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FLOWS METHOD IN GLOBAL ANALYSIS

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Abstract. We study the gradient flows method for $W^{r,p}(\mathcal{M},\mathcal{N})$, where $\mathcal{M}$ and $\mathcal{N}$ are Riemannian manifolds and rp may be less than the dimension of $\mathcal{M}$.

Dedicated to Professor Dang Dinh Ang on the occasion of his 70th birthday

Introduction

Let $(\mathcal{M},g)$ be a compact, connected and orientable Riemannian manifold of class C^∞ and of dimension $m \geq 1$, possibly with boundary $\partial\mathcal{M}$. $\mathcal{N}$ shall denote a complete Riemannian manifold of class C^∞ and of dimension n_1, without boundary. We assume that $\mathcal{N}$ is isometrically imbedded into $\mathbb{R}^n$. Denote by $W^{r,p}(\mathcal{M},\mathbb{R}^n)$ the usual Sobolev space and put

$$||u||_{r,p} = \left\{ \sum_{0\leq k\leq r} \int_{\mathcal{M}} |D^k u|^p \nu_g \right\}^{1/p},$$

$$W^{r,p}(\mathcal{M},\mathcal{N}) = \{u \in W^{r,p}(\mathcal{M},\mathbb{R}^n) : u(x) \in \mathcal{N} \text{ a.e. on } \mathcal{M}\},$$

where ν_g is the volume element on $\mathcal{M}$.

If $r > \frac{m}{p}$ and $\mathcal{N}$ is compact, then $W^{r,p}(\mathcal{M},\mathcal{N})$ has the Finsler manifold structure (a proof is in [19]), and we can study variation problem in $W^{r,p}(\mathcal{M},\mathcal{N})$ (see [13, 14, 19]). When $r \leq \frac{m}{p}$ and $\mathcal{N}$ is not flat, we have the following difficulties:

(*i*) $W^{r,p}(\mathcal{M},\mathcal{N})$ may not have any manifold structure.

(*ii*) Some arcwisely connected component of $W^{r,p}(\mathcal{M},\mathcal{N})$ may not be open in $W^{r,p}(\mathcal{M},\mathcal{N})$.

Therefore in this case it is not easy to construct flows for deformations used in the variational method, and an extremal point of a functional f in a component may not be a critical point of f.

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To overcome these difficulties one has used the following methods:

(i) The heat flow method: Eells, Sampson and Hamilton have considered evolution equations to get the existence of harmonic maps into manifolds having non-positive sectional curvature (see [10, 16]). For the case of general target manifolds Chen and Struwe have proved the existence of heat flows in the weak sense (see [2, 4, 22]). But Chen and Ding showed that this method may not work in every case because of the blow-up of heat-flows in [3].

(ii) Perturbation method: Using the perturbation of functionals, Sacks and Uhlenbeck [21] have obtained the existence of minimal immersions of 2-spheres. But with this method we can only study the global analysis problem at the border-line case $m = rp$.

(iii) Weakly lower semicontinuity: Using the lower semicontinuity of the functional with respect to the weak topology of $W^{r,p}(\mathcal{M},\mathcal{N})$ one can get the existence of its extremal points. But we may not have this weakly lower semicontinuity when $r \geq 2$ (see [18]).

(iv) Constructive methods: Eells, Lemaire, Ratto, Wood and many mathematicians have constructed harmonic maps in special cases (see [8, 9]).

