ACTA MATHEMATICA VIETNAMICA
Volume 20, Number 2, 1995, pp. 257-264

ON BING'S QUESTION ABOUT FIXED POINT PROPERTY

LE HOANG TRI

Abstract. Bing [1] constructed a compactum X in $\mathbb{R}^3$ which has the fixed point property but $X \cup D$ does not, where D is a rectangle and $X \cap D$ is an interval. He also asked whether $X \times [0,1]$ has the fixed point property. In [4] Young gave a positive answer to this question. The aim of this note is to extend Young's result to the product $X \times A$ where A is a compact AR-space. The result does not hold if A is a compact fixed point space.

1. Introduction

We say that a space X has the fixed point property (or a fixed point space) if each continuous map $f : X \longrightarrow X$ has a fixed point.

It is well-known that there exists a compactum X which the fixed point property but $X \times I$ does not [3].

In [1] Bing constructed a compactum X in $\mathbb{R}^3$ which has the fixed point property but $X \cup D$ does not, where D is a rectangle and $X \cap D$ is an interval. He asked whether $X \times I$ has the fixed point property (see [1], question 5). Young gave a positive answer to this question [4]. In this note we prove the following theorem extending Young's result.

THEOREM MAIN. *Let X be the compactum constructed by Bing. Then $X \times A$ has the fixed point property for any compact AR-space A.*

Moreover we shall show that the theorem does not hold if A is a fixed point space.

Bing's compactum X is the union of three segments $[p_1, p_4]$, $[p_4, p_5]$, $[p_5, p_9]$, two curves G_1, G_2 and a spiral S (see Figure 1). They are described precisely as follows.

The points $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9$ have the coordinates $(-2,3), (-2,1), (-2,-2-\sin\frac{1}{4}), (-2,-4), (2,-4), (2,-3), (2,-1), (2, 2+\sin\frac{1}{4}), (2,3)$ respectively, (Figure 1).

Received May 23, 1994; in revised form May 23, 1995.

The curve G_1 is given by the equation

$$y = 2 + \sin\frac{1}{x+2} \quad (-2 < x \leq 2)$$

while G_2 is the reflection of G_1 through the origin (0,0).

To describe S we select a sequence $\{q_n\}$ of points in G_1 such that the length of the arc $[q_i, p_8] \subset G_1$ is $2+\sum_{j=1}^{i}\frac{1}{j}$. Since it is a harmonic sequence, the points $q_1, q_2, \ldots$ approach the segment $[p_1, p_2]$. On the vertical line passing through q_i, let a_i and b_i be the points $\frac{1}{i}$ below G_1 and $\frac{1}{i+1}$ above G_2, respectively. Let d_i be the reflection of b_i through the origin and c_i be the point of the vertical line passing through d_i which is $\frac{1}{i+1}$ units above G_2. The spiral S starts from p_9, runs to a_1 along a semi-circle in a plane normal to the plane containing G_1, follows a straight path to b_1, goes from b_1 to c_1 along an arc parallel to G_2, then straight to d_1, then to a_2 along an arc parallel to G_1, ... In general, the vertical segments $[a_i, b_i]$ and $[c_i, d_i]$ are in S as arcs from b_i to c_i parallel to G_2 and arcs from d_i to a_{i+1} parallel to G_1.

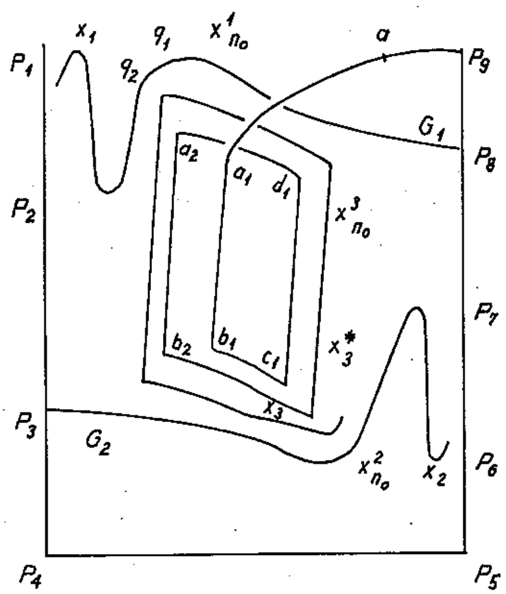

Fig. 1

2. Proof of Main Theorem

The main tool for the proof is the following theorem of Borsuk [2].

