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# POLYHEDRAL ANNEXATION VS OUTER APPROXIMATION FOR THE DECOMPOSITION OF MONOTONIC QUASICONCAVE MINIMIZATION PROBLEMS

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**Abstract.** We discuss polyhedral annexation and outer approximation methods for the decomposition of linearly constrained quasiconcave minimization problems with a special structure of monotonicity. We show that polyhedral annexation is generally superior, except when the problem has some additional structure.

## 1. Introduction

Consider the problem:

(P) minimize $f(x)$ subject to $x \in D$.

where $D$ is a polyhedron in $R^n$, and $f : \Omega \to R$ is a quasiconcave (not necessarily continuous) function on a closed convex set $\Omega \supset D$, satisfying the following *rank k monotonicity condition:*

(*) There exists a polyhedral cone $K$ of dimension $n-k$ such that

$$x \in \Omega,\ x' - x \in K \Rightarrow x' \in \Omega, f(x') \geq f(x). \tag{1}$$

As shown in [18] this monotonic structure occurs in a wide class of problems of relevant practical interest. From (1) it immediately follows that the cone $K$ is contained in the recession cone of $\Omega$. Two special cases that have received particular attention in the literature are the following:

(P1) minimize $f(y) + dz$ subject to $(y, z) \in D$,

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where $y \in R^p$, $z \in R^q$, $d \in R^q$, $(p+q=n)$, $f(y)$ is a concave function and $D$ is a polyhedron in $R^n$ [10], [13], [15], [4]. Here $k = p+1$, $\Omega = R^n$, $K = \{(v,w) \in R^p \times R^q : v_1 = \ldots = v_p = 0,\, dw \geq 0\}$.

(P2) $$\text{minimize } \prod_{i=1}^{k}(c^i x + d^i) \text{ subject to } x \in D,$$

where $D$ is a polyhedron contained in $\Omega := \{x : c^i x + d^i \geq 0, i = 1, \ldots, k\}$ (linear multiplicative programming [6], [7], [8] and the vectors $c^1, \ldots, c^k$ are linearly independent. Here $K = \{u : c^1 u \geq 0, \ldots, c^k u \geq 0\}$. Other examples of problems that can be reduced to the form (P) with property (*) can be found in [17], [18] and references therein.

When $k = 1$, (P) is an extremely easy problem which reduces to solving one or two linear programs (see Remark 3 below). When $k = 2$, the most efficient methods for solving (P) are parametric methods as developed in [7], [6] for linear multiplicative programs, [5], [20] for special minimum concave cost network flow problems and [26] for the general case. However, parametric methods become impracticable for $k > 2$, except in some rare cases when the constraint polyhedron has a very special structure [22], [23], [24], [25]. Therefore, outer approximation and decomposition methods of Benders-Geoffrion's type [13],[15], [8] polyhedral annexation procedures [17], [18] and branch and bound methods [2], [18], [4] have been developed to cope with the difficulty. It should be noted, however, that, except for [17] and [18], most of the mentioned papers only deal with the special cases (P1) and (P2) mentioned above. To the best of our knowledge, nowhere in the literature to date the general problem (P) has been treated in sufficient detail and a study has been carried out about the relative efficiency of different decomposition methods.

The aim of this paper is to discuss and to compare two decomposition approaches based on primal and dual outer approximation principles for the general problem (P) under assumption (*), with $k > 2$ but relatively small. To exploit the monotonic structure the general idea of decomposition is to convert (P) into a quasiconcave minimization problem in $R^k$. Various decomposition methods differ by the reduction technique and also the way to handle the reduced problem. Practical computational experience suggests that, for the above specified values of $k$, outer approximation procedures should be at least competitive with branch and bound algorithms. Also these methods do not require any continuity property for the objective function. There are, however, two different decomposition methods of problem (P) via outer approximation: the ordinary (primal) and the dual. The former is a variant of Benders-Geoffrion's procedure tailored to the special monotonic structure; the latter is a direct application of polyhedral annexation technique earlier developed in [16]. For the sake of simplicity, we will assume that $D$ is a polytope (bounded polyhedron) and

$$K = \{u : \; Cu \geq 0\},$$

where $C$ is a $k \times n$ matrix, but with a little more effort the methods can be extended to the general case when $D$ may be unbounded while $K$ is an arbitrary polyhedral cone. Although no restriction is set on the value of $k$, our approach should be practical mainly for values of $k$ within the above specified range (typically for $2 < k < 10$ on current microcomputers). Computational experience indicates that for these values of $k$ both primal and dual outer approximation procedures are easy to implement and perform quite well even for fairly large $n$. In any case, since the hardest part of the global search in these methods operates in dimension $k$ rather than $n$, their advantage is obvious if $k \ll n$.

