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MINIMIZING THE PRODUCT OF TWO DISCRETE CONVEX FUNCTIONS

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Abstract. A method for solving a discrete convex multiplicative programming (referred to as DCMP) is proposed. It is shown that DCMP can be solved by parametric discrete convex minimization. The method is specialized for the problem of minimizing the product of two discrete separable convex functions over a general supermatroid. The results of computational experiments for the last method are reported. An approximation algorithm is also proposed for solving the problem in a more general case.

1. Introduction

In recent years discrete nonlinear optimization problems have attracted the attention of many authors [1–8]. Minimizing a discrete nonconvex function is a very difficult problem. Even for the simplest case, when the function to be minimized is supermodular over a lattice, the problem is shown to be NP–hard [2]. So it is of particular interest to find special classes among those problems for which efficient solution methods may be developed. The present paper concerns the discrete minimization of the product of two convex positive functions with separable variables. The product of two convex separable functions might not be convex. Such a function may have multiple local minimizers. In [1] and, recently, in [3] the continuous case of this problems has been investigated. For its solution the authors proposed efficient algorithms, based on the use of linear parametric programming technique. Using the same approach in [5] a polynomial algorithm has been developed for the problem of finding a spanning tree with minimal product on a two-weighted graph.

In this paper we propose an efficient algorithm for finding the global minimum of the product of two convex positive separable functions over a general supermatroid. The paper consists of 5 sections. After the introduction, in Section 2 we show that the problem of minimizing the product of two positive functions

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over an arbitrary discrete set can be solved by parametric programming technique. Section 3 is devoted to the description of an algorithm for solving discrete programming problems on a general supermatroid. An efficient algorithm for solving the problem of minimizing the product of two discrete separable convex functions over general supermatroids is described in Section 4 together with the computational experiments made on test problems. In the last section, Section 5, an approximation method is developed for solving the problem of minimizing the product of two discrete-convex functions over a discrete set.

2. Minimizing the product of two positive functions over an arbitrary discrete set

Let us consider the following discrete optimization problem:

$$\min\{f(x).g(x) \ : \ x \in \mathrm{D}\}, \tag{1}$$

where $f(x), g(x)$ are positive and finite over a finite discrete set D. To solve the problem (1), consider the next parametric problem $P(t)$:

$$\varphi(t) = \min\{f(x) + t\, g(x) \ : \ x \in \mathrm{D}\}, \tag{2}$$

where t is a nonnegative real parameter.

First, we establish some interesting and important properties of the objective curve $\varphi(t)$.

LEMMA 1. *The function $\varphi(t)$ is nondecreasing, continuous, concave and piecewise linear on $[0, +\infty)$ with finite number N of breaking points*

$$0 < t_1 < t_2 < \cdots < t_N < +\infty.$$

PROOF. Because of the finiteness of the set D, the function $\varphi(t)$ is piecewise linear. Furthermore, for arbitrary $0 \leq \alpha \leq 1, t' > 0, t'' > 0$, from the finiteness of the functions $f(x), g(x)$ it follows that

$$\varphi(\alpha t' + (1-\alpha)t'') = \min\{f(x) + (\alpha t' + (1-\alpha)t'')g(x) \ : \ x \in \mathrm{D}\}$$
$$= \min\{\alpha(f(x) + t'\, g(x)) + (1-\alpha)(f(x) + t''\, g(x)) : x \in \ \mathrm{D}\ \}$$
$$\geq \min\{\alpha(f(x) + t'\, g(x)) \ : \ x \in \mathrm{D}\}$$
$$+ \min\{(1-\alpha)(f(x) + t''\, g(x)) \ : \ x \in \ \mathrm{D}\ \} = \alpha\varphi(t') + (1-\alpha)\varphi(t'')$$

This inequality shows that $\varphi(t)$ is concave.

Since $g(x)$ is finite, it is obvious that $\varphi(t)$ is continuous at the point $t = 0$. The concavity of $\varphi(t)$then ensures its continuity at other points of D.

The fact that $\varphi(t)$ is nondecreasing follows from the positivity of $g(x)$ over D. The lemma is then proved.