We can find more details of the above methods and other methods in [23]. The purpose of the present paper is to extend the gradient flows method in critical point theory for the case $r \leq \frac{m}{p}$. Let f be a real functional on $W^{r,p}(\mathcal{M},\mathcal{N})$. Extending f into $W^{r,p}(\mathcal{M},\mathbb{R}^n)$, we can define the differentiability and a vector field corresponding to f without taking care of the smoothness of $W^{r,p}(\mathcal{M},\mathcal{N})$. Then we prove that the restriction on $C^r(\mathcal{M},\mathcal{N})$ of this vector field is a vector field on $C^r(\mathcal{M},\mathcal{N})$. Using the smooth manifold structure of $C^r(\mathcal{M},\mathcal{N})$ we get a flow on $C^r(\mathcal{M},\mathcal{N})$ corresponding to this vector field. But this vector field may not be bounded on $C^r(\mathcal{M},\mathcal{N})$ and some curve of the flow may be only defined in finite time, e.g. there may be u in $C^r(\mathcal{M},\mathcal{N})$ such that the curve starting at u is only defined on a bounded interval $[0, t_u)$. Therefore we could not use this flow for the deformation lemma in the critical point theory. But we observe that in this case f decreases very fast along this curve and we can get a critical point of f. Thus we can apply the flow method to the case in which $W^{r,p}(\mathcal{M},\mathcal{N})$ is not a smooth mannifold but $C^r(\mathcal{M},\mathcal{N})$ is dense in it.

Let $\mathcal{H}$ be an arcwisely connected component in $W^{r,p}(\mathcal{M},\mathcal{N})$ and $\mathcal{K}$ its closure in $W^{r,p}(\mathcal{M},\mathcal{N})$. Using the flow constructed as above we can study the following problems:

(P_1) *Let* x *be in* $\mathcal{K}$ *such that* $f(x) = \inf f(\mathcal{K})$. *Is* x *a critical point of* f*?*

(P_2) *Let* $\{u_k\}$ *be a minimizing sequence of* f *in* $\mathcal{H}$. *When does* $\{f'(u_k)\}$ *converge to* 0 *as* k *tends to* ∞*?*

If the metric on $\mathcal{N}$ is not euclidean, the problem (P_1) is not trivial (see [17]). It is related to a question of J. Eells: When does a minimizer of a real functional f become a weak solution of the Lagrange-Euler equation associated to f? If f is continuously differentiable on $W^{r,p}(\mathcal{M}, \mathbf{R}^n)$ and $C^r(\mathcal{M}, \mathcal{N})$ is dense in $\mathcal{H}$, then we get an affirmative answer for (P_1) (see Theorem 3.1). If f belongs to a class of functionals, we get an affirmative answer for (P_2) when $\{u_k\}$ is in $C^r(\mathcal{M}, \mathcal{N})$ (see Theorem 3.2). Using the Ekeland variational principle we only can find a sequence $\{v_k\}$ associated to $\{u_k\}$ in (P_2) such that $\{v_k\}$ is a minimizing sequence of f and $\{f'(u_k)\}$ converges to 0 as k tends to ∞. The replacement of $\{u_k\}$ by $\{v_k\}$ is inconvenient in some cases because we can choose $\{u_k\}$ with some special properties but $\{v_k\}$ may not have these properties.

As a result of our paper we see that the main difficulty of the flows method is not the smoothness structure of the set of maps but the Palais–Smale condition of functionals, which seems to have relations not only with the dimension of source manifolds but also to the geometry of target manifolds (see [13, 14]). As in [24] our approach to global analysis is simpler than those in [7, 13, 14, 19]. We have studied this method in [6], when the functional satisfies the Palais–Smale condition.

1. Notations and definitions

In [5] we have observed that we could use a very small part of tangent space at any point in $\overset{\circ}{W}{}^{1,2}(\mathcal{M}, \mathbf{R}^n)$ to establish flows for deformations in the Lusternik–Schnirelman theory. On the other hand, some nice properties of the tangent space of target manifolds can help us to get such a part. Now, combining these ideas we request the target manifold $\mathcal{N}$ has the following properties:

$(T1)$ *There are local charts* $\{(\psi_j, \mathcal{O}_j)\}$ *of* $\mathcal{N}$ *and a positive real number* η *such that* $\|\psi_j\|_{C^r} + \|\psi_j^{-1}\|_{C^r} \leq \eta^{-1}$ *and* $\{x \in \mathcal{N} : |x - a| < \eta\}$ *is contained in some* $\mathcal{O}_j$ *for any* $a \in \mathbf{R}^n$.

