_2$ holds for all $\alpha \in I_2$. We define the mapping $E^{\pm} : I_2 \times U^* \to R^{2p}, E^{\pm} = (E_1^{\pm}, \dots, E_{2p}^{\pm})$, by

$$E_k^{\pm}(\alpha, x) = \begin{cases} < (\pm 1 + |\alpha|^a) T(\sum_{j=1}^{2p} |\alpha| x_j \phi^j + w_{\pm}(|\alpha^a, |\alpha|^a, |\alpha| x)) + \\ +(1 \pm |\alpha|^a)(1 \pm |\alpha|^a) H\left(\frac{\bar{\lambda}}{(1\pm|\alpha|^a)^b}, \sum_{j=1}^{2p} x_j \phi^j + \frac{w_{\pm}(|\alpha|^a, |\alpha|^a, |\alpha| x)}{|\alpha|}\right) + \\ +(1 \pm |\alpha|^a)(1 \pm |\alpha|^a)|\alpha|^{-a} K\left(\frac{\bar{\lambda}}{(1\pm|\alpha|^a)^b}, \sum_{j=1}^{2p} |\alpha| x_j \phi^j + \right. \\ \left. + w_{\pm}(|\alpha|^a, |\alpha|^a, |\alpha| x)\right), \psi^k > & \text{for } \alpha \neq 0, \\ \mathcal{A}_k(x), & \text{for } \alpha = 0. \end{cases}$$

By choosing $I_2' \subset I_2$ if necessary we may assume that $E^{\pm}(t\alpha, x) \neq 0$ holds for all $\alpha \in I_2, t \in [0, 1]$ and $x \in \partial U^*$. Indeed, by contrary we assume that for any n there exist $\alpha_n \in I_n, t_n \in [0, 1]$ and $x_n \in \partial U^*$ such that $\alpha_n \to 0$ and $E^{\pm}(t_n \alpha_n, x_n) = 0$. Since ∂U^* is a compact set, therefore we may assume $x_n \to x_0 \in \partial U^*$ and $\alpha_n t_n \to 0$. Using 3) of Lemma 3.1 we deduce $E(0, x_0) = 0$ or $\mathcal{A}(x_0) = 0$. This contradicts the definition of $\deg(\mathcal{A}, U^*, o)$. Further, for any fixed $\alpha \in I_2, \alpha \neq 0$, we define the mapping $\xi^{\pm} : [0, 1]_{\times} U^* \to R^{2p}$ by

$$\xi^{\pm}(t, x) = E^{\pm}(t\alpha, x), \quad (t, \alpha) \in [0, 1] \times U^*.$$

Hence, $\xi^{\pm}(t, x) \neq 0$ for all $t \in [0, 1], x \in \partial U^*$ and then we have

$$\begin{aligned}\deg(E^{\pm}(\alpha,.),U^*,0) &= \deg(\xi^{\pm}(1,.),U^*,0)\\ &= \deg(\xi^{\pm}(0,.),U^*,0)\\ &= \deg(\mathcal{A},U^*,0) \neq \{0\}.\end{aligned}$$

It then follows that for any $\alpha \in I_2, \alpha \neq 0$, there exists $x^{\pm}(\alpha) \in U^*$ such that $E^{\pm}(\alpha, x^{\pm}(\alpha)) = 0$. Further, the proof proceeds exactly as the one of Theorem 1 in [11]. This completes the proof of the theorem.

The proofs of the following corollaries are similar to the ones of Corollaries 2 and 3 in [11].

COROLLARY 3.3. *Let Hypotheses 1, 2, be satisfied and the mapping $\mathcal{A}$ defined as in (3.12) be a potential operator with potential h, which possesses a local minimum and isolated critical point $\bar{x} \neq 0$. Then the conclusions of Theorem 3.2 continue to hold.*

COROLLARY 3.4. *Let Hypotheses 1, 2 be satisfied. In addition, assume that there exists a point $\bar{x} \in R^{2p}, \bar{x} \neq 0$, such that $\mathcal{A}(\bar{x}) \neq 0$ and*

$$\det\left(\frac{\partial \mathcal{A}_k}{\partial x_j}(\bar{x})\right)_{k,j=1,\dots,2p} \neq 0.$$

Then the conclusions of Theorem 3.2 continue to hold.

LEMMA 3.5. *Under Hypotheses 1, 2 there exist neighborhoods I_2 of zero in R, D_2 of the origin in $\mathcal{X}_1$ such that for any fixed $\alpha \in I_0, \alpha \neq 0, (|\alpha|^{a-1}, |\alpha|^a, |\alpha|x) \in I_2 \times I_2 \times U_1$, $x = (x_1, \dots, x_{2p}) \in U_1$, one can find a point $w_{\pm}(|\alpha|^{a-1}, |\alpha|^a, |\alpha|x) \in D_2$ satisfying 1) - 3) of Lemma 3.1 with $(|\alpha|^a, |\alpha|^a, |\alpha|x)$ replaced by $(|\alpha|^{a-1}, |\alpha|^a, |\alpha|x)$.*

PROOF. The proof of this lemma proceeds exactly as in Lemma 2.1.