THEOREM 2.1. *Every compact AR-space has the fixed point property.*

We shall need the following Lemma.

LEMMA 2.2. *Let K be a compactum homeomorphic to a compact, convex set of a Banach space, and g a continuous map from K into X. Then there exist points $x_1 \in G_1$, $x_2 \in G_2$, $x_3 \in S$ such that*

$$g(X) \subset [p_1,p_3] \cup [p_3,p_4] \cup [p_4,p_5] \cup [p_5,p_8] \cup [p_8,p_9] \cup [p_9,x_3) \cup [p_8,x_1) \cup [p_3,x_2).$$

PROOF. Without loss of generality, we may assume that K is a convex set. We will prove the existence of x_3. The existence of x_1,x_2 can be proved similarly.

Assume on the contrary that such a point x_3 does not exist, i.e. for each $x \in S$, there exists $k \in K$ such that g(k) belongs to the "twist" side with respect to x. Then by the path connectedness of g(K) there is $n_0 \in N$ such that $b_m \in g(K)$ for every $m \geq n_0$. Take a sequence $\{k_m\} \subset K$ such that $g(k_m) = b_m$ for each $m \geq n_0$. By the compactness of K we may assume that $k_m \longrightarrow k^*$. Select $k'_m \in [k_m, k_{m+1}]$ such that $g(k'_m) = d_m$. Note that $k_m \longrightarrow k^*$ and $k'_m \longrightarrow k^*$ while $\{g(k_m)\} = \{b_m\} \longrightarrow p_3$ and $\{g(k'_m)\} = \{d_m\} \longrightarrow p_8$. This contradicts the continuity of g, hence the lemma is proved.

PROPOSITION 2.3. *Let*

$$X_1 = [p_3,p_4] \cup [p_4,p_5] \cup [p_5,p_6] \cup [p_6,p_7] \cup G_2.$$

Then for every $N \in \mathbb{N} \cup \{\infty\}, X_1 \times I^N$ has the fixed point property.

PROOF. Assume on the contrary that there is a continuous map

$$f : X_1 \times I^N \longrightarrow X_1 \times I^N$$

which has no fixed point. Then by Borsuk's theorem we have

$$f([p_6,p_7] \times I^N) \not\subset [p_6,p_7] \times I^N.$$

Take $(x_0,t_0) \in [p_6,p_7] \times I^N$ such that $f(x_0,t_0) \notin [p_6,p_7] \times I^N$. Select a sequence $\{x_n\} \subset G_2$ such that each x_n has the same ordinate as x_0 and $x_n \longrightarrow x_0$ (see Fig. 2). We will prove that there exists n_0 such that

$$(*) \qquad \pi_1 \circ f(\{x_{n_0}\} \times I^N) \subset [p_7, x_{n_0}]$$

Here $\pi_1 : X \times I^N \longrightarrow X$ denotes the natural projection of $X \times I^N$ onto X and

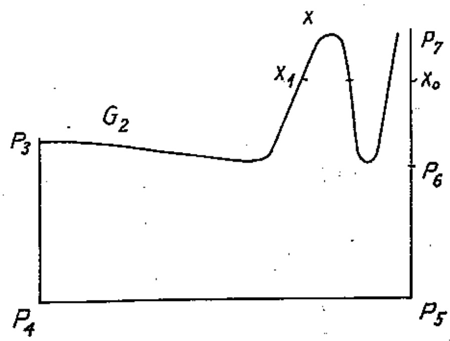

Fig. 2

$[p_7, x] = [p_7, p_5] \cup [p_5, p_4] \cup [p_4, p_3] \cup [p_3, x]$. If this is not the case, we have