$(T2)$ *There is a* C^{r+1}*-map* Θ *from* $\mathbf{R}^n$ *into the space* $L(\mathbf{R}^n, \mathbf{R}^n)$ *of continuous linear maps from* $\mathbf{R}^n$ *into* $\mathbf{R}^n$ *with usual norm such that* $\sup_{y \in \mathbf{R}^n} \|D^s\Theta(y)\| < \infty$

for any s in $\{0,1,...,r+1\}$, and $\Theta(y)$ is a projection from $\mathbf{R}^n$ onto the tangent space $T_y(\mathcal{N})$ of $\mathcal{N}$ at any y in $\mathcal{N}$, which is identified as a linear subspace of $\mathbf{R}^n$ parallel to an affine linear subspace of $\mathbf{R}^n$ passing through y.

If $\mathcal{N}$ is compact, then it has the two above properties. Note that such a map Θ need to be defined only on $\mathcal{N}$, then we extend it into $\mathbf{R}^n$. For example, let $\langle,\rangle$ be the scalar product in $\mathbf{R}^n$ and φ in $C_0^\infty(\mathbf{R}^n,\mathbf{R})$ such that $\varphi(y)=1$ in a neighborhood of S^{n-1}. Then we can choose the map Θ for S^{n-1} in $(T2)$ as follows

$$\Theta(x)z = z - \langle z, \varphi(x)x\rangle x \quad \forall(x,z)\in \mathbf{R}^n\times\mathbf{R}^n.$$

For any (u,φ) in $W^{r,p}(\mathcal{M},\mathbf{R}^n)\times C^r(\mathcal{M},\mathbf{R}^n)$ and z in $\mathcal{M}$, put $\Theta_u(\varphi)(z) = \Theta(u(z))\varphi(z)$. We assume throughout this paper the following condition

$(T3)$ *Fix φ in $C^{r+1}(\mathcal{M},\mathbf{R}^n)$. Then the map $u\longmapsto\Theta_u(\varphi)$ is continuous from $W^{r,p}(\mathcal{M},\mathbf{R}^n)$ into $W^{r,p}(\mathcal{M},\mathbf{R}^n)$.*

If $r=1$, by $(T2)$ there is a constant K such that $\forall u,v\in W^{r,p}(\mathcal{M},\mathcal{N})$, $\varphi\in C^2(\mathcal{M},\mathbf{R}^n)$,

$$|D\Theta_u(\varphi)-D\Theta_v(\varphi)|\le K(|u-v||D\varphi|+|u-v||Du||\varphi|+|Du-Dv||\varphi|).$$

Thus, by the Hölder theorem we have

$$\begin{aligned}&||D\Theta_u(\varphi)-D\Theta_v(\varphi)||_{L^p}\le\\ &\qquad\le K||\varphi||_{W^{1,\infty}}(1+||u||_{1,p})\left(||u-v||_{L^{\frac{p}{p-1}}}+||Du-Dv||_{L^p}\right).\end{aligned}$$

Therefore we have $(T3)$ when $r=1$ and $\dfrac{p-1}{p}\ge\dfrac{m-p}{mp}$ or $p\ge\dfrac{2m}{m+1}$. Similarly, the condition $(T3)$ is satisfied when $r=2$ and $p\ge\max\{\dfrac{2m}{m+2},\dfrac{3m}{m+4}\}$.

DEFINITION 1.1. Let $\{(\phi_j,\Omega_j)\}_{j\in J}$ be a family of local charts of $\mathcal{M}$ such that $\cup_{j\in J}\Omega_j=\mathcal{M}$, and let $L(W^{r,p}(\mathcal{M},\mathbf{R}^n),\mathbf{R})$ be the space of linear mappings from $W^{r,p}(\mathcal{M},\mathbf{R}^n)$ into $\mathbf{R}$. For any j in J and any T in $L(W^{r,p}(\mathcal{M},\mathbf{R}^n),\mathbf{R})$ put