LEMMA 2. *If x^k is an optimal solution of problem $P(t)$ for $t \in [t_{k-1}, t_k]$, where $1 \le k \le N+1, t_0 = 0, t_{N+1} = +\infty$, then $g(x^k) \ge g(x^{k+1}), f(x^k) \le f(x^{k+1})$.*

PROOF. From the definition of t_k, x^k and $1 \le k \le N+1$ it follows that $f(x^k) + t\,g(x^k) = f(x^k)+t_k g(x^k)+(t-t_k)g(x^k) \ge f(x^{k+1})+t_k g(x^{k+1})+(t-t_k)g(x^{k+1})$, for $t_k \le t \le t_{k+1}$ Since both x^k and x^{k+1} are optimal solutions of problem $P(t_k)$, we have

$$f(x^k) + t_k\, g(x^k) = f(x^{k+1}) + t_k\, g(x^{k+1}).$$

It means that the first inequality of the lemma holds. Together with the last equality it immediately implies the second inequality in the statement of the lemma.

Denote the set of all optimal solutions of problem $P(t)$ by $\pi(t)$.

LEMMA 3. *There exists a point $x^N \in \pi(t_N)$, which is an optimal solution for the following problem*

$$\min\{g(x) \ :\ x \in D\} \tag{3}$$

PROOF. Let x' be an optimal solution of problem (3). Assuming the contrary, we have

$$g(x') < g(x), \quad \text{for every } \ x \in \pi(t_N). \tag{4}$$

Let $x^N \in \pi(t)$ for any fixed t such that $t > t_N$. Hence $x^N \in \pi(t_N)$ and $x^N \in \pi(t)$ for every $t > t_N$. Together with (4) this implies $g(x') < g(x^N)$. Furthermore, since $f(x)$ is finite, there exists a value $t' > t_N$ such that $f(x') + t'\,g(x') < f(x^N)+t'\,g(x^N)$. This contradicts the assumption that x^N is an optimal solution to problem $P(t')$.

The following theorem shows the relationship between problem (1) and $P(t)$.

THEOREM 1. *Let x^i be an optimal solution of the problem (2) for every $t \in [t_{i-1}, t_i], i = 1, \dots, N+1$ ($t_0 = 0; t_{N+1} = +\infty$). Then the optimal solution x^* of the problem (1) can be determined as :*

$$x^* = \ \arg\min\{f(x^i) \cdot g(x^i) \ :\ i = 1, \dots, N+1\}.$$

PROOF. Since $f(x), g(x)$ are positive for every $x \in$ D, we have

$$\min\left(\frac{[f(x)+tg(x)]}{\sqrt{t}}\right) = 2[f(x)\cdot g(x)]^{\frac{1}{2}}$$

Hence, we have

$$\begin{aligned}
\min_{x\in \mathrm{D}} \{f(x)\cdot g(x)\} &= \frac{1}{4}\min_{x\in \mathrm{D}}\{\min_{t\geq 0}\{\frac{[f(x)+tg(x)]}{\sqrt{t}}\}\}^2 \\
&= \frac{1}{4}\{\min_{t\geq 0}\{\frac{\min_{x\in \mathrm{D}}[f(x)+tg(x)]}{\sqrt{t}}\}\}^2 \\
&= \frac{1}{4}\{\min_{1\leq i\leq N+1}\min_{t_{i-1}\leq t\leq t_i}\{\frac{[f(x^i)+t\,g(x^i)]}{\sqrt{t}}\}\}^2 \\
&\geq \min_{1\leq i\leq N+1}\{f(x^i)\cdot g(x^i)\} = f(x^*)\cdot g(x^*).
\end{aligned}$$

The theorem is proved.

REMARK 1. In the case when D is a polytope, $f(x)$, $g(x)$ are linear, the result was given in [1]; in other cases, when D is the set of integer points of a convex integer polytope, $f(x), g(x)$ are linear; D is a polytope and $f(x), g(x)$ are convex functions, the similar results was given in [4,5] and, later, in [2].