$$(**) \qquad \pi_1 \circ f(\{x_n\} \times I^N) \not\subset [p_7, x_n]$$

for every $n \in \mathbf{N}$. By Lemma 2.2, there exists $x \in G_2$ such that

$$\pi_1 \circ f(\{x_0\} \times I^N) \subset [p_7, x).$$

Let $\epsilon = d(x, \pi_1 \circ f(\{x_0\} \times I^N))$, then $\epsilon > 0$ because $\{x_0\} \times I^N$ is compact. By the uniform continuity of f, there exists $n \in N$ such that

(1) $x_n \notin [p_7, x]$,

(2) $\pi_1 \circ f(x_n, t_0) \in [p_7, x)$,

(3) $d(\pi_1 \circ f(x_n, t), \pi_1 \circ f(x_0, t)) < \frac{\epsilon}{2}$ for every $t \in I^N$; By $(**), (1), (2)$ and by the path connectedness of $\pi_1 \circ f(\{x_n\} \times I^N)$ there exists $t_1 \in I^N$ such that $x = \pi_1 \circ f(x_n, t_1)$. By (3) we have $d(x, \pi_1 \circ f(x_0, t_1)) < \frac{\epsilon}{2}$, therefore

$$\epsilon = d(x, \pi_1 \circ f(\{x_0\} \times I^N) \leq d(x, \pi_1 \circ f(x_0, t_1)) < \frac{\epsilon}{2}.$$

This contradiction proves $(*)$.

We now define a map $\varphi : X_1 \times I^N \longrightarrow [p_7, x_{n_0}] \times I^N$ by the formula

$$\varphi(x,t) = \begin{cases} (x,t) & \text{if } x \in [p_7, x_{n_0}] \\ (x_{n_0}, t) & \text{if } x \notin [p_7, x_{n_0}] \end{cases}$$

Note that φ is not continuous, but the map

$$g = \varphi \circ f\Big|_{[p_7, x_{n_0}] \times I^N} : [p_7, x_{n_0}] \times I^N \to [p_7, x_{n_0}] \times I^N$$

is continuous by Lemma 2.2. Therefore, by Borsuk's theorem there exists $(x^1, t^1) \in [p_7, x_{n_0}] \times I^N$ such that $\varphi \circ f(x^1, t^1) = (x^1, t^1)$. We shall show that

(x^1, t^1) is a fixed point of f. This will contradict our assumption.

Consider two cases:

Case 1: $x^1 \in [p_7, x_{n_0})$. Then $\pi_1 \circ f(x^1, t^1) \in [p_7, x_{n_0})$. Therefore $\varphi \circ f(x^1, t^1) = f(x^1, t^1) = (x^1, t^1)$.

Case 2: $x^1 = x_{n_0}$. Then $\varphi \circ f(x_{n_0}, t^1) = (x_{n_0}, t^1)$. Thus, $\pi_1 \circ f(x_{n_0}, t^1) \notin [p_7, x_{n_0})$. By $(*)$ we have $\pi_1 \circ f(x_{n_0}, t^1) = x_{n_0}$. Therefore $\varphi \circ f(x_{n_0}, t^1) = f(x_{n_0}, t^1) = (x_{n_0}, t^1)$.

In this way Proposition 2.3 is proved.

PROPOSITION 2.4. *Let*

$$D = [p_1, p_3] \cup [p_3, p_4] \cup [p_4, p_5] \cup [p_5, p_6] \cup [p_6, p_7] \cup [p_7, p_8]$$

and $X_2 = D \cup G_1 \cup G_2$. *Then for every* $N \in \mathbb{N} \cup \{\infty\}$, $X_2 \times I^N$ *has the fixed point property.*