$$\begin{aligned}C_j^r(\mathcal{M},\mathbf{R}^n)&=\{u\in C^r(\mathcal{M},\mathbf{R}^n): \text{support of } u \text{ is contained in } \Omega_j\},\\ W_j^{r,p}(\mathcal{M},\mathbf{R}^n)&=\{u\in W^{r,p}(\mathcal{M},\mathbf{R}^n): \text{support of } u \text{ is contained in } \Omega_j\},\\ ||T||_{u,j,r,p}&=\sup\left\{|T\Theta_u(\varphi)| : \varphi\in C_j^r(\mathcal{M},\mathbf{R}^n),\ ||\Theta_u(\varphi)||_{r,p}\le 1\right\}\\ ||T||_{u,r,p}&=\sup_{j\in J}||T||_{u,j,r,p},\end{aligned}$$

where $||T||_{u,j,r,p}$ and $||T||_{u,r,p}$ may be equal to ∞.

DEFINITION 1.2. Let f be a continuous mapping from $W^{r,p}(\mathcal{M},\mathbf{R}^n)$ into $\mathbf{R}$. We say f is weakly continuously differentiable on $W^{r,p}(\mathcal{M},\mathbf{R}^n)$ if and only if two following conditions are satisfied:

(i) For any $u \in W^{r,p}(\mathcal{M},\mathbf{R}^n)$ there exists a $f'(u)$ in $L(W^{r,p}(\mathcal{M},\mathbf{R}^n),\mathbf{R})$ such that

$$\lim_{t\to 0}\frac{f(u+tv)-f(u)}{t} = f'(u)v \quad \forall v \in W^{r,p}_j(\mathcal{M},\mathbf{R}^n),\ \forall j \in J.$$

(ii) The map $(u,\varphi) \longmapsto f'(u)\varphi$ is continuous from $W^{r,p}(\mathcal{M},\mathbf{R}^n)\times W^{r,p}_j(\mathcal{M},\mathbf{R}^n)$ into $\mathbf{R}$ for any j in J.

In this paper we denote the set $\{x \in W^{r,p}(\mathcal{M},\mathbf{R}^n) : ||x-a||_{r,p} < s\}$ by $B(a,s)$.

2. Flows on $W^{r,p}(\mathcal{M},\mathcal{N})$

Let f be a continuously differentiable real function on $W^{r,p}(\mathcal{M},\mathbf{R}^n)$. In this section we establish a flow associated to f on $W^{r,p}(\mathcal{M},\mathcal{N})$ and study its properties. First we obtain a vector field corresponding to f as follows.

LEMMA 2.1. *Let j in J, μ be in the open interval $(0,1)$ and f be a continuously differentiable real function on $W^{r,p}(\mathcal{M},\mathbf{R}^n)$. Put*

$$A_j = \{x \in W^{r,p}(\mathcal{M},\mathbf{R}^n) : ||f'(x)||_{x,j,r,p} > 0\}.$$

Then there is a C^1-map v from an open neighborhood $\mathcal{V}$ of A_j in $W^{r,p}(\mathcal{M},\mathbf{R}^n)$ into $W^{r,p}(\mathcal{M},\mathbf{R}^n)$ such that

(i) $||v(u)||_{r,p} \le 1$ and $f'(u)v(u) \le -\mu||f'(u)||_{u,j,r,p}$ for any u in A_j and

(ii) the restriction of v on $C^r(\mathcal{M},\mathcal{N})\cap\mathcal{V}$ is a C^1-map from $(C^r(\mathcal{M},\mathcal{N})\cap \mathcal{V},\ ||\,||_{C^r})$ into $(C^r_j(\mathcal{M},T_\mathcal{N}),\ ||\,||_{C^r})$.