After the introduction, we present the primal and dual outer approximation methods in Sections 2 and 3. In Section 4 we discuss the results of computational experiments. With all their preliminary character and their limited value, they clearly indicate that the dual method should outperform the primal in most problems where no additional structure intervenes.

## 2. Decomposition by primal outer approximation

Since rank $C = k$, the matrix $C$ defines an affine mapping (also denoted by $C$ for the sake of simplicity) from $R^n$ onto $R^k$ and $C(\Omega)$ can be identified with a convex set in $R^k = C(R^n)$, containing $C(K) = \{t \in R^k : t = Cx \geq 0\}$. For every $t \in C(\Omega)$ if $Cx' = t$ and $Cx = t$ for $x', x \in \Omega$, then by virtue of (1), $f(x') = f(x)$, therefore, $F(t) = f(x)$ for $Cx = t$ is unambiguously defined.

From the quasiconcavity of $f(x)$ on $\Omega$ it then follows that for any $\gamma \in R$, the level set $\{t \in C(\Omega) : F(t) \geq \gamma\} = \{t : t = Cx,\ x \in \Omega,\ f(x) \geq \gamma\}$ is convex. Hence, the function $F(t)$ is quasiconcave on $C(\Omega)$.

Thus problem (P) can be reformulated as:

(RP) $\qquad$ minimize $F(t)$ subject to $t \in G$,

where $G = C(D)$ is a polytope of dimension $k$ in $R^k$ and $F(t)$ is a quasiconcave function on $C(\Omega) \supset G$.

Note that by writing $C = [C_B, C_N]$, where $C_B$ is a $k \times k$ nonsingular matrix, and accordingly, $x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$ we have

$$x = \begin{bmatrix} C_B^{-1}C_N \\ 0 \end{bmatrix} t + \begin{bmatrix} -C_B^{-1}C_N \\ I \end{bmatrix} x_N = Zt + y,$$

with $Cy = x_N - x_N = 0$, hence

$$F(t) = f(Zt). \tag{2}$$

To solve (RP) (which is now a problem in $R^k$ with $k$ small) a method which suggests itself is by outer approximation of $G = C(D)$ (see [12] , [2]). Following this method, a key point is: given a point $\bar{t} \in C(R^n) \subset R^k$, determine whether $\bar{t} \in G = C(D)$, and if $\bar{t} \notin G$ then construct a linear inequality (cut) $L(t) \leq 0$ that excludes $\bar{t}$ without excluding any point of $G$. This is an easy matter even though $G$ is not given explicitly as the solution set of a system of linear inequalities.

Of course, $\bar{t} \notin C(D)$ if and only if the linear system $Cx = \bar{t}$ has no solution in $D$, so the existence of $L(t)$ is ensured by the separation theorem or any of its equivalent forms (such as Farkas-Minkowski lemma). However, we are not so much interested in the existence as in the effective construction of $L(t)$. Therefore, the best way to deal with the above mentioned point is via the duality theorem of linear programming (which is another form of the separation theorem). Specifically, assuming $D = \{x : Ax \leq b,\ x \geq 0\}$, with $A \in R^{m\times n}$, $b \in R^m$, consider the dual pair of linear programs

$LP(\bar{t}) \qquad \max\{0 : Cx = \bar{t}, Ax \leq b, x \geq 0\},$

$LP^*(\bar{t})$ $\qquad \min\{\langle \bar{t}, v\rangle + \langle b, w\rangle : C^T v + A^T w \geq 0,\ w \geq 0\}.$

PROPOSITION 1. *1) If $v = 0, w = 0$ is an optimal solution of $LP^*(\bar{t})$ then $\bar{t} \in G$.*

*2) Otherwise there exists an extreme direction $(\bar{v}, \bar{w})$ of the cone $C^T v + A^T w \geq 0,\ w \geq 0$, such that $\langle \bar{t}, \bar{v}\rangle + \langle b, \bar{w}\rangle < 0$. Then the affine function*

$$L(t) = -\langle t, \bar{v}\rangle - \langle b, \bar{w}\rangle \tag{3}$$

*satisfies*

$$L(\bar{t}) > 0, \quad L(t) \leq 0 \quad \forall t \in G. \tag{4}$$

PROOF. 1) If $v = 0, w = 0$ is optimal for LP*($\bar{t}$) then by the duality theory of linear programming, LP($\bar{t}$) is feasible, i.e. $\bar{t} \in G$.