PROOF. Assume that there is a continuous map $f : X_2 \times I^N \longrightarrow X_2 \times I^N$ which has no fixed point. By Borsuk's theorem we have

$$f([p_1, p_2] \times I^N) \not\subset [p_1, p_2] \times I^N,$$

and

$$f([p_6, p_7] \times I^N) \not\subset [p_6, p_7] \times I^N.$$

Thus, there exist $(x_1, t_1) \in [p_1, p_2] \times I^N$ and $(x_2, t_2) \in [p_6, p_7] \times I^N$ such that $\pi_1 \circ f(x_1, t_1) \notin [p_1, p_2]$ and $\pi_1 \circ f(x_2, t_2) \notin [p_6, p_7]$. By Lemma 2.2 there exist $x_i^* \in G_i$, i = 1, 2, such that

$$\pi_1 \circ f([p_1, p_2] \times I^N) \subset D \cup [p_8, x_1^*) \cup G_2,$$

and that

$$\pi_1 \circ f([p_6, p_7] \times I^N) \subset D \cup [p_3, x_2^*) \cup G_1.$$

As in the proof of Proposition 2.3 there exists $x_{N_0}^1 \in G_1$ whose abscissa is smaller than the abscissa of x_1^* such that

$$\pi_1 \circ f(\{x_{n_0}^1\} \times I^N) \subset D \cup [p_8, x_{n_0}^1] \cup G_2,$$

and there exists $x_{n_0}^2 \in G$ having abscissa greater than the abscissa of x_2^* such that

$$\pi_1 \circ f(\{x_{n_0}^2\} \times I^N) \subset D \cup [p_3, x_{n_0}^2] \cup G_1$$

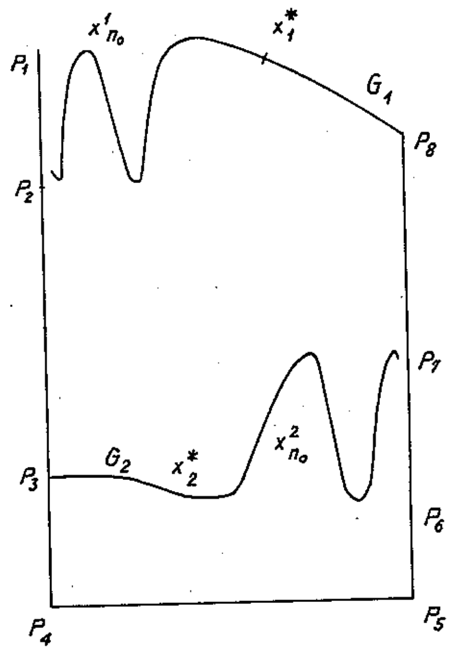

Fig. 3

(see Fig. 3)

Let $Q = D \cup [p_8, x^1_{n_0}] \cup [p_3, x^2_{n_0}]$ and define a map

$$\varphi : X_2 \times I^N \longrightarrow Q \times I^N$$

by the formula

$$\varphi(x,t) = \begin{cases} (x,t) & \text{if } x \in Q, \\ (x^1_{n_0}, t) & \text{if } x \in G_1 \setminus Q, \\ (x^2_{n_0}, t) & \text{if } x \in G_2 \setminus Q. \end{cases}$$

Note that φ is not continuous but $\varphi \circ f\big|_{Q\times I^N}$ is continuous by Lemma 2.2. Since $Q \times I^N$ is a compact AR-space, by Borsuk's theorem there exists $(x^1, t^1) \in Q \times I^N$ such that

$$\varphi \circ f(x^1, t^1) = (x^1, t^1).$$

As in the proof of Proposition 2.3 we can show that

$$f(x^1, t^1) = (x^1, t^1).$$

This completes the proof.