PROOF. For any u in A_j there is $\varphi(u)$ in $C^r_j(\mathcal{M},\mathbf{R}^n)$ such that

$$||\Theta_u(\varphi(u))||_{r,p} < 1 \text{ and } f'(u)(\varphi(u)) < -\mu||f'(u)||_{u,j,r,p}.$$

By ($T3$) and the continuity of f' we can find a positive real number d_u such that for any y in $B(u,2d_u)$

$$f'(y)\Theta_y(\varphi(u)) < -\mu||f'(y)||_{y,j,r,p} \quad \text{and} \quad ||\Theta_y(\varphi(u))||_{r,p} \le 1.$$

Since $C^r(\mathcal{M},\mathbf{R}^n)$ is dense in $W^{r,p}(\mathcal{M},\mathbf{R}^n)$, we can find $\bar{u}$ in $B(u,d_u)\cap C^r(\mathcal{M},\mathbf{R}^n)$. Then $B(\bar{u},d_u)\subset B(u,2d_u)$ and the family $\{B(\bar{u},d_u) : u\in A_j\}$ covers A_j. Therefore there exists a locally finite refinement $\{B(\bar{u}_i,d_{u_i})\}$ of that family, which covers A_j. Denote $\bigcup_i B(\bar{u}_i,d_{u_i})$ by $\mathcal{V}$. For any i put

$$q_i(u)=\begin{cases}\left(d^p_{u_i}-\|u-\bar{u}_i\|^p_{r,p}\right)^r & \forall u\in B(\bar{u}_i,d_{u_i})\\ 0 & \forall u\in W^{r,p}(\mathcal{M},\mathbf{R}^n)\setminus B(\bar{u}_i,d_{u_i}).\end{cases}$$

Then q_i is of class C^1 on $W^{r,p}(\mathcal{M},\mathbf{R}^n)$. Set

$$v(u)=\left(\sum_j q_zj(u)\right)^{-1}\Theta_u\left(\sum_i q_zi(u)\varphi(u_i)\right)\quad\forall u\in\mathcal{V},$$

which satisfies (i). Note that $\bar{u}_i$ belongs to $C^r(\mathcal{M},\mathbf{R}^n)$ for any i. Since $\|\,\|_{C^r}$ is stronger than $\|\,\|_{r,p}$, the restriction of q_i on $C^r(\mathcal{M},\mathbf{R}^n)$ is also of class C^1 on $C^r(\mathcal{M},\mathbf{R}^n)$. Thus, by $(T2)$ we get (ii).

Note that $C^r(\mathcal{M},\mathcal{N})\cap\mathcal{V}$ is open in $C^r(\mathcal{M},\mathcal{N})$. By the results of [7, 15, 19] $v|_{C^r(\mathcal{M},\mathcal{N})\cap\mathcal{V}}$ in Lemma 2.1 is a vector field on $C^r(\mathcal{M},\mathcal{N})\cap\mathcal{V}$. Thus for any u in $C^r(\mathcal{M},\mathcal{N})\cap\mathcal{V}$ there is a C^1-curve $w(u,.)$ from an interval $[0,t_1)$ into $C^r(\mathcal{M},\mathcal{N})$ such that

$$\begin{cases}\dfrac{dw(u,t)}{dt}=v(w(u,t)) & \forall t\in(0,t_1)\\ w(u,0)=u.\end{cases}$$

We study w by the following lemmas.

LEMMA 2.2. *Let u_0 be in A_j and s a positive real number such that $B(u_0,3s)\subset A_j$ and there is only a finite number of $B(\bar{u}_i,d_{u_i})$ having a non-empty intersection with $B(x_0,3s)$. Then there is a positive real number t_0 such that $w(u,.)$ is defined on $[0,t_0)$ for any u in $B(x_0,s)\cap C^r(\mathcal{M},\mathcal{N})$.*

PROOF. Note that we only use a finite family $\{\bar{u}_i,\varphi(u_i)\}$ in $C^r(\mathcal{M},\mathbf{R}^n)$ to define $v(u)$ for any u in $B(u_0,3s)$. Thus, by $(T2)$ there is a constant C_0 such that

$$\|v(a)\|_{C(\mathcal{M},\mathbf{R}^n)}\le C_0 \text{ and } \|v(a)-v(b)\|_*\le C_0\|a-b\|_*$$

for any a and b in $B(u_0,s)\cap C^r(\mathcal{M},\mathbf{R}^n)$, where $\|\,\|_*$ is $\|\,\|_{C(\mathcal{M},\mathbf{R}^n)}$ or $\|\,\|_{C^r(\mathcal{M},\mathbf{R}^n)}$.