2) If an extreme direction $(\bar{v}, \bar{w})$ of the cone $C^T v + A^T w \geq 0,\ w \geq 0$ satisfies

$$\langle \bar{t}, \bar{v}\rangle + \langle b, \bar{w}\rangle < 0, \tag{5}$$

then LP*($\bar{t}$) is unbounded, hence LP($\bar{t}$) is infeasible, i.e. $\bar{t} \notin G$. From (5) we have $L(\bar{t}) > 0$, while for any $t \in G$, since LP($t$) is feasible, LP*($t$) must have optimal value zero, hence $\langle t, \bar{v}\rangle + \langle b, \bar{w}\rangle \geq 0$, i.e. $L(t) \leq 0$.

On the basis of the above proposition, we can state the following

ALGORITHM 1 (Primal OA Decomposition Method).

0. Select an initial polytope $S_1 \subset C(R^n) \subset R^k$ with known vertex set $V_1$ and such that $C(D) \subset S_1 \subset C(\Omega)$. Set $r = 1$.

1. Compute $t^r \in \operatorname{argmin}\{F(t) : t \in V_r\}$.

2. Solve the linear porgam LP*($t^r$), starting with the basic feasible solution $v = 0, w = 0$.

a) If $v = 0, w = 0$ is optimal to LP*($t^r$) then terminate: $t^r \in G$, hence $t^r$ solves (RP) (the corresponding basic optimal solution $x^r$ of LP ($t^r$) satisfies $Cx^r = t^r$ and solves (P)).

b) Otherwise, an extreme direction $(v^r, w^r)$ of the constraint set of LP*($t^r$) is found such that (5) holds for $\bar{v} = v^r, \bar{w} = w^r$ (so by Lemma 1 the affine function $L(t)$ satisfies (4) for $\bar{t} = t^r$).

3. Let $L(t)$ be defined by (3), where $\bar{v} = v^r, \bar{w} = w^r$. Form the polytope

$$S_{r+1} = S_r \cap \{t : L(t) \leq 0\}$$

and compute the vertex set $V_{r+1}$ of $S_{r+1}$ (from knowledge of the vertex set $V_r$ of $S^r$). Set $r \leftarrow r+1$ and go back to Step 1.

THEOREM 1. *Algorithm 1 terminates after finitely many steps.*

PROOF. From its definition in Step 1 $F(t^r) = \min\{F(t) : t \in S_r\}$. If Step 2a) occurs then $t^r \in G$ and since $S_r \supset G$, it follows that $F(t^r) = \min\{F(t) : t \in G\}$. Clearly, $(v^1, w^1), (v^2, w^2), \dots$ are all distinct because for any $r, L_r(t^r) > 0$ while $L_r(t^{r+s}) \leq 0 \ \forall s \geq 1$. Since each $(v^r, w^r)$ is an extreme direction of a fixed cone (the constraint set of $\mathrm{LP}^*(t^r)$, finiteness of the algorithm is ensured.

REMARK 1. In many problems (as in (P1), (P2) mentioned in the introduction) we have

$$D \subset \{x : Cx + d \geq 0\} \subset \Omega \text{ for some } d \in R^n. \tag{6}$$

In this case, as initial polytope one can take $S_1 = \{t \in C(R^n) : \alpha_i \leq t_i \leq \beta_i, \ i = 1, \dots, k\}$, where $\alpha_i = \min\{c^i x : x \in D\}$, $\beta_i = \max\{c^i x : x \in D\}$ and $c^i$ is the $i$-th row of $C$. Indeed, clearly $C(D) \subset S_1$. To see that $S_1 \subset C(\Omega)$, note that if $t \in S_1$, then $t_i \geq \alpha_i, \ i = 1, \dots, k$, hence $t = Cx$ for $x$ satisfying $c^i x \geq \alpha_i \ \forall i$; this implies $Cx \geq -d$ (because $D \subset \{x : c^i x + d_i \geq 0 \ \forall i\}$), hence $x \in \Omega$ and $t \in C(\Omega)$.