Let $\bar{\lambda}, \phi^1, \dots, \phi^{2p}, \psi^1, \dots \psi^{2p}$ and $\mathcal{A}$ be as above. We define the mapping $\mathcal{B}^{\pm} : R^{2p} \to R^{2p}, \mathcal{B}^{\pm} = (\mathcal{B}_1^{\pm}, \dots, \mathcal{B}_{2p}^{\pm})$, by

$$\mathcal{B}_k^{\pm}(x) = \mp < \frac{d\sum_{j=1}^{2p}(x_j\phi^j)}{dt}, \psi^k > + \mathcal{A}_k(x),\ x = (x_1, \dots, x_{2p}) \in R^{2p},$$

for $k = 1, \dots, 2p$, and make

HYPOTHESIS 5. *Hypothesis 4 is satisfied with $\mathcal{A}, \bar{x}, U^*$ replaced by $\mathcal{B}^{\pm}, \bar{x}^{\pm}, U^{*\pm}$, respectively.*

We have

THEOREM 3.6. *Under hypotheses 1, 2, and 5, $(\bar{\lambda}, 0)$ is a Hopf bifurcation point of periodic solutions of the equation (3.10). More precisely, the conclusions of Theorem 3.2 continue to hold with*

$$\varrho_{\pm}(\alpha = \pm\, |\alpha|^{a-1}$$
$$\lambda_{\pm}(\alpha) = \frac{\bar{\lambda}}{(1 \pm |\alpha|^{a})^{b}}$$

and

$$v_{\pm}(\alpha) = \sum_{j=1}^{2p} |\alpha| x_j^{\pm}(\alpha)\phi^j + w_{\pm}(|\alpha|^{a-1}, |\alpha|^a, |\alpha| x^{\pm}(\alpha))$$

PROOF. The proof of this theorem proceeds exactly as the one of Theorem 3.2 with $E^{\pm}$ replaced by $\mathcal{J}^{\pm} = (\mathcal{J}_1^{\pm}, \ldots, \mathcal{J}_{2p}^{\pm})$, where

$$\mathcal{J}_k^{\pm}(\alpha, x) = \begin{cases} < (\pm 1 + |\alpha|) \left(\frac{d(\sum_{j=1}^{2p} x_j \phi^j + \frac{w(|\alpha|^{a-1}, |\alpha|^a, |\alpha| x)}{\alpha}}{dt} \right) + \\ +(\pm|\alpha| + |\alpha|^a) T \left(\sum_{j=1}^{2p} x_j \phi^j + \frac{w(|\alpha|^{a-1}, |\alpha|^a, |\alpha| x)}{\alpha} \right) \\ +(1 \pm |\alpha|^a)(1 \pm |\alpha|^{a-1})|\alpha|^{-a} H \left(\frac{\bar{\lambda}}{(1 \pm |\alpha|^a)^b}, \sum_{j=1}^{2p} x_j \phi^j + \right. \\ \qquad \left. + \frac{w(|\alpha|^{a-1}, |\alpha|^a, |\alpha| x)}{\alpha} \right) \\ +(1 \pm |\alpha|^a)(1 \pm |\alpha|^{a-1})|\alpha|^{-a} K \left(\frac{\bar{\lambda}}{(1 \pm |\alpha|^a)^b}, \sum_{j=1}^{2p} |\alpha| x_j \phi^j + \right. \\ \left. + w(|\alpha|^{a-1}, |\alpha|^a, |\alpha| x) \right), \psi^k > & \text{for } \alpha \neq 0, \\ \mathcal{B}_k(x), & \text{for } \alpha = 0. \end{cases}$$

This completes the proof of the theorem.

By the same methods as in the proofs of Corollaries 2, 3, in [11], we have

COROLLARY 3.7. *Let Hypotheses 1, 2 be satisfied and let the mapping $\mathcal{B}^{\pm}$ be a potential operator with potential $h^{\pm}$ which possesses a local minimum and isolated critical point $\bar{x} \neq o$. Then the conclusions of Theorem 3.6 continue to hold*

COROLLARY 3.8. *Let Hypotheses 1, 2 be satisfied. In addition, assume that* *there exists a point* $\bar{x}^{\pm} \in R^{2p}, \bar{x}^{\pm} \neq 0$, *such that* $\mathcal{B}^{\pm}(\bar{x}^{\pm}) = 0$ *and*

$$\det \left(\frac{\partial \mathcal{B}_k^{\pm}}{\partial x_j}(\bar{x}^{\pm}) \right)_{j,k=1,\dots,2p} \neq 0.$$

Then the conclusions of Theorem 3.6 continue to hold.

LEMMA 3.9. *Under Hypotheses 1, 2 there exists a neighborhoods* I_2 *of zero in* R, D_2 *of the origin in* $\mathcal{X}_1$ *such that for any fixed* $\alpha \in I_2, \alpha \neq 0, (|\alpha|^{a-1}, |\alpha|^{a-1}, |\alpha|x) \in I_2 \times I_2 \times U_1, x = (x_1, \dots, x_{2p}) \in U_1$ *one can find a point* $w_{\pm}(|\alpha|^{a-1}, |\alpha|^{a-1}, |\alpha|x) \in D_2$ *satisfying 1) - 3) of Lemma 3.1 with* $(|\alpha|^a, |\alpha|^a, |\alpha|x)$ *replaced by* $(|\alpha|^{a-1}, |\alpha|^{a-1}, |\alpha|x)$.

PROOF. The proof of this lemma proceeds exactly as the one of Lemma 2.1.

Let $\bar{\lambda}, \phi^1, \dots, \phi^{2p}, \psi^1, \dots, \psi^{2p}$ and α be as above. We define the mapping $\mathcal{C}^{\pm} : R^{2p} \to R^{2p}, \mathcal{C}^{\pm} = (\mathcal{C}_1^{\pm} = (\mathcal{C}_1^{\pm}, \dots, \mathcal{C}_{2p}^{\pm}))$, by

$$\mathcal{C}_k^{\pm}(x) = \pm < T \left(\sum_{j=1}^{2p} x_j \phi^j \right), \psi^k > + \mathcal{A}_k(x), \quad x = (x_1, \dots, x_{2p}) \in R^{2p},$$

and make

HYPOTHESIS 6. *Hypothesis 4 is satisfied with* $\alpha, \bar{x}, U^*$ *replaced by* $\mathcal{C}^{\pm}, \bar{x}^{\pm}$ *and* $U^{*\pm}$, *respectively.*