Fix u in $B(u_0,s)\cap C^r(\mathcal{M},\mathbf{R}^n)$. Let η be the positive real number in the condition $(T1)$. Let (φ, Ω) be a local chart of $\mathcal{M}$ such that $u(\Omega) \subset B(z, \frac{\eta}{4})$ for some z in $\mathbf{R}^n$. Thus we can choose a local chart $(\psi, \mathcal{O})$ of $\mathcal{N}$ such that

$$\psi \circ u \circ \varphi^{-1}(U) \subset B(0, \eta') \subset B(0, 3\eta') \subset \psi(\mathcal{O}) \subset \mathbf{R}^{n_1},$$

where $U = \varphi(\Omega)$ and $\eta\prime$ depends only on η.

Therefore we can reduce the problem to the case of euclidean metrics. Let $U \subset \mathbf{R}^m$, $\bar{u} \in C^r(U, B(0, \eta\prime))$ and $\bar{v} \in C^1(C^r(U, B(0, 3\eta')), C^r(U, \mathbf{R}^{n_1}))$ such that

$$||\bar{v}(a)||_{C(\mathcal{M}, \mathbf{R}^{n_1})} \leq C_1 \text{ and } ||\bar{v}(a) - \bar{v}(b)||_* \leq C_1 ||a - b||_*$$

for any a and b in $Y = C^1(U, B(0, 3\eta'))$, where $||\,||_*$ is $||\,||_{C(U, \mathbf{R}^{n_1})}$ or $||\,||_{C^r(U, \mathbf{R}^{n_1})}$ and the constant C_1 depends only on C_0 and η. Choose $t_1 = \min\left\{\frac{\eta'}{C_1 + 1}, \frac{1}{C_1 + 1}\right\}$. For any ϕ in $X \equiv C([0, t_1], Y)$ put

$$(T\phi)(t) = \bar{u} + \int_0^t \bar{v}(\phi(s))ds \quad \forall t \in [0, t_1].$$

It is easy to prove that T is a contraction on X and that it has a unique fixed point there. Therefore we can choose t_0 for the lemma.

LEMMA 2.3. *Let $u \in A_j \cap C^r(\mathcal{M}, \mathcal{N})$ and $[0, t_u)$ the maximal interval where $w(u, .)$ can be defined. We have*

(i) $w(u, .)$ is a C^1-curve from $[0, t_u)$ into Z, where Z is $C(\mathcal{M}, \mathcal{N})$ or $C^r(\mathcal{M}, \mathcal{N})$ or $W^{r,p}(\mathcal{M}, \mathcal{N})$,

(ii) $||w(, u, t) - u||_{r,p} \leq t$ for any $t \in [0, t_u)$,

(iii) If t_u is finite, then $\{w(u, t)\}$ converges to u_0 in $W^{r,p}(\mathcal{M}, \mathcal{N})$ as $t \to t_0$ and $||f'(u_0)||_{u_0, j, r, p} = 0$,

(iv) If there is a positive real number s such that $B(u, s)$ is contained in A_j, then $t_u \geq s$,

(v) If $||f'(x)||_{x, j, r, p} > b$ for any x in $B(u, s)$, then $f(u) - f(w(u, s)) \geq \mu b s'$ $\forall s' \in (0, s)$, where μ is as in Lemma 2.1.