Also if (6) holds then (RP) is equivalent to

$$\text{minimize } F(t) \text{ subject to } t \in \tilde{G},$$

where $\tilde{G} = G + R^k_+ = \{t : t \geq Cx, \ x \in D\}$. Indeed, from (1) $F(t') \geq F(t)$ for any $t' \geq t = Cx$ with $x \in D$ because then $t' = Cx'$ with $Cx' \geq Cx \geq -d$, i.e. $x \in \Omega$ by (6). Since $\bar{t} \in \tilde{G}$ if and only if $Cx \leq \bar{t}, \ x \in D$, in $\mathrm{LP}(\bar{t})$ one can relax the constraint $Cx = \bar{t}$ to $Cx \leq \bar{t}$, and hence include the inequality $v \geq 0$ in the constraints of $\mathrm{LP}^*(\bar{t})$.

REMARK 2. There are several ways to choose $\mathrm{LP}(\bar{t})$ in order that its solution helps to recognize whether $\bar{t} \notin G$ and to construct $L(t)$ satisfying (4). In practice $\mathrm{LP}(\bar{t})$ should be chosen such that it can be solved efficiently. For instance, consider problem (P1) when $f(y) = f(Mz)$ while $D = \{(y, z) : \ Mz = y, \ Nz = b, \ z \geq 0\}$ and for fixed $y$ linear programs with constraints $(y, z) \in D$

can be solved efficiently. Then to check whether $\bar{t} = (\bar{y}, \bar{t}_0) \notin G$, one should solve LP($\bar{t}$) of the form

$$\min\{dz : (\bar{y}, z) \in D\},$$

(see e.g. [2], Chapter VIII): $(\bar{y}, \bar{t}_0) \in G$ if and only if this subproblem has an optimal value not exceeding $\bar{t}_0$ .

REMARK 3. For $k = 2$, (RP) is a two-dimensional problem which can be solved efficiently by parametric linear programming methods as mentioned in the introduction. For $k = 1$ , i.e. $f(x) = F(l(x))$, where $l(x)$ is an affine function, (RP) is a one-dimensional problem (find the minimum of the univariate quasiconcave function $F(t)$ over the line segment $G = C(D)$) and hence can be solved by just comparing $F(t)$ at the two endpoints of the line segment $G$ (these endpoints are found by minimizing and maximizing $l(x)$ over $D$). If in addition $F(t)$ is a monotonic increasing function then (RP) has a single (local and global) minimum which is achieved simply at any minimizer of $l(x)$ over $D$. Thus, such problems are extremely easy. Of course it does not make sense to use these problems for testing general purpose concave minimization algorithms, as has been done unfortunately in [3] (see also [2], pages 252-253).

## 3. Decomposition by dual outer approximation

The above decomposition method is an outer approximation performed on (RP), the reduced primal problem. Although it looks simple, it requires the availability of a simple polytope $S_1$ such that $C(D) \subset S_1 \subset C(\Omega)$, a condition which may fail to hold for some important problems. Furthermore, as we will see in the discussion (Section 4), its implementation poses in certain circumstances some numerical problems.

An alternative decomposition method free from these limitations is by polyhedral annexation as developed in [16], [17]. This method can also be interpreted as an inner approximation or dual outer approximation procedure [2], [11].

Without loss of generality we can assume that the origin $0 \in R^n$ is a vertex of $D$. For any $\gamma \in R$ define $X_\gamma = \{x \in \Omega : f(x) \geq \gamma\}$. By quasiconcavity of

$f(x)$, $X_\gamma$ is a convex set. Clearly,

$$\gamma_0 = \min\{f(x) : x \in D\} \Rightarrow D \subset X_{\gamma_0} \Rightarrow X^*_{\gamma_0} \subset D^*, \tag{7}$$

where $E^* = \{y : \langle y, x\rangle \le 1\ \forall x \in E\}$ is the polar of $E$. By assumption $K := \{x : Cx \ge 0\}$ is contained in the recession cone of $\Omega$, hence $K \subset \Omega$ since $0 \in \Omega$. In view of (1), if $Cx \ge 0$ then $f(x) \ge f(0) \ge \gamma_0$, hence $x \in X_{\gamma_0}$. Thus, $K \subset X_{\gamma_0}$, and if for some $\gamma \in f(D)$ there is a polyhedron $P \subset X_\gamma$ such that $P^* \subset D^*$ then surely $\gamma = \gamma_0$, because then $D \subset P \subset X_\gamma$. Based on this observation, the idea of polyhedral annexation is to construct a sequence of expanding polyhedrons

$$K \subset P_1 \subset P_2 \subset \dots \tag{8}$$

together with a nonincreasing sequence $\gamma_1 \ge \gamma_2 \ge \dots$ such that

$$P_r \subset X_{\gamma_r}, \quad r = 1, 2, \dots \tag{9}$$

in such a way that eventually, at some iteration $s$ :

$$P^*_s \subset D^*. \tag{10}$$

Then, since $D \subset P_s \subset X_{\gamma_s}$, the last value $\gamma_s$ will yield the optimal value.