We have

THEOREM 3.10. *Under Hypotheses 1, 2 and 6,* $(\bar{\lambda}, 0)$ *is a Hopf bifurcation point of periodic solutions of the equation (3.10). More precisely, the conclusions of Theorem 3.2 continue to hold with*

$$\varrho_{\pm}(\alpha) = \pm |\alpha|^{a-1}$$
$$\lambda_{\pm}(\alpha) = \frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}$$

and

$$v_{\pm}(\alpha) = \sum_{j=1}^{2p} |\alpha| x_j^{\pm}(\alpha) \phi^j + w_{\pm}(|\alpha|^{a-1}, |\alpha|^{a-1}, |\alpha| x^{\pm}(\alpha)).$$

The proof of this theorem proceeds exactly as the one of Theorem 3.2 with

$E^{\pm}$ replaced by $M^{\pm} = (M_1^{\pm}, \ldots, M_{2p}^{\pm})$, where

$$M_k^{\pm}(\alpha, x) = \begin{cases} < (\pm 1 + |\alpha|^{a-1}) T\left(\sum_{j=1}^{2p} x_j \phi^j + \frac{w_{\pm}(|\alpha|^{a-1}, |\alpha|^{a-1}, |\alpha| x)}{\alpha}\right) + \\ \quad + (1 \pm |\alpha|^{a-1})(1 \pm |\alpha|^{a-1}) H\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}, \sum_{j=1}^{2p} x_j \phi^j + \right. \\ \quad \left. + \left(\frac{w_{\pm}(|\alpha|^{a-1}, |\alpha|^{a-1}, |\alpha| x)}{\alpha}\right)\right) + (1 \pm |\alpha|^{a-1})(1 \pm |\alpha|^{a-1}). \\ \quad |\alpha|^{-a} K\left(\frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}, \sum_{j=1}^{2p} |\alpha| x_j \phi^j + \right. \\ \quad \left. w_{\pm}(|\alpha|^{a-1}, |\alpha|^{a-1}, |\alpha| x)\right), \psi^k >, & \text{for } \alpha \neq 0, \\ \mathcal{C}_k^{\pm}(x), & \text{for } \alpha = 0. \end{cases}$$

This completes the proof of the theorem.

The proofs of the following corollaries are similar to the ones of Corollaries 2 and 3 in [11].

COROLLARY 3.11. *Let Hypotheses 1, 2 be satisfied and the mapping $\mathcal{C}$ defined as above be a potential operator with potential $h^{\pm}$, which possesses a local minimum and isolated critical point $\bar{x}^{\pm} \neq 0$. Then the conclusions of Theorem 3.10 continue to hold.*

COROLLARY 3.12. *Let Hypotheses 1 and 2 be satisfied. In addition, assume that there exists a point $\bar{x}^{\pm} \in R^{2p}, \bar{x}^{\pm} \neq 0$ such that $\mathcal{C}^{\pm}(\bar{x}^{\pm}) = 0$ and*

$$\det\left(\frac{\partial \mathcal{C}_k^{\pm}}{\partial x_j}(\bar{x}\pm)\right)_{k,j=1,\ldots,2p} \neq 0.$$

Then the conclusions of the Theorem 3.10 continue to hold.

LEMMA 3.13. *Under Hypotheses 1, 2 there exist neighborhoods I_2 of zero in R, D_2 of the origin in $\mathcal{X}_1$ such that for any fixed $\alpha \in I_2, \alpha \neq 0, (|\alpha^a, |\alpha^{a-1}, |\alpha| x) \in I_2 \times I_2 \times U_1, x = (x_1, \ldots, x_{2p})$ one can find a point $w_{\pm}(|\alpha|^a, |\alpha|^{a-1}, |\alpha| x) \in D_2$ satisfying 1) - 3) of Lemma 3.1 with $(|\alpha|^a, |\alpha|^a, |\alpha| x)$ replaced by $(|\alpha|^a, |\alpha|^{a-1}, |\alpha| x)$.*

PROOF. The proof of this lemma proceeds exactly as the one of Lemma 3.1.

Next, let $\bar{\lambda}, \phi^1, \ldots, \phi^{2p}, \psi^1, \ldots, \psi^{2p}$ and be α as above. We define the mapping $\mathcal{D}^{\pm} : R^{2p} \to R^{2p}, \mathcal{D}^{\pm} = (\mathcal{D}_1^{\pm}, \ldots, \mathcal{D}_{2p}^{\pm})$, by

$$\mathcal{D}_k^{\pm}(x) = \mp < L(\bar{\lambda}, \sum_{j=1}^{2p} x_j \phi^j), \psi^k > + \mathcal{A}_k(x), \quad x = (x_1, \ldots, x_{2p}) \in R^{2p},$$

for $k = 1, \ldots, 2p$, and make

HYPOTHESIS 7. *Hypothesis 4 is satisfied with $\alpha, \bar{x}, U^*$ replaced by $\mathcal{D}^{\pm}, \bar{x}^{\pm}$ and $U^{*\pm}$, respectively.*

We have

THEOREM 3.14. *Under Hypotheses 1, 2, and 7, $(\bar{\lambda}, 0)$ is a Hopf bifurcation point of periodic solutions of the equation (3.10). More precisely, the conclusions of Theorem 3.2 continue to hold with*

$$\varrho_{\pm}(\alpha) = \pm\, |\alpha|^a$$
$$\lambda_{\pm}(\alpha) = \frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}$$

and

$$v_{\pm}(\alpha) = \sum_{j=1}^{2p} |\alpha| x_j^{\pm}(\alpha)\phi^j + w_{\pm}(|\alpha|^a, |\alpha|^{a-1}, |\alpha| x)).$$