THEOREM 2.5. *$X \times I^N$ has the fixed point property for any $N \in \mathbb{N} \cup \{\infty\}$.*

PROOF. Set $D_1 = (p_3, p_4] \cup [p_4, p_5] \cup [p_5, p_6)$ and $X_3 = X_2 \setminus D_1$. First, we show that $f(X_3 \times I^N) \not\subset X_3 \times I^N$. In fact, assume on the contrary that $f(X_3 \times I^N) \subset X_3 \times I^N$. Then by Lemma 2.2, there exists $x \in S$ belonging to the twist side with respect to the set $\pi_1 \circ f(D_1^* \times I^N)$, where $D_1^* = D_1 \cup \{p_3\} \cup \{p_6\}$ (see Fig. 1). Define $\varphi : X \times I^N \longrightarrow X_2 \times I^N$ by the formula

$$\varphi(x,t) = \begin{cases} (x,t) & \text{if } x \in X_2, \\ (p_8, t) & \text{if } x \notin X_2. \end{cases}$$

Then the map $\varphi \circ f\Big|_{X_2 \times I^N} : X_2 \times I^N \to X_2 \times I^N$ is continuous. By Proposition 2.4 this map has a fixed point. As in the proof of Proposition 2.3, we can show that any fixed point of $\varphi \circ f\Big|_{X_2 \times I^N}$ is a fixed point of f. This contradicts our assumption. Hence $f(X_3 \times I^N) \not\subset X_3 \times I^N$.

Let $(x_0, t_0) \in X_3 \times I^N$ such that $\pi_1 \circ f(x_0, t_0) \notin X_3$. Then we have three cases:

a) There exists x_3^* belonging to the twist side with respect to the set $\pi_1 \circ f(\{x_0\} \times I^N)$ (by Lemma 2.2).

b) There exists $x_{n_0}^3 \in S$ belonging to the twist side with respect to x_3^* and

$$\pi_1 \circ f(\{x_{n_0}^3\} \times I^N) \subset X_2 \cup [p_8, x_{n_0}^3].$$

c) There exist $x_{n_0}^i \in G_i$, i = 1, 2, such that

$$\pi_1 \circ f(\{x_{n_0}^1\} \times I^N) \subset (X \setminus G_1) \cup [p_8, x_{n_0}^1],$$

$$\pi_1 \circ f(\{x_{n_0}^2\} \times I^N) \subset (X \setminus G_2) \cup [p_3, x_{n_0}^2].$$

Set

$$Q^* = D \cup [p_8, x_{n_0}^1] \cup [p_3, x_{n_0}^2] \cup [p_8, p_9] \cup [p_9, x_{n_0}^3]$$

and define $H : X \times I^N \longrightarrow Q^* \times I^N$ by the formula

$$H(x,t) = \begin{cases} (x,t) & \text{if } x \in Q^*, \\ (x_{n_0}^1, t) & \text{if } x \in G_1 \setminus Q^*, \\ (x_{n_0}^2, t) & \text{if } x \in G_2 \setminus Q^*, \\ (x_{n_0}^3, t) & \text{if } x \in S \setminus Q^* \end{cases}.$$

Note that $G = H \circ f\Big|_{Q^* \times I^N}$ is continuous. Since $Q^* \times I^N$ is compact, AR, G has a fixed point (Theorem 2.1). As before we can see that this point is a fixed point of f. Thus the theorem is proved.

It is well-known that every compact AR-space is homeomorphic to a retract of I^∞ and every retract of a fixed point space is a fixed point space. Therefore by Theorem 2.5 we obtain our main result.

THEOREM 2.6. *$X \times A$ has the fixed point property for any compact AR-space A.*

Our next result shows that in the above theorem "compact AR-space" cannot be replaced by "compact fixed point space".

PROPOSITION 2.7. *Let A be any compactum such that $[0,1] \times A$ does not have fixed point property. Then neither does $X \times A$, where X is the Bing set.*

PROOF. Since X retracts onto I, we infer that $X \times A$ retracts onto $I \times A$, hence the assertion follows from [1].

REMARK. Knill [3] has constructed a compactum A which has the fixed point property but $I \times A$ does not. Therefore, by Proposition 2.7 $X \times A$ does not have the fixed point property.

Acknowledgements. I would like to express my thanks to Pham Ngoc Thao and Nguyen To Nhu for many helpful discussions during the preparation of this note.

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HANOI INSTITUTE OF MATHEMATICS
P. O. BOX 631, BOHO, 10000 HANOI VIETNAM