Theorem 1 shows that the problem of minimizing the product of two positive finite functions over an arbitrary bounded discrete set can be solved by solving the parametric problem (2) (namely by finding all its optimal solutions for every nonnegative parameter t).

Thus, to solve the problem (1) we have to study the parametric problem (2). It is clear that if the functions $f(x), g(x)$ are linear and D is an integer polytope, problem (2) can be solved by ordinary parametric linear programming. But, when the functions $f(x)$ and $g(x)$ are nonlinear, the problem (2) is rarely studied. To our knowledge, in recent years, only the following simple problem has been studied in [9]:

$$\begin{aligned}
\min\{\sum_{j=1}^{n}(f_j(x_j)+t\,g_j(x_j))\} : \sum_{j=1}^{n} x_j = M,& \\
u_j \leq x_j \leq v_j \;;\; x_j \text{ are integers}; j = 1,\dots,n\},&
\end{aligned}$$

where $f_j(x_j), g_j(x_j)$ are convex functions.

In [7] we have proposed an algorithm for solving problem (2) with D being a general supermatroid and $f(x)$, $g(x)$ being separable convex. This algorithm may be used for solving problem (1) after some modifications based on Theorem 1.

3. Description of the algorithm for solving the convex parametric discrete problem over a general supermatroid

Let us recall the definition of a general supermatroid [8]. A general supermatroid is a finite subset D of Z^n with the following property: For arbitrary two points $x, y \in$ D there exists a sequence of points $x = x^1, x^2, \dots, x^{r(x,y)+1} = y$, such that $x^i \in$ D$, r(x^i, x^{i+1}) = 1, x + (x^{i+1} - x^i) \in O(x, 1) \cap$ D$, i = 1, \dots, r(x, y)$; where $O(x, 1)$ denotes the unit ball in Z^n with the metric

$$r(x,y) = \frac{1}{2}\{\sum_{j=1}^{n} |x_j - y_j| + |\sum_{j=1}^{n}(x_j - y_j)|\}$$

One of the typical examples of such sets is the general supermatroid of the tree type which is defined as follows

$$\mathrm{D}[a] = \{x \in Z^n \ : \ c_i \leq \sum_{j=1}^{n} a_{ij}\, x_j \leq b_i, \quad i = 1, \dots, m\},$$

where $c,\ b \in Z^m$, (a_{ij}) is a (m, n)–matrix of the tree type, i.e., a binary matrix such that the sets of indices of nonzero elements of any pair of its rows have the following property: they have an empty intersection or one of them contains the other.

Now, we determine the set of feasible directions from a point x of D:

$$\mathrm{fes}(x, D) = \{s \ : \ x + s \in \mathrm{D}, \quad s \in \{e_i, -e_i, e_i - e_j\}, \quad i, j = 1, \dots, n,\ i \neq j\},$$

where $e_i(e_j)$ is the $i(j)$–th identity vector. Denote

$$\Delta^s f(x) = f(x+s) - f(x);$$
$$\Delta^s g(x) = g(x+s) - g(x);$$
$$F(x,t) = f(x) + tg(x);$$
$$\mathrm{fes}_-(x, \mathrm{D}) = \{s \ : \ s \in \mathrm{fes}(x, \mathrm{D}), \Delta^s g(x) < 0\};$$
$$\mathrm{fes}_+(x, \mathrm{D}) = \{s \ : \ s \in \mathrm{fes}(x, \mathrm{D}), \Delta^s g(x) > 0\};$$
$$t(s,x) = -\frac{\Delta^s f(x)}{\Delta^s g(x)};$$

Let D be a general supermatroid, $f(x), g(x)$ be convex separable functions. Then the point x^t will be an optimal solution to problem $P(t)$ if and only if the following optimality criterion is satisfied [8]

$$\Delta^s F(x^t,t) \geq 0 \text{ for every } s \in \text{fes}\,(x^t, \text{D})\,. \tag{5}$$

For an optimal solution x^t of problem $P(t)$ let denote

$$t_+(x^t) = t(s^+, x) = \min\{t(s,x^t) \ : \ t(s,x^t) \geq t, s \in \text{fes}_-(x^t, \text{D})\}\,;$$
$$t_-(x^t) = t(s^-, x) = \max\{t(s,x^t) \ : \ t(s,x^t) \leq t, s \in \text{fes}_+(x^t, \text{D})\}.$$

We adopt the convention that $\max\{\emptyset\} = 0$ and $\min\{\emptyset\} = +\infty$.