PROOF. The proof of this theorem proceeds exactly as the one of Theorem 3.2 with $E^{\pm}$ replaced by $F^{\pm} = (F_1^{\pm}, \dots, F_{2p}^{\pm})$, where

$$F_k^{\pm}(\alpha, x) = \begin{cases} < (\mp 1 \pm |\alpha|) \dfrac{d(\sum_{j=1}^{2p}(x_j\phi^j + \frac{w(|\alpha|^a, |\alpha|^{a-1}, |\alpha| x)}{|\alpha|})}{dt} + \\ \quad +(\pm\alpha + |\alpha|^a) T\left(\sum_{j=1}^{2p} x_j\phi^j + \frac{w(|\alpha|^a, |\alpha|^{a-1}, |\alpha| x)}{|\alpha|}\right) + \\ \qquad +(1 \pm |\alpha|^{a-1})(1 \pm |\alpha|^a) \\ \quad H\left(\frac{\bar{\lambda}}{1 \pm |\alpha|^{a-1})^b}, \sum_{j=1}^{2p} x_j\phi^j + \frac{w(|\alpha|^a, |\alpha|^{a-1}, |\alpha| x)}{|\alpha|}\right) + \\ \qquad +(1 \pm |\alpha^{a-1})(1 \pm |\alpha|^a)|\alpha|^{-a} \\ \quad H\left(\frac{\bar{\lambda}}{1 \pm |\alpha|^{a-1})^b}, \sum_{j=1}^{2p} |\alpha| x_j\phi^j + w(|\alpha|^a, |\alpha|^{a-1}, |\alpha| x)\right), \psi^k >, \\ \qquad\qquad \text{for } \alpha \neq 0, \\ \mathcal{D}_k^{\pm}(x) \qquad\qquad \text{for } \alpha = 0. \end{cases}$$

This completes the proof of the theorem.

The following corollaries are proved by the same methods as the one of Corollaries 2, 3 in [11].

COROLLARY 3.15. *Let Hypotheses 1, 2 be satisfied and let the mapping $\mathcal{D}^{\pm}$ defined as above be a potential operator with potential $h^{\pm}$, which possesses* *a local minimum and isolated critical point* $\bar{x}^{\pm} \neq 0$. *Then the conclusions of Theorem 3.10 continue to hold.*

COROLLARY 3.16. *Let Hypotheses 1, 2 be satisfied. In addition, assume that there exists a point* $\bar{x}^{\pm} \in R^{2p}, \bar{x}^{\pm} \neq 0$, *such that*

$$\det\left(\frac{\partial \mathcal{D}_k^{\pm}}{\partial x_j}(\bar{x}^{\pm})\right)_{k,j=1,\ldots,2p} \neq 0.$$

Then the conclusions of Theorem 3.10 continue to hold.

REMARK 3.17. Similarly, we also make Remarks 2.3 - 2.8 in Section 2 for the equation (3.10).

EXAMPLE 3.18. We consider the existence of Hopf bifurcation points of periodic solutions of the equation

$$\frac{du}{dt} + T(u) + \lambda L(u) + H(\lambda, u) + K(\lambda, u)\ ,\ (\lambda, u) \in R \times R^4 \tag{3.13}$$

where

$$T = \begin{pmatrix} -1, & -1, & 0, & 0 \\ 1, & -1, & 0, & 0 \\ 0, & 0, & -1, & 1 \\ 0, & 0, & -1, & -1 \end{pmatrix}, \qquad L = \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix}$$

$$H(\lambda, u) = \lambda\left(\sum u_j\right)^3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad , \quad u = (u_1, \ldots, u_4),$$

and K satisfies Hypothesis 2.

A simple calculation shows that if $\bar{\lambda} = 1$, the mapping $T + L$ has $\pm i$ as eigenvalues with multiplicity 2.

We have

$$v^1 = \gamma^1 = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} 1 \\ \text{-i} \\ 1 \\ \text{-i} \end{pmatrix} = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \frac{i}{2\sqrt{2\pi}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$v^2 = \gamma^2 = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} -i \\ 1 \\ -i \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \frac{i}{2\sqrt{2\pi}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

If then follows,

$$\phi^1 = \psi^1 = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} cost \\ -sint \\ cost \\ sint \end{pmatrix} \quad , \quad \phi^2 = \psi^2 = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} -sint \\ -cost \\ -sint \\ cost \end{pmatrix}$$

$$\phi^3 = \psi^3 = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} sint \\ cost \\ -sint \\ cost \end{pmatrix} \quad , \quad \phi^4 = \psi^4 = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} cost \\ -sint \\ -cost \\ -sint \end{pmatrix}.$$

One can easily see that

$$\mathcal{D}_1^{\pm}(x) = \mp x_1 + \frac{3}{8\pi}\left(\frac{1}{2}(x_1+x_3)^3 + (x_2+x_4)^2(x_1+x_3)\right)$$

$$\mathcal{D}_2^{\pm}(x) = \mp x_2 - \frac{3}{8\pi}\left(\frac{1}{2}(x_2+x_4)^3 + (x_1+x_3)^2(x_2+x_4)\right)$$

$$\mathcal{D}_3^{\pm}(x) = \mp x_3 + \frac{3}{8\pi}\left(\frac{1}{2}(x_1+x_3)^3 + (x_2+x_4)^2(x_1+x_3)\right)$$

$$\mathcal{D}_4^{\pm}(x) = \mp x_4 - \frac{3}{8\pi}\left(\frac{1}{2}(x_2+x_4)^3 + (x_1+x_3)^2(x_2+x_4).\right).$$

Taking $\mathcal{D}^+$ and $\bar{x}_2^{\pm} = \bar{x}_4^{\pm} = 0$, $\bar{x}_1^{\pm} = \bar{x}_3^{\pm} = \pm\left(\frac{2\pi}{3}\right)^{\frac{1}{2}}$, we can easily verify $\mathcal{D}^+(\bar{x}^{\pm}) = 0$ and

$$\left(\frac{\partial \mathcal{D}_k^+}{\partial x_j}(\bar{x}^{\pm})\right) = \begin{pmatrix} \frac{1}{2}, & 0, & \frac{3}{2}, & 0 \\ 0, & 0, & 0, & 1 \\ \frac{3}{2}, & 0, & \frac{1}{2}, & 0 \\ 0, & 1, & 0, & 0 \end{pmatrix}$$

a nonsingular matrix. Therefore, we apply Corollary 3.16, to conclude that $(1,0)$ is a Hopf bifurcation point of periodic solutions of the equation (3.13). Analogously, if we chose $\bar{x}_2^{\pm} = \bar{x}_4^{\pm} = \pm\left(\frac{2\pi}{3}\right)^{\frac{1}{2}}$ and $\bar{x}_1^{\pm} = \bar{x}_3^{\pm} = 0$, we can also prove that the conditions of Corollary 3.16 are satisfied. Consequently, using Remark 2.4 for the equation (3.13), we deduce that there exist at least 8 distinct parameter families of nontrivial periodic solutions of this equation.