Specifically, let $L$ be the linearity space of $K$, i.e. $L = \{x : Cx = 0\}$. Then $K^* \subset L^\perp$ and from convex analysis [9] it is well known that $K^* = \text{cone}\{-c^1, \dots, -c^k\}$, while $L^\perp$ is the linear space spanned by $c^1, \dots, c^k$ ($c^i$ is the $i$-th row of $C$). Since rank $C = k$, $L^\perp$ is a linear space of dimension $k$. Let $\pi : R^k \to R^n$ be the linear mapping defined by $\pi(t) = \sum_{i=1}^{k} t_i c^i$ for every $t = (t_1, \dots, t_k)$, so that $L^\perp = \pi(R^k)$, $K^* = \pi(R^k_-)$.

Now, since for any $\gamma$ the set $X_\gamma$ contains $K$, we can always take, as starting polyhedron in the sequence (8), $P_1 = K \subset X_{\gamma_1}$, where $\gamma_1$ is the function value at any available feasible solution. Letting $Q_1 := P^*_1$ this implies $Q_1 = K^* = \pi(R^k_-)$. If $Q_1 \subset D^*$, or equivalently, if

$$\max\{\langle y, x\rangle : x \in D\} \le 1 \quad \forall y \in Q_1 \tag{11}$$

then $P_1^* = Q_1 \subset D^*$ and we are done((10) holds, so $\gamma_1$ is the optimal value). Otherwise, there is $y^1 \in Q_1 \setminus D^*$ and we can try to find a cut to exclude $y^1$ and determine a polyhedron $Q_2 \subset Q_1$ satisfying (9), i.e. such that $P_2 := Q_2^* \subset X_{\gamma_2}$ for some $\gamma_2 \leq \gamma_1$. If this can be done, then we can repeat the procedure with $P_2$ in place of $P_1$.

A key point in this scheme is: given at iteration $r$ a polyhedron $Q_r$, determine whether $Q_r \subset D^*$, and if there is $y^r \in Q_r \setminus D^*$ then construct a cut to exclude this point and determine a polyhedron $Q_{r+1}$ satisfying (9).

Let $\tilde{Q}_r \subset R^k$ be a polyhedron such that $Q_r = \pi(\tilde{Q}_r)$. For every $t \in \tilde{Q}_r$ we have $\pi(t) = \sum_{i=1}^k t_i c^i \in Q_r$. Define then

$$\mu(t) = \sup\{\sum_{i=1}^k t_i \langle c^i, x\rangle : x \in D\}. \tag{12}$$

and let $x(t)$ be a basic optimal solution of the linear program in (12). Denote the vertex set and the extreme direction set of $\tilde{Q}_r$ by $V_r, U_r$ respectively.

PROPOSITION 2. *1) If*

$$\mu(t) \leq 1\ \forall t \in V_r, \quad \mu(t) \leq 0\ \forall t \in U_r, \tag{13}$$

*then $Q_r \subset D^*$, hence $D \subset P_r$.*

*2) If $\mu(t^r) > 1$ for some $t^r \in V_r$ or $\mu(t^r) > 0$ for some $t^r \in U_r$ and $x^r := x(t^r)$ then $x^r \in D \setminus P_r$ and for any $\theta_r \geq 1$ and $\gamma_{r+1} \in f(D)$ such that $\gamma_{r+1} \leq \min\{\gamma_r, f(\theta_r x^r)\}$, the polyhedron*

$$\tilde{Q}_{r+1} = \tilde{Q}_r \cap \{t : \sum_{i=1}^k t_i \langle c^i, x^r\rangle \leq 1/\theta_r\} \tag{14}$$

*satisfies*

$$P_{r+1} := [\pi(\tilde{P}_{r+1})]^* = conv(P_r \cup \{\theta_r x^r\}) \subset X_{\gamma_{r+1}} \tag{15}$$