THEOREM 2. *Let D be a general supermatroid, $f(x)$ and $g(x)$ be separable convex functions. Then*

a) x^t is an optimal solution of $P(t)$ for every $t \in [t_-(x^t),\ t_+(x^t)]$;

b) If $t_-(x^t) > 0$ then $x^t_- = x^t + s^-$ is an optimal solution of $\mathbf{P}(t_-(x^t))$;

c) If $t_+(x^t) < +\infty$ then $x^t_+ = x^t + s^+$ is an optimal solution of $\mathbf{P}(t_+(x^t))$.

PROOF. a) Since x^t is an optimal solution of problem $\text{P}(t)$, from the optimality criterion (5) we get

$$\Delta^s F(x^t,t) \geq 0,\ \forall s \in \text{fes}(x^t, \text{D}). \tag{6}$$

Together with the definition of $t_+(x^t)$ this yields $\Delta^s F(x^t,t') \geq 0,\ s \in \text{fes}_-(x^t, \text{D}), t' \in [t, t_+(x^t)]$. Combining $\Delta^s g(x^t) \geq 0,\ \forall s \in \text{fes}(x^t,\text{D}) \setminus \text{fes}_-(x^t,\text{D})$ with (6) we obtain

$$\Delta^s F(x^t,t') = \Delta^s F(x^t,t) + (t'-t)\cdot \Delta^s g(x^t) \geq 0,$$

for arbitrary $t' \in [t, t_+(x^t)],\ s \in \text{fes}(x^t,\text{D})$. Therefore, the optimality criterion (5) is satisfied for the solution x^t and the parameter t'. This means that x^t is an optimal solution of the problem $\text{P}(t')$ for every $t' \in [t, t_+(x^t)]$.

Further, from the definition of $t_-(x^t)$ and (6) it follows that

$$\Delta^s F(x^t,t') \geq 0,\ \forall s \in \text{fes}_+(x^t,\text{D}),\ t' \in [t_-(x^t), t]$$

and

$$\Delta^s g(x^t) < 0 \qquad \text{for } s \in \text{fes}(x^t,\text{D}) \setminus \text{fes}_+(x^t,\text{D}).$$

Taking (6) into account we obtain

$$\Delta^s F(x^t, t') = \Delta^s F(x^t, t) + (t' - t) \cdot \Delta^s g(x^t) \geq 0,$$

for every $t' \in [t_-(x^t), t]$, $\forall s \in \text{fes}(x^t, \text{D})$. This means that the optimality criterion (5) is satisfied for the solution x^t and the parameter t'.

As a consequence, x^t is an optimal solution of problem $P(t')$ for every $t' \in [t_-(x^t), t]$.

b) If $t_-(x^t) > 0$, then from the definition of $t(s^-, x^t)$ we get

$$t(s^-, x^t) = \frac{-\Delta^{s^-} f(x^t)}{\Delta^{s^-} g(x^t)}$$

or

$$\Delta^{s^-} f(x^t) + t_-(x^t) \cdot \Delta^{s^-} g(x^t) = 0.$$

This yields $F(x^t_-, t_-(x^t)) = F(x^t, t_-(x^t))$. However, according to part a), x^t is an optimal solution to problem $P(t_-(x^t))$. Therefore, x^t_- is also its optimal solution.

c) may be proved by similar arguments.

Since $g(x^t)$ is a t–gradient of function $F(x^t, t)$ at t we shall also refer to this as a gradient of the optimal solution x^t. It is obvious that if t is not a breaking point of the function $\varphi(t)$, then all the optimal solutions of problem P(t) should have the same gradient which is equal to $g(x^t)$. The definition of $x^t_+, x^t_-, t_+(x^t), t_-(x^t)$ yields that the gradient of the solution x^t_+ (x^t_-) is smaller (greater) than the gradient of the solution x^t. Therefore the points $t_+(x^t), t_-(x^t)$ are breaking points of the function $\varphi(t)$. This implies the following corollary.