4. The degenerate cases of Hopf bifurcation

Let the mappings $\mathcal{A}, \mathcal{B}^{\pm}, \mathcal{C}^{\pm}$ and $\mathcal{D}^{\pm}$ be as in Section 3. We begin this section by making the following hypotheses.

HYPOTHESIS 8. *There exists a point $\bar{x} \in R^{2p}, \bar{x} \neq 0$, such that $\mathcal{A}(\bar{x}) = 0$ and*

$$\det\left(\frac{\partial \mathcal{A}_k}{\partial x_j}(\bar{x})\right)_{k,j=1,\ldots,2p} = 0.$$

HYPOTHESIS 9. *Hypothesis 8 with $\bar{x}$ and $\mathcal{A}$ replaced by $\bar{x}^{\pm}$ and $\mathcal{B}^{\pm}$, respectively.*

HYPOTHESIS 10. *Hypothesis 8 with $\bar{x}$ and $\mathcal{A}$ replaced by $\bar{x}^{\pm}$ and $\mathcal{C}^{\pm}$, respectively.*

HYPOTHESIS 11. *Hypothesis 8 with $\bar{x}$ and $\mathcal{A}$ replaced by $\bar{x}^{\pm}$ and $\mathcal{D}^{\pm}$, respectively.*

The case when one of the above hypotheses occurs is said to be a degenerate case. In what follows we consider Hopf bifurcation points of the equation (3.10) in degenerate cases, assuming that Hypotheses 1, 2 in Section 2 are satisfied. We only investigate the case when Hypothesis 8 is fulfilled, the other cases should be studied similarly.

Now, let

$$X_0^r = \left\{x \in R^{2p} / \left(\frac{\partial \mathcal{A}_k}{\partial x_j}(\bar{x})\right) x = 0\right\} = [\xi^1, \ldots, \xi^r]$$
$$X_0^{*r} = \left\{x \in R^{2p} / \left(\frac{\partial \mathcal{A}_k}{\partial x_j}(\bar{x})\right)^* x = 0\right\} = [\xi^{*1}, \ldots, \xi^{*r}]$$

and

$$X_1^r = \{x \in R^{2p} / < x, \xi^j > = 0, \ j = 1, \ldots, r\}$$
$$X_1^{*r} = \{x \in R^{2p} / < x, \xi^{*j} > = 0, \ j = 1, \ldots, r\}$$

One can verify that $R^{2p} = X_0^r \oplus X_1^r = X_0^{*r} \oplus X_1^{*r}$. Further, let P_r, Q_r be the projections of R^{2p} into X_0^{*r}, X_1^{*r}, respectively. By $D^j \mathcal{A}(\bar{x}) = \mathcal{A}_{x \ldots x}(\bar{x}), (j-$ times), we denote the j-th derivative of the mapping $\mathcal{A}$ at the point $\bar{x}, j = 1, 2, \ldots$. Let c be the smallest natural number such that $P_r D^c \mathcal{A}(\bar{x}) \not\equiv$ on X_0^r. We define the mapping $f : R^r \to R^r, f = (f_1, \ldots, f_2)$, by

$$f_k(t) = < D^c \mathcal{A}(\bar{x}) \left(\sum_{j=1}^{r} t_j \xi^j\right), \xi^{*k} > , \ t = (t_1, \ldots, t_r) \in R^r, \ k = 1, \ldots, r, \quad (4.14)$$

Analogously, we define the mappings $g^{\pm}, h^{\pm}, q^{\pm} : R^r \to R^r$, with $\mathcal{A}$ replaced by $\mathcal{B}^{\pm}, \mathcal{C}^{\pm}$ and $\mathcal{D}^{\pm}$, respectively.

We make the following hypotheses on these mappings

HYPOTHESIS 12. *There exists a point $t^* \in R^r, t^* = (t_1^*, \ldots, t_r^*)$ and a neighborhood U^* of t^* in R^r such that the topological degree, $\deg(f, U^*, 0)$, of f with respect to U^* and the origin in R^r is defined and different from zero.*

HYPOTHESIS 13. *Hypothesis 12 with t^*, U^* and f^* replaced by $t^{*\pm}, U^{*\pm}$ and $g^{\pm}$, respectively.*

HYPOTHESIS 14. *Hypothesis 12 with t^*, U^* and f^* replaced by $t^{*\pm}, U^{*\pm}$ and $h^{\pm}$, respectively.*

HYPOTHESIS 15. *Hypothesis 12 with t^*, U^* and f^* replaced by $t^{*\pm}, U^{*\pm}$ and $q^{\pm}$, respectively.*