PROOF. Since $w(u, .)$ is a C^1-curve from $[0, t_u)$ into $C^r(\mathcal{M}, \mathcal{N})$ and the topology of $C^r(\mathcal{M}, \mathcal{N})$ is stronger than those of $C(\mathcal{M}, \mathcal{N})$ and $W^{r,p}(\mathcal{M}, \mathcal{N})$, we get (i). By (i) of Lemma 2.1, we have

$$||w(u, s) - w(u, t)||_{r,p} \leq |s - t| \quad \forall s, t \in [0, t_u), \tag{2.1}$$

which yields (ii). When t_u is finite, by (2.1) $\{w(u,t)\}$ converges to u_0 in $W^{r,p}(\mathcal{M},\mathcal{N})$ as $t \to t_u$. Since $[0,t_u)$ is the maximal domain of $w(u,.)$, by Lemma 2.2 u_0 does not belong to A_j. Thus we have (iii). By Lemma 2.2, (ii) and (iii) we get (iv). Put $\bar{f}(t) = f(w(u,t))$ for any $t \in [0,t_u)$. By Lemma 2.2 and (i) of Lemma 2.1 we see that $s \le t_u$ and

$$\bar{f}'(t) = f'(w(u,t))v(w(u,t)) \le -\mu b \quad \forall t \in (0,s).$$

Therefore we get (v).

REMARK. The flow $w(u,.)$ only changes the value of u inside Ω_j. Therefore we can use special local properties of $\mathcal{M}$ to study global problems on $W^{r,p}(\mathcal{M},\mathcal{N})$.

3. Applications

Let f be a continuously differentiable real function on $W^{r,p}(\mathcal{M},\mathbf{R}^n)$. Let $\mathcal{H}$ be an arcwisely connected component in $W^{r,p}(\mathcal{M},\mathcal{N})$, $\mathcal{K}$ be its closure and x in $\mathcal{K}$. Then we have the following results.

THEOREM 3.1. *Assume that $f(x) = \inf f(\mathcal{H})$ and $C^r(\mathcal{M},\mathcal{N}) \cap \mathcal{H}$ is dense in $\mathcal{H}$. Then $\|f'(x)\|_{x,r,p} = 0$.*

PROOF. Let $\{x_k\}$ be a sequence in $C^r(\mathcal{M},\mathcal{N}) \cap \mathcal{H}$ such that $\{x_k\}$ converges to x in $W^{r,p}(\mathcal{M},\mathcal{N})$. Assume that $\|f'(x)\|_{x,j,r,p} > 2b > 0$ for some j and b. Then we can find a real number s such that $\|f'(y)\|_{y,j,r,p} > b$ for any $y \in B(x,3s)$. When k is greater than some k_0, x_k belongs to $B(x,s)$. Replacing x_k by $w(x_k,s)$ and applying (v) of Lemma 2.3 we see that $f(x)$ can not be $\inf f(\mathcal{H})$. This contradiction proves the theorem.

THEOREM 3.2. *Assume that*

(F) For any minimizing sequence $\{v_k\}$ of f in $\mathcal{H}$ such that $\|f'(v_k)\|_{v_k,j,r,p} > 2b$ for some j and some positive real number b, we can find a positive real number s such that $\|f'(x)\|_{x,j,r,p} > b$ for any x in $\bigcup_k B(v_k,s)$.

Then $\{\|f'(u_k)\|_{u_k,j,r,p}\}$ converges to 0 when $\{u_k\}$ is a minimizing sequence of f in $\mathcal{H}$ and belongs to $C^r(\mathcal{M},\mathcal{N})$.

PROOF. Using (F) and arguing as in the proof of Theorem 4.1 we get the theorem.

REMARK 3.1. The density of $C^\infty(\mathcal{M},\mathcal{N})$ has been studied (see [1] and its references). If f is the functional corresponding to the harmonic maps, Theorem 3.1 has been proved in [17] for some special cases.

REMARK 3.2. With the Ekeland variational principle [11, 12] we have to replace $\{u_k\}$ by another sequence to get the result of Theorem 3.2.

REMARK 3.3. If f is the functional corresponding to the harmonic maps, then f satisfies (F). In general, f satisfies (F) if the mappings f and $x \to ||f'(x)||_{x,r,p}$ are bounded and uniformly continuous on $\left(f^{-1}(B), || \, ||_{r,p}\right)$ for any bounded subset B of $\mathbf{R}$.

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