COROLLARY 1. *The points $t_+(x^t), t_-(x^t)$ given by Theorem 2 are breaking points of the function $\varphi(t)$.*

Combining Theorem 2 and Corollary 1 we obtain the following algorithm for finding out all breaking points $t_1, \ldots, t_N$ of the function $\varphi(t)$ and, consequently, all optimal solutions $x^1, \ldots, x^{N+1}$ of problem P(t).

ALGORITHM 1.

Initialization. After solving P(0) we obtain the optimal solution x^1. Set $k = 1, p = 1, t_0 = 0$.

k–iteration:

1. If $\Delta^s g(x^k) \geq 0,\ s \in \text{fes}(x^k, \text{D})$, all breaking points $t_1, \dots, t_{p-1}$ of function $\varphi(t)$ and, respectively, the optimal solutions $x^1, \dots, x^p$ are found. Stop. Else calculate $t_+(x^k) = t(s^+, x^k) = \min\{t(s, x^k)\colon\ t(s, x^k) \geq t_{k-1},\ s \in \text{fes}_-(x^k, \text{D})\}$ and set $t_k := t_+(x^k), x^{k+1} := x^k + s^+$.
2. If $t_k < +\infty$ and $t_k \neq t_{k-1}$, set $x^{p+1} := x^{k+1}, p := p + 1$.
3. If $t_k < +\infty$, then go to $(k+1)$–iteration else Stop.

THEOREM 3. *Algorithm 1 stops after a finite number of iterations, yielding all breaking points $t_1, \dots, t_{p-1}$ of the objective curve $\varphi(t)$ and, as a consequence, all the optimal solutions $x^1, \dots, x^p$ of the problem $P(t)$.*

PROOF. Theorem 2 and Corollary 1 imply that $t_1, \dots, t_{p-1}$ are breaking points for the function $\varphi(t)$ and $x^1, \dots, x^p$ are optimal solutions of problem $\text{P}(t)$. We need only to show the finiteness of the proposed algorithm. Indeed, assuming the contrary yields the existence of a number k_0 such that for every $k \geq k_0, t_k = t_{k-1}$. This produces an infinite sequence of optimal solutions $\{x^k\}$ with different gradients (from the definitions of x^{k+1} and t_k it's easily seen that the gradient of x^{k+1} must be smaller than that of x^k for every $k > k_0$). The uniqueness of the gradient of x^k then follows from the infiniteness of D, which is a contradiction. The proof of the theorem is complete.

We note that the Algorithm 1 requires solving the problem of minimizing a convex function over a general supermatroid only once (problem $\text{P}(0)$). The last problem can be solved efficiently by the greedy algorithm proposed in [8].

4. The Algorithm for solving the problem of minimizing the product of two positive separable convex functions over a general supermatroid. Computational experiments

Let us consider problem (1) in the case when D is a general supermatroid and $f(x), g(x)$ are positive separable convex functions over D. A special case of this problem when $f(x)$ and $g(x)$ are linear was investigated in [5]. In this work a polynomial algorithm with running time $\mathbf{O}(m^3 \log(\log n))$ has been proposed for solving the problem of finding a spanning tree with a minimal product on the two-weighted graph having n vertices and m arcs.

Combining Theorem 1 and Algorithm 1 we obtain the following algorithm for solving the considered problem.

ALGORITHM 2.

Initialization. Solve **P**(0). Denote by x^1 the obtained optimal solution. Set $k = 1, t_0 = 0, x^* = x^1$.

k–iteration:

1. If $\Delta^s g(x^k) \geq 0, s \in \text{fes}(x^k, \mathrm{D})$, the solution x^* is an optimal solution of the problem (1), Stop. Else calculate

$$t_+(x^k) = t(s^+, x^k) = \min\{t(s, x^k) : \ t(s, x^k) \geq t_{k-1}, s \in \text{fes}_-(x^k, \mathrm{D})\},$$

   and set$t_k := t_+(x^k), x^{k+1} := x^k + s^+$
2. If $t_k < +\infty$ and $t_k \neq t_{k-1}$, then set

$$x^* := \arg\min\{f(x^*).g(x^*), f(x^{k+1}) \cdot g(x^{k+1})\}.$$

3. If $t_k < +\infty$, then go to $(k+1)$–iteration else Stop.

Using Theorem 1 and Theorem 2 it is not difficult to prove the following theorem.