We have

THEOREM 4.1. *Under Hypotheses 1, 2, 8 and 12, $(\bar{\lambda}, 0)$ is a Hopf bifurcation point of periodic solutions of the equation (3.10). More precisely, to given $d > 0$ with $dc < a - 1$, there exists a neighborhood I_0 of zero in R such that for each $\alpha \in I_2, \alpha \neq 0$, one can find $t^{\pm}(\alpha) = (t_1^{\pm}(\alpha), \ldots, t_r^{\pm}(\alpha)) \in U^*$ such that $(\varrho_{\pm}(\alpha), \lambda_{\pm}(\alpha), v_{\pm}(\alpha))$ with*

$$\varrho_{\pm}(\alpha) = \pm |\alpha|^a$$
$$\lambda_{\pm}(\alpha) = \frac{\bar{\lambda}}{(1 \pm |\alpha|^a)^b}$$

and

$$v_{\pm}(\alpha) = \sum_{k=1}^{2p} |\alpha| (\bar{x}_k + \sum_{j=1}^{r} |\alpha|^d t_j^{\pm}(\alpha) \xi_k^j) \phi^k + o(|\alpha|) \text{ as } |\alpha| \to 0$$

satisfies the equation (3.10), $\varrho_{\pm}(\alpha) \to 0$, $\lambda_{\pm}(\alpha) \to \bar{\lambda}$, $v_{\pm}(\alpha) \to 0$ as $\alpha \to 0$, $v_{\pm}(\alpha) \neq 0$ for $\alpha \neq 0$, and $u_{\pm}(\alpha)(t) = v_{\pm}(\alpha)\left(\frac{t}{1+\varrho_{\pm}(\alpha)}\right)$ is a $(1 + \varrho_{\pm}(\alpha)2\pi$ - periodic.

PROOF. Let I_2, U_1 and $w_{\pm}$ be from Lemma 3.1. Without loss of generality we may assume $\bar{x} \notin U_1$. The proof of this theorem proceeds exactly as the one of Theorem 1 in [12] with A replaced by $\mathcal{A}$ and C replaced by $\mathcal{N}^{\pm} = (\mathcal{N}_1^{\pm}, \ldots, \mathcal{N}_{2p}^{\pm})$

where

$$\mathcal{N}_k^{\pm}(\alpha,x)=\begin{cases} < (1\pm|\alpha|^a)^2|\alpha|^{-a}M\left(\frac{\bar\lambda}{(1\pm|\alpha|^a)^b},\sum_{j=1}^{2p}|\alpha|x_j\phi^j+w_\pm(|\alpha|^a,|\alpha|^a,|\alpha|x)\right) & \\ \qquad -H\left(\bar\lambda,\sum_{j=1}^{2p}x_j\phi^j\right),\phi^k > \quad , & \text{for } \alpha\neq 0 \\ 0 & \text{for } \alpha=0. \end{cases}$$

PROOF. This completes the proof of the theorem.

COROLLARY 4.2. *Let Hypotheses 1, 2 and 8 be satisfied and c be an odd number. In addition, assume that $f(t)\neq 0$ for all $t\in R^r, |t|=1$. Then the conclusions of Theorem 4.1 continue to hold for $U^*=\{\xi\in R^r/|\xi|<1\}$.*

PROOF. Since c is an odd number, it follows that the mapping f is an odd mapping. The condition of the corollary implies $f(t)\neq 0$ for all $t\in\partial U^*$. So, by the Borsuk Theorem (see, for example, [4, Theorem 4.1]), the topological degree $\deg(f,U^*,0)$, of f with respect to U^* and the origin in R^r is defined and different from zero. Consequently, Hypothesis 12 is also satisfied. Therefore, to complete the proof of the corollary, it remains to apply Theorem 4.1.

COROLLARY 4.3. *Let Hypotheses 1, 2 and 8 be satisfied. Let $r=1$ and c be an odd number. Then the conclusions of Theorem 4.1 continue to hold.*

PROOF. Since $r=1$, it follows $f(t)=< D^c\mathcal{A}(\bar x)(t\,\xi^1),\xi^{*1} >=t^c< D^c\mathcal{A}(\bar x)(\xi^1),\xi^{*1} >\neq 0$ for $t\in R, |t|=1$. Consequently, this corollary follows immediately from Corollary 4.2.

THEOREM 4.4. *Under Hypotheses 1, 2, 9 and 13, the same conclusions of Theorem 4.1 continue to hold with $\varrho_\pm,\lambda_\pm$ replaced by*

$$\varrho_\pm(\alpha)=\pm|\alpha|^{a-1}$$

and

$$\lambda_\pm(\alpha)=\frac{\bar\lambda}{(1\pm|\alpha|^a)^b},$$

respectively.

PROOF. The proof of this theorem proceeds exactly as the one of Theorem 1 in [12] with A replaced by $\mathcal{B}^{\pm}$ and C replaced by $\mathcal{I}^{\pm} = (\mathcal{I}_1^{\pm}, \ldots, \mathcal{I}_{2p}^{\pm})$, where

$$\mathcal{I}_k^{\pm}(\alpha, x) = \begin{cases} < (1 \pm |\alpha|^a)(1 \pm |\alpha|^{a-1})|\alpha|^{-a} M \left(\frac{\bar{\lambda}}{(1 \pm |\alpha|^a)^b}, \sum_{j=1}^{2p} |\alpha| x_j \phi^j + \right. \\ \left. + w_{\pm}(|\alpha|^a, |\alpha|^{a-1}, |\alpha| x)\right) - H(\bar{\lambda}, \sum_{j=1}^{2p} x_j \phi^j), \psi^k >, & \text{for } \alpha \neq 0 \\ 0 & \text{for } \alpha \neq 0. \end{cases}$$

PROOF. This completes the proof of the theorem.

The following corollaries are proved by the same ways as Corollaries 4.2, 4.3.