PROOF. For every $y \in Q_r$ we have $y = \pi(t)$ for some $t \in \tilde{Q}_r$. But every $t \in \tilde{Q}_r$ is of the form

$$t = \sum_{v \in V_r} \lambda_v v + \sum_{u \in U_r} \lambda_u u, \quad \lambda_v \geq 0, \lambda_u \geq 0, \sum_{v \in V_r} \lambda_v = 1.$$

Condition (13) then implies that

$$\langle \pi(t), x\rangle = \sum_{v\in V_r} \lambda_v \langle \pi(v), x\rangle + \sum_{u\in U_r} \lambda_u \langle \pi(u), x\rangle \le 1 \quad \forall x \in D.$$

Hence $\langle y, x\rangle \le 1\ \forall x \in D$, i.e. $Q_r \subset D^*$, or equivalently, $D \subset X_{\gamma_r}$. To prove the second part of the proposition, observe that the relation $\langle t^r, x^r\rangle = \mu(t^r) > 1$ (for $t^r \in V_r$) or $\langle t^r, x^r\rangle = \mu(t^r) > 0$ (for $t^r \in U_r$ ) implies that $x^r \notin [Q_r]^* = P_r$. Since $\tilde{Q}_{r+1} = \tilde{Q}_r \cap \{t : \langle \sum_{i=1}^{k} t_i c^i, \theta_r x^r\rangle \le 1\}$ it follows from well known properties of polars [9] that $Q^*_{r+1} =$ conv $(Q^*_r \cup \{\theta_r x^r\})$ hence (15) because $Q^*_{r+1} = P_{r+1}, Q^*_r = P_r$ and $P_r \subset X_{\gamma_r}$ while $f(\theta_r x^r) \ge \gamma_{r+1}$.

We can thus formulate:

ALGORITHM 2 ( Dual OA Decomposition Method).

0. Make sure that the origin 0 is a vertex of $D$. Set $\bar{x}^1$ = best feasible solution available, $\gamma_1 = f(\bar{x}^1), \tilde{Q}_1 = R^k_-,\ V_1 = V'_1 = \{0\},\ U_1 = U'_1 = \{-e^1, \ldots, -e^k\}$ where $e^i$ is the $i$-th unit vector of $R^k$). Set $r = 1$.

1. For every $t \in (V'_r \cup U'_r) \setminus \{0\}$ solve the linear program

$$\max\{\sum_{i=1}^{k} t_i \langle c^i, x\rangle : x \in D\}$$

to obtain $\mu(t)$. While solving this program, update $\bar{x}^r$ and $\gamma_r$ whenever possible.

2. Check condition (13).

a) If (13) holds then terminate: $\bar{x}^r$ is an optimal solution.

b) If $\mu(t^r) > 1$ for some $t^r \in V_r$ or $\mu(t^r) > 0$ for some $t^r \in U_r$ then let $x^r$ be a basic optimal solution of the linear program defining $\mu(t^r)$. Compute

$$\theta_r = \sup\{\theta : f(\theta x^r) \ge \gamma_r\} \tag{16}$$

and define

$$\tilde{Q}_{r+1} = \tilde{Q}_r \cap \{t : \sum_{i=1}^{k} t_i \langle c^i, x^r\rangle \le 1/\theta_r\}.$$

3. Compute the vertex set $V_{r+1}$ and the extreme direction set $U_{r+1}$ of $\tilde{Q}_{r+1}$, and let $V'_{k+1} = V_{r+1} \setminus V_r,\ U'_{r+1} = U_{r+1} \setminus U_r$. Set $r \leftarrow r + 1$ and go back to Step 1.

THEOREM 2. *Algorithm 2 terminates after finitely many steps.*

PROOF. From Proposition 2, $P_{r+1} = \text{conv}\,(P_r \cup \{\theta_r x^r\})$, while $x^{r+1} \notin P_{r+1}$, hence $x^{r+1}$ is distinct from all $x^1, \dots, x^r$. Since every $x^r$ is a vertex of $D$, the number of iterations is thus bounded by the number of vertices of $D$.

REMARK 4. If $f(x)$ is quadratic then the computation of $\theta_r$ in (16) merely reduces to solving a quadratic equation of one variable.