THEOREM 3. *The solution x^* given by the above described algorithm is an optimal solution of the problem (1).*

The implementation of the described algorithm requires the use of an algorithm available for solving the minimization problem of separable convex function over a general supermatroids. The last problem can be efficiently solved by the greedy algorithm proposed in [8]. Note also that we have to solve this problem (problem P(0)) only once. The above algorithm was coded in PASCAL and ran on a IBM PC AT 80286. It has been successfully used to solve the following problem

Table 1. Computational results

n	10	10	10	15	15	15	20	20	20	25	25	25
m	5	10	15	10	15	20	10	15	20	10	25	30
(*)	23	27	17	86	96	78	274	289	386	962	714	549
(**)	41	23	13	102	86	59	240	290	266	673	337	280
(***)	17	13	10	31	27	24	43	54	42	68	62	63

(*) Average time used in *Stage 1* (solving P(0)) (in sec.)
(**) Average time used in *Stage 2* (solving the parametric Problem) (in sec.)
(***) Average number of the breaking points

$$f(x).g(x) \to \min\,,$$
$$b_i \le \sum_{j=1}^{n} a_{ij}x_j \le a_i, \quad i = 1, \dots, m\,,$$
$$x_j \ge 0, \;\text{ integer}, \; j = 1, \dots, n\,,$$

where $f(x)$ and $g(x)$ are separable convex quadratic nonnegative functions over the feasible set, $A = \{a_{ij}\}$ is a matrix of the tree type; $a_i, b_i \in Z$, $i = 1, \dots, m$.

The elements of the matrix A and vectors a, b, are randomly generated in the interval $[0, 300]$. For each size, ten problems were tested. To solve $\mathbf{P}(0)$ we have used the greedy algorithm from [8].

The computational results show that the average computational time for solving the parametric problem is often not greater than that one required for problem P(0). Moreover, the ratio between them is decreasing with the increase of the number of its variables. We also note that the number of breaking points of the problem under consideration approximately equals twice the number of its variables.

5. Minimizing the product of two positive discrete convex functions

Let us consider problem (1) in the case when D is a set of integer points of a bounded convex set in $\mathbf{R}^n$ and $f(x), g(x)$ are positive discrete-convex functions.

First, we recall that a function $f(x)$ is called a discrete-convex function on Z^n if all its second order gradients

$$\begin{aligned}\Delta_i\Delta_j f(x) &= \Delta_i(\Delta_j f(x)) = \Delta_j(\Delta_i f(x))\\ &= f(x + e_i + e_j) - f(x + e_i) - f(x + e_j) + f(x)\end{aligned}$$

are nonnegative.

An important example for the discrete-convex function is the restriction to Z^n of a separable convex function on $\mathbf{R}^n$.

Some main properties of discrete–convex functions are established in the following theorem

THEOREM 4. [7] *Given a function $f(x)$ defined over Z^n, the following statements are equivalent:*

1) $f(x)$ is discrete-convex on Z^n_+ ;

2) $\Delta_i f(x)$ *is nondecreasing on* Z_+^n *;*

3) $f(y) - f(x) \geq \sum_{i \in N'} (x_i - y_i) \cdot \Delta_i f(x) + \sum_{i \in N''} (x_i - y_i) \cdot \Delta_i f(x \vee y - e_i)$, *where* $N' = \{i \, : \, x_i > y_i\}$, $N'' = \{i \, : \, y_i > x_i\}$ *;*

4) If $x \geq y$ *, then* $f(y) - f(x) \geq \sum_{i=1}^n (y_i - x_i) \Delta_i f(x_i)$.