COROLLARY 4.5. *Let Hypotheses 1, 2 and 9 satisfied and c be an odd number. In addition, assume that $g^{\pm}(t) \neq 0$ for all $t \in R^r, |t| = 1$. Then the conclusions of Theorem 4.4 continue to hold for $U^{*\pm} = \{\xi \in R^r / |\xi| < 1\}$.*

COROLLARY 4.6. *Let Hypotheses 1, 2 and 9 be satisfied. Let $r = 1$ and c be an odd number. Then the conclusions of Theorem 4.4 continue to hold.*

THEOREM 4.7. *Under Hyptheses 1, 2, 10 and 14, the conclusions of Theorem 4.1 continue to hold with $\varrho_{\pm}, \lambda_{\pm}$ replaced by*

$$\varrho_{\pm}(\alpha) = \pm |\alpha|^{a-1}$$
$$\lambda_{\pm}(\alpha) = \frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}.$$

PROOF. The proof of this theorem proceeds exactly as the one of Theorem 1 in [12] with A replaced by $\mathcal{C}^{\pm}$ and C replaced by $\mathcal{P}^{\pm} = (\mathcal{P}_1^{\pm}, \ldots, \mathcal{P}_{2p}^{\pm})$, where

$$\mathcal{P}_k^{\pm}(\alpha, x) = \begin{cases} < (1 \pm |\alpha|^a)(1 \pm |\alpha|^{a-1})|\alpha|^{-a} M \left(\frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}, \sum_{j=1}^{2p} |\alpha| x_j \phi^j + \right. \\ \left. + w_{\pm}(|\alpha|^{a-1}, |\alpha|^{a-1}, |\alpha| x)\right) - K \left(\bar{\lambda}, \sum_{j=1}^{2p} x_j \phi^j \right), \psi^k >, & \text{for } \alpha \neq 0 \\ 0 & \text{for } \alpha = 0. \end{cases}$$

This completes the proof of the theorem.

The following corollaries are proved by the same ways as Corollaries 4.2, 4.3, respectively.

COROLLARY 4.8. *Let Hypotheses 1, 2 and 9 be satisfied and let c be an odd number. In addition, assume that $h^{\pm}(t) \neq 0$ for all $t \in R^r, |t| = 1$. Then the conclusions of Theorem 4.7 continue to hold for $U^{*\pm} = \{\xi \in R^r / |\xi| < 1\}$.*

COROLLARY 4.9. *Let Hypotheses 1, 2 and 10 be satisfied. Let $r = 1$ and c be an odd number. Then the conclusions of Theorem 4.7 continue to hold.*

THEOREM 4.10. *Under Hypotheses 1, 2, 11 and 15, the same conclusions of Theorem 4.1 continue to hold with $\varrho_\pm, \lambda_\pm$ replaced by*

$$\varrho_\pm(\alpha) = \pm |\alpha|^a$$
$$\lambda_\pm(\alpha) = \frac{\bar{\lambda}}{(1 \pm |\alpha|^{a-1})^b}.$$

PROOF. The proof of this theorem procceds exactly as the one of Theorem 1 in [12] with A replaced by $\mathcal{D}^\pm$ and C replaced by $\mathcal{Y}^\pm = (\mathcal{Y}_1^\pm, \ldots, \mathcal{Y}_{2p}^\pm)$, where

$$\mathcal{Y}_k^\pm(\alpha, x) = \begin{cases} < (1 \pm |\alpha|^{a-1})(1 \pm |\alpha|^a)|\alpha|^{-a} M\left(\frac{\bar{\lambda}}{(1\pm|\alpha|^{a-1})^b}, \sum_{j=1}^{2p} |\alpha| x_j \phi^j + \right. \\ \left. + w_\pm(|\alpha|^{a-1}, |\alpha|^a, |\alpha| x)\right) - K(\bar{\lambda}, \sum_{j=1}^{2p} x_j \phi^j), \psi^k >, & \text{for } \alpha \neq 0 \\ 0 & \text{for } \alpha = 0. \end{cases}$$

PROOF. This completes the proof of the theorem.

The following corollaries are proved by the same ways as Corollaries 3.16, 4.3 respectively.

COROLLARY 4.11. *Let Hypotheses 1, 2 and 11 be satisfied and let c be an odd number. In addition, assume $q^\pm(t) \neq 0$ for all $t \in R^r, |t| = 1$. Then the conclusions of Theorem 4.10 continue to hold for $U^{*\pm} = \{\xi \in R^{2p} / |\xi| < 1\}$.*

COROLLARY 4.12. *Let Hypotheses 1, 2 and 10 be satisfied. Let $r = 1$ and c be an odd number. Then the conclusions of Theorem 4.7 continue to hold.*

To illustrate the above results we now consider some special cases of the equation (3.10). Let $T, L, K, \bar{\lambda}, \phi^1, \ldots, \phi^{2p}$ and $\psi^1, \ldots, \psi^{2p}$ be as above. Let $g : \Lambda \times X \times X \to R$ be a mapping such that for any fixed $\lambda \in \Lambda, g(\lambda, ., .)$ is a bilinear form. Let L_1 be a linear mapping from X into X. In addition, we assume

1) $< T(\phi^j), \psi^k >= c_k \delta_j k$,

2) $< g(\bar{\lambda}, \phi^j, \phi^k) L_1(\phi^m), \psi^n >= a_k b_n \delta_{jk} \delta_{mn}, a_m b_n \neq 0, j, k, m, n = 1, \ldots, 2p$, $g(\bar{\lambda}, \phi^m, \phi^m) = a_m > 0$ for all $m = 1, \ldots, 2p$.

We investigate Hopf bifurcation points of the equation

$$\frac{dv}{dt} + T(v) + L(\lambda, v) + g(\lambda, v, v) L_1(v) + K(\lambda, v) = 0, \ (\lambda, v) \in \Lambda \times \bar{D}.$$

We have:

THEOREM 4.13. *If the above conditions are satisfied, then* $(\bar{\lambda}, 0)$ *is a Hopf bifurcation point of periodic solutions of the equation (4.15). More precisely, the same conclusions of Theorem 4.7 continue to hold for* $a = 3$.