If $f(x)$ has the form

$$f(x) = \max\{dy : \; Mx + Ny \leq q, y \geq 0\}.$$

( as it occurs in the $(MP)$ reformulation of bilevel linear programs or max-min problems), then the computation of $\theta_r$ reduces to solving a linear program

$$\max_{\theta,y}\{\theta : \; dy \geq 0, M(\theta x^r) + Ny \leq q, y \geq 0, \theta \geq 0\}$$

(see [25]).

REMARK 5. To have an initial polyhedron more tightly approximating $X_{\gamma_1}$ and to avoid computing the extreme direction set $U_r$ of each current polyhedron $\tilde{Q}_r$ we can proceed as follows.

Assume that $\gamma_1 = f(\bar{x}^1) < f(0)$, hence $0 \in intX_{\gamma_1}$. Let $c^0 = -\sum_{i=1}^k c^i$, and for each $i = 0, 1, \dots, k$ compute the point $\hat{c}^i = \theta_i c^i$ such that $\theta_i > 0, f(\hat{c}^i) = \gamma_1$ (agree to take $\theta_i$ equal to an arbitrarily large positive number $\theta_\infty$ if $f(\theta c^i) > \gamma_1 \, \forall \theta > 0$).

PROPOSITION 3. *Let $L$ be the lineality space of $K$. If*

$$P_1 = S + L, \quad S = [\hat{c}^0, \hat{c}^1, \dots, \hat{c}^k],$$

*then*

$$P_1^* = S^* \cap L^\perp = \{y \in L^\perp : \langle \hat{c}^i, y\rangle \leq 1, \; i = 0, 1, \dots, k\}$$

*satisfies (9) and the polyhedron $\tilde{Q}_1$ such that $\pi(\tilde{Q}_1) = P_1^*$ is a $k$-simplex in $R^k$ determined by the system of inequalities*

$$\sum_{i=1}^k t_i \langle c^i, c^j\rangle \leq \frac{1}{\theta_j}, \; j = 0, 1 \dots, k.$$

*The vertex set of this simplex can be computed by solving, for each* $q = 0, 1, \ldots, k$ *the system*

$$\sum_{i=1}^{k} t_i \langle c^i, c^j \rangle = \frac{1}{\theta_j},\ j \in \{0, 1, \ldots, k\} \setminus \{q\}.$$

A proof of this proposition can be found in [17].

## 4. Discussion and computational experience

For simplicity of exposition we have assumed that rank $C = k$, i.e. the rows $c^1, \ldots, c^k$ of $C$ are linearly independent. This condition is satisfied in most cases of practical interest. However, if rank $C = h < k$, and $c^1, \ldots, c^h$ are linearly independent then the function $f(x)$ is monotonic with respect to $H = \{u : \langle c^i, u \rangle = 0, i = 1, \ldots h\}$ (which is actually the constancy space of $f(\bar{x})$), so with minor changes the above method can be applied, with $h, H$ replacing $k, K$. Alternatively, one can work in $R^k$ by considering the mapping $t \in R^k \mapsto \pi(t) = \sum_{i=1}^{k} t_i c^i$, which maps $R^k$ onto a $h$-dimensional subspace of $R^n$.

The two above algorithms have been coded in C language and run on a PC AT 80486 DX, with numeric processor. As test problems we used linear mutiplicative programming problems of the form (P2):

$$\min\{\prod_{i=1}^{k}(c^i x + d_i) : \ Ax \leq b,\ x \geq 0\}, \tag{17}$$

where the vectors $c^i$ and coefficients $a_{ij}$ of the matrix $A \in R^{m \times n}$ were randomly generated in the segment [-2,2], the vectors $b \in R^m$ and $d \in R^k$ were chosen so as to ensure that $\emptyset \neq D := \{x \in R^n : Ax \leq b, x \geq 0\} \subset \{x : c^i x + d_i > 0, i = 1, \ldots, k\}$. Since the matrix $C$ in (17) may be arbitrary, testing on problems (17) does provide information on how the algorithm works for general matrix $C$.

To compare the two algorithms, we solved 10 problems (17) with $n = 10, m = 10, k = 3$ and 10 other problems with $n = 15, m = 10, k = 3$ by each algorithm. The results reported in Table 1 clearly speak in favour of the dual OA algorithm.