Property 2) is similar to the monotone property of the derivative of a convex differentiable function on $\mathbf{R}^n$. It should be used for constructing approximate solutions as well as for estimating its quality in the discrete-convex programming by a similar way as it has been done in the convex programming.

Now let us introduce an order on D as follows

$$x \leq y \quad \text{if and only if} \quad x_i \leq y_i, \; i = 1, \ldots, n.$$

We call a neighbourhood of a point $x^o \in \mathrm{D}$ (and denote by $O(x^0)$) the set:

$$O(x^0) = \{x \; : \; x \in \mathrm{D}, x \leq x^0 \;\; > \text{or} \; > x \geq x^0\}.$$

From theorem 4 (statements 2. and 3.) we immediately obtain the following

COROLLARY 2. *A point* $x^0 \in D$ *is a minimizer of a discrete-convex function in* $O(x^0)$ *if and only if it is its minimizer in* $O'(x^0, 1)$ *equipped with the metric*

$$r'(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

A point $x^0 \in \mathrm{D}$ is called a local minimum of $f(x)$ over D if $f(x^0) \leq f(x)$, for every $x \in O(x^0, 1) \cap \mathrm{D}$. It is obvious that $O'(x^0, 1) \subset O(x^0, 1)$. Hence, from Corollary 2 it follows that a local minimum may be considered as a good approximate solution.

It is not difficult to show that a necessary and sufficient condition for a local minimum $x^0 \in \mathrm{D}$ of $f(x)$ is

$$\Delta^s F(x^0) \geq 0, \quad \forall s \in \mathrm{fes}(x^0, \mathrm{D}).$$

By an argument analogous to that used for the proof of Theorem 2 we obtain the following theorem

THEOREM 5. *Let* x^t *be a local minimum of problem* $P(t)$ *(for fixed* $t \geq 0$*). Then* x^t *is a local minimum of problem* $\mathrm{P}(t')$ *for every* $t' \in [t_-(x^t), t_+(x^t)]$.

Set

$$S = \{w \; : \; w = -\frac{\Delta^s f(x)}{\Delta^s g(x)}, \; x \in \mathrm{D}, \; s \in \mathrm{fes}(x, \mathrm{D})\};$$

and

$$0 < w_0 < \min\{|w' - w''| \ : \ w' \neq w'', \ w', w'' \in S\}.$$

Using Theorems 1 and 5 we develop an approximation algorithm for solving problem (1) in the mentioned above case.

ALGORITHM 3.

Initialization. Choose any approximation solution x^0 ∈D, set $k = 1$, $> t^0 = 0, x^* = x^0$.

k-iteration:

1. Solve P(t) with $t = t^{k+1} + w_0$ by the algorithm of [8] with starting solution $x^0(t^{k+1}) = x^{k+1}$, we obtain a local minimum x^k. Calculate $t^k = t_+(x^k)$ (x^k is a local minimum P(t) for every $t \in [t^{k-1}, t^k]$), then set $x^* = \arg\min\{f(x^*)\cdot g(x^*), f(x^k).g(x^k)\}$.
2. If $t(x^k) = +\infty$,
then algorithm terminates and gives x^* as an approximation solution; else go to $(k+1)$–iteration.

The finiteness of Algorithm 3 is implied from the finiteness of the feasible set D and the fact that $|\text{fes}(x, \text{D})| \leq n^2 + n$, for every $x \in$ D. In addition, the definition of $t_+(x^k)$ and w_0 shows that the number of iterations required by Algorithm 3 is bounded by $(|S| + 1)$.

It is obvious that when f and g are both linear functions then $|S| \leq \frac{n(n+1)}{2}$. We then obtain

COROLLARY 3. *The number of iterations required by Algorithm 3 in the case when f, g are linear never exceeds $(n+1)(n+2)/2$.*

We note that an obvious shortcoming of Algorithm 3 is that it doesn't allow us to estimate the quality of the obtained approximate solutions. To overcome this shortcoming we have developed an ε-approximation algorithm for solving combinatorial multiplicative programming problems. It will be reported in a subsequent paper.

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