PROOF. In this case we have

$$\begin{aligned}\mathcal{A}_k(x) &= < g(\bar{\lambda}, \sum_{j=1}^{2p} x_j\phi^j, \sum_{j=1}^{2p} x_j\phi^j)L_1(\sum_{j=1}^{2p} x_j\phi^j), \psi^k > \\ &= \sum_{j,q,l=1}^{2p} x_j x_q x_s < g(\bar{\lambda}, \phi^j, q^q)L(\phi^s), \psi^k > \\ &= \sum_{j=1}^{2p} a_j b_k x_j^2 x_k,\end{aligned}$$

and hence

$$\mathcal{C}_k^{\pm}(x) = \pm c_k x_k + (-1)^{k-1} \sum_{j=1}^{2p} a_j b_k x_j^2 x_k.$$

We assume $\frac{-c_1}{a_1 b_1} > 0$. Consider the mapping $\mathcal{C}^+$ and take $\bar{x}^{\pm} = (\bar{x}_1^{\pm}, \ldots, \bar{x}_{2p}^{\pm})$ with $\bar{x}_1^{\pm} = \pm\sqrt{\frac{-c_1}{a_1 b_1}}, \bar{x}_2^{\pm} = \cdots = \bar{x}_{2p}^{\pm} = 0$. Then $\mathcal{C}^+(\bar{x}^{\pm}) = 0$ and

$$C = \left(\frac{\partial \mathcal{C}_k^+}{\partial x_j}(\bar{x}^{\pm})\right) = \begin{pmatrix} -2\sigma c_1, 0, \ldots, & 0 \ldots 00 \\ 0, \frac{c_2 b_1 - b_2 c_1}{b_1} & 0, \ldots 00 \\ 0, 0, & 0 \\ 0, & \frac{c_{2p} b_1 - b_{2p} c_1}{b_1} \end{pmatrix}$$

If $c_j b_1 \neq b_j c_1$ for all $j = 2, \ldots, 2p$, the matrix C is nonsingular. In this case, we apply Corollary 3.12 to complete the proof of the theorem.

Now, we suppose that $c_{j1} b_1 = b_{j1} c_1 = \cdots = b_{j_r} c_1$ for $r \in \{2, 3, \ldots, 2p\}$. Without loss of generality we may assume $j_l = 2p - r + l, \quad l = 1, \ldots, r$. Further, we can easily see that

$$R_0 = R_0^* = \{x \in R^{2p} / Cx = 0\} = [\xi^1, \ldots, \xi^r]$$

with

$$xi^1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \xi^r = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

(1 is in the $2p - r + j$–th row of ξ^j). We have

$$DC^+(\bar{x}^\pm)h = \left(-2c_1h_1, \frac{(c_2b_1 - c_1b_2)}{b_1}h_2, \ldots, \frac{c_{2p-r}b_1 - b_{2p-r}c_1}{b_1}h_{r-1}, 0, \ldots, 0\right)$$

$$D^2C^{+=}(\bar{x}^\pm)h = \left(-\left(6a_1b_1h_1 + \sum_{j=2}^{2p} 2a_jb_1h_j\right)\bar{x}_1^\pm, \sum_{j=2}^{2p} 2a_1b_j\bar{x}_1^\pm h_1h_j, \ldots, \sum_{j=1}^{2p} 2a_1b_j\bar{x}_1^\pm h_1h_j\right)$$

This implies $P_0DC^+(\bar{x}^\pm)h = P_0D^2C^+(\bar{x}^\pm)h = 0$ for all $h \in R_0$. By a simple calculation we can verify

$$\begin{aligned} D^3C^+(\bar{x}^\pm)h =& (6a_1b_1h_1^3 + 2a_2b_1h_1h_2^2 + \cdots + 2a_{2p}b^2h^2h_{2p}^2, \\ & 2a_1b_2h_2h_1^2 + 6a_2b_2h_2^3 + 2a_3b_3h_2h_3^2 + \cdots + 2a_{2p}b_2h_2h_{2p}^2, \ldots, \\ & 2a_1b_{2p}h_{2p}h_1^2 + \cdots + 2a_{2p-1}b_{2p}h_{2p}h_{2p-1}^2 + 6a_{2p}b_{2p}h_{2p}^3). \end{aligned}$$

Hence,

$$\begin{aligned} h_k^+(t) &= < D^3C^+(\bar{x}^\pm)(\sum_{j=1}^r t_j\xi^j), \xi^k > \\ &= 2b_{2p-r+k}t_k(3a_{2p-r+k}t_k^2 + \sum_{j=1, j\neq k}^r a_{2p-r+j}t_j^2). \end{aligned}$$

We now claim that $h^+(t) \neq 0$ for all $t \in R^r, |t| = 1$. By contrary, assume that there exists $\bar{t} \in R^r, |\bar{t}| = 1$ and $h^+(\bar{t}) = 0$. This yields

$$2b_{2p-r+k}\bar{t}_k(3a_{2p-r+k}\bar{t}_k^2 + \sum_{j=2}^r a_{2p-r+j}\bar{t}_j^2 = 0,$$

for all $k = 1, \ldots, r$. Since $|\bar{t}| = 1$, we deduce that fore some $\bar{k} = 1, \ldots, r$,

$$(3a_{2p-r+\bar{k}}\bar{t}_{\bar{k}}^2 + \sum_{j \neq = \bar{k}}^{r} a_{2p-r+j}\bar{t}_j^2) = 0,$$

a contradiction. Consequently,

$$\deg(h^+, U, 0) \neq 0 \text{ for } U = \{t \in R^r / |t| < 1\}.$$

Applying Corollary 4.8, we obtain the proof of the theorem.

EXAMPLE 4.14. Let T, L and K be as in Example 3.18, and $g(u,v) = (u,v)$, $L_1 = u, \bar{\lambda} = 1$. We have

$$< g(\phi^j, \phi^k)L_1, (\phi^m), \psi^n > = < (\phi^j, \phi^k)\phi^m, \phi^n > \frac{6}{64}\delta_{jk}\delta_{mn}.$$

Hence $(1, 0)$ is a Hopf bifurcation point of the equation

$$\frac{du}{dt} + T(u) + \lambda L(u) + \lambda g(u,u)u + K(\lambda, u) = 0, (\lambda, u) \in R \times R^r.$$

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