The superiority of Algorithm 2 can be explained in part by the following observations:

1) The primal OA method constructs a sequence approximating the optimal solution from outside the set $C(D) \subset R^k$. As already mentioned, this requires the availability of a simple polytope $S_1$ contained in $C(\Omega)$ and containing $C(D)$. Therefore, it cannot be applied if the function $f(x)$ is not defined on certain points outside $D$. Furthermore, at every iteration, this method involves the computation of the values of the function $F(t)$ at the points $t \in V_r$, which in turn requires the computation of the matrix $Z$ in (2) unless the latter is readily available as in (P1) and (P2). By contrast, in the polyhedral annexation method the optimal solution is approximated by a sequence of feasible solutions $x(t)$, so there is no need to know the explicit value of $F(t)$ at any $t$ and this method works even if the function $f(x)$ is only defined on $D$.

2) The linear program $\text{LP}^*(t^r)$ in Step 2 of Algorithm 1 is often highly degenerated, and is not very easy to solve (cycling is more likely to occur than usually). It has $n$ constraints and for large $n$ its solution usually requires a large number of iterations (of the order of $2n$ iterations). The linear program in Step 1 of Algorithm 2 has the same constraint set as the original problem ($m$ constraints with usually $m < n$) and does not present any particular unpleasant feature not inherent to $D$ itself.

3) From computational experiments it seems that the primal OA method tends to generate more cuts than the dual OA, thus implying a faster growth of the vertex set of the current polytope.

Note, however, that for certain problems (such as (P1); see Remark 2) with a constraint set $D$ and the matrix $C$ having a special structure the linear program $\text{LP}^*(t^r)$ can be chosen so that it can be solved efficiently. For these problems the primal method may be quite competitive with the dual.

To see how Algorithm 2 works for different values of $m, n, k$ we also tested it on a number of problems (P2) with $k$ from 3 to 5, $m$ from 10 to 30 and $n$ from 50 to 100. The results, reported in Table 2, indicate that this algorithm is quite practical for the considered values of $k$, even if $m, n$ are fairly large.

Table 1

| Methods | Characteristics (average values) | $n = 10$ | $n=15$ |
|---|---|---|---|
| Primal OA | Number of iterations | 8.9 | 11.3 |
| | Number of linear programs | 61.3 | 90.7 |
| | Maximal number of stored vertices | 27.2 | 36.8 |
| | Time (seconds) | 3.76 | 7.77 |
| Dual OA | Number of iterations | 5.8 | 5.1 |
| | Number of linear programs | 25.1 | 33 |
| | Maximal number of stored vertices | 13.3 | 20.7 |
| | Time (seconds) | 2.79 | 4.5 |

Table 2

| Problem | $m$ | $n$ | $k$ | Number of iterations | Number of LP's | Max number stored vertices | Time (seconds) |
|---|---|---|---|---|---|---|---|
| 1 | 10 | 50 | 3 | 7 | 44 | 29 | 13 |
| 2 | 10 | 80 | 3 | 7 | 29 | 17 | 12 |
| 3 | 10 | 100 | 3 | 8 | 34 | 19 | 11 |
| 4 | 20 | 20 | 3 | 5 | 22 | 12 | 12 |
| 5 | 20 | 50 | 3 | 5 | 23 | 12 | 21 |
| 6 | 20 | 80 | 3 | 6 | 25 | 15 | 20 |
| 7 | 25 | 100 | 3 | 7 | 40 | 27 | 47 |
| 8 | 30 | 20 | 3 | 8 | 40 | 20 | 36 |
| 9 | 30 | 30 | 3 | 6 | 31 | 16 | 30 |
| 10 | 30 | 50 | 3 | 7 | 36 | 18 | 67 |
| 11 | 30 | 80 | 3 | 8 | 37 | 23 | 74 |
| 12 | 30 | 100 | 3 | 10 | 46 | 25 | 109 |
| 13 | 10 | 30 | 4 | 8 | 78 | 38 | 23 |
| 14 | 10 | 100 | 4 | 6 | 52 | 28 | 16 |
| 15 | 20 | 80 | 4 | 5 | 38 | 20 | 28 |
| 16 | 20 | 100 | 4 | 11 | 107 | 55 | 111 |
| 17 | 30 | 50 | 4 | 8 | 68 | 38 | 106 |
| 18 | 30 | 80 | 4 | 8 | 70 | 39 | 137 |
| 19 | 10 | 50 | 5 | 11 | 324 | 233 | 123 |
| 20 | 20 | 80 | 5 | 9 | 139 | 75 | 107 |
| 21 | 30 | 50 | 5 | 10 | 182 | 99 | 292 |
| 22 | 30 | 100 | 5 | 13 | 337 | 201 | 711